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BOUNDS FOR ODD *k*-PERFECT NUMBERS

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Abstract

Let $k \geq 2$ be an integer. A natural number n is called k-perfect if $\sigma(n) = kn$. For any integer $r \geq 1$, we prove that the number of odd k-perfect numbers with at most r distinct prime factors is bounded by $(k-1)4^{r^3}$.

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1. Introduction

Let $\sigma(n)$ be the sum of the positive divisors of a natural number n. For a rational number $k > 1$, if $\sigma(n) = kn$ then n is called *multiperfect* (or *k-perfect*). In the special case when $k = 2$, n is called a *perfect number*. No odd k-perfect numbers are known for any integer $k \geq 2$.

Let $\omega(n)$ denote the number of distinct prime factors of the positive integer n. In 1913, Dickson [4] proved that for any natural number r, there are only finitely many odd perfect numbers n with $\omega(n) \leq r$. Pomerance [9] gave an explicit upper bound of such n in 1977 and proved that

$$n \leq (4r)^{(4r)^{2^{r^2}}}.$$

Heath-Brown [5] later improved the bound to $n < 4^{4^r}$. Cook [3] refined this to $n < 195^{4^r/7}$. In 2003, Nielsen [6] improved the bound further and proved that for any integer $k \geq 2$, if n is an odd k-perfect number with r distinct prime factors then

$$n \leq 2^{4^r}. \tag{1}$$

Recently, Pollack [8] bounded the number of such n by modifying Wirsing's method [10]. He showed that for each positive integer r, the number of odd perfect numbers n with $\omega(n) \leq r$ is bounded by 4^{r^2}.

In this paper we will study the analogous problem for the odd k-perfect numbers. Our main result is the following theorem.

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THEOREM 1. *Let $k \geq 2$ be an integer. Then for any integer $r \geq 1$, the number of odd k-perfect numbers n with $\omega(n) \leq r$ is bounded by $(k-1)4^{r^3}$.*

REMARK. Our bound $(k-1)4^{r^3}$ is much smaller than the bound (1). In the case $k = 2$, Theorem 1 reduces to a weaker result than Pollack's bound 4^{r^2}, while the following Lemma 3 will yield Pollack's result.

2. Proofs

If n_1 is k_1-perfect, n_2 is k_2-perfect and $(n_1, n_2) = 1$, then n_1n_2 is k_1k_2-perfect. In view of this fact, we make the following definition.

DEFINITION 2. A multiperfect number n is called primitive if for any divisor d of n with $1 < d < n$ and $(d, n/d) = 1$,

$$d \nmid \sigma(d).$$

For example, if n is an odd perfect number, then n is primitive. To see why, we observe that if there is a divisor d of an odd perfect number n with $1 < d < n$, $d|\sigma(d)$, then $\sigma(d)/d \geq 2$. Therefore

$$2 = \frac{\sigma(n)}{n} = \sum_{m|n} \frac{m}{n} = \sum_{m|n} \frac{1}{m} > \sum_{m|d} \frac{1}{m} = \frac{\sigma(d)}{d} \geq 2,$$

which is absurd.

LEMMA 3. *Let $x \geq 1$ and $\alpha > 1$ be a positive rational number. Let I be the number of odd primitive α-perfect numbers $n \leq x$ with $\omega(n) \leq r$. Then*

$$I \leq 2.62 \frac{1}{\alpha^2 - 1} (\log x)^r.$$

If α is an integer, then

$$I \leq 0.02(\log x)^r.$$

PROOF. Let $n \leq x$ be an odd primitive α-perfect number and $\omega(n) = s \leq r$. We denote by $v_p(n)$ the highest power of prime p dividing n. Suppose that p_1 is the smallest positive prime factor of n and $e_1 := v_{p_1}(n)$. Let $\alpha = u/v$ with u, v positive integers. Then $\sigma(n) = \alpha n$ implies that

$$up_1^{e_1} \cdot \frac{n}{p_1^{e_1}} = v\sigma(p_1^{e_1})\sigma\left(\frac{n}{p_1^{e_1}}\right). \tag{2}$$

Since n is primitive,

$$\frac{n}{p_1^{e_1}} \nmid \sigma\left(\frac{n}{p_1^{e_1}}\right).$$

By (2), we deduce that

$$v\sigma(p_1^{e_1}) \nmid (up_1^{e_1}).$$

It follows that there exists at least one prime $p_2|(v\sigma(p_1^{e_1}))$ such that

$$v_{p_2}(v\sigma(p_1^{e_1})) > v_{p_2}(up_1^{e_1}). \tag{3}$$

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By (2) and (3) we know that

$$p_2 \Big| \frac{n}{p_1^{e_1}}.$$

We may assume without loss of generality that p_2 is the smallest such prime and denote $e_2 := \nu_{p_2}(n)$. Replacing $n/p_1^{e_1}$ by $n/(p_1^{e_1} p_2^{e_2})$ and iterating the argument above, we can determine prime p_3 with $p_3 | (n/p_1^{e_1} p_2^{e_2})$. Write $e_3 = \nu_{p_3}(n)$. Continuing in this way, we can obtain primes p_i and exponents $e_i = \nu_{p_i}(n)$, $i = 4, \ldots, s$. Hence the standard factorization of n can be written as follows:

$$n = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}.$$

We need to count the number of possibilities for such primes p_i and exponents e_j. The algorithm shows that p_2 is determined only by p_1 and e_1, p_3 is determined by p_1, p_2 and e_1, e_2, and for each $1 \le i \le s$, p_i is determined by $p_1, \ldots, p_{i-1}$ and $e_1, \ldots, e_{i-1}$. Therefore it is sufficient to count the number of possibilities of p_1 and $e_1, e_2, \ldots, e_s$. Cohen and Hendy (see [2, (20)]) proved that

$$p_1 < \frac{2s}{\alpha^2 - 1} + 2.$$

Hence the number of such prime p_1 is at most $s/(\alpha^2 - 1)$. Since $p_i^{e_i} \| n$, $n \le x$,

$$e_i \le \frac{\log x}{\log p_i}.$$

We conclude that the number of possibilities for the sequence $p_1, e_1, \ldots, e_s$ is bounded by

$$\frac{s}{\alpha^2 - 1} \prod_{i=1}^{s} \frac{\log x}{\log p_i}.$$

Recall that $1 \le s = \omega(n) \le r$. It follows that

$$\begin{aligned} I &\le r \cdot \frac{s}{\alpha^2 - 1} \prod_{i=1}^{s} \frac{\log x}{\log p_i} \\ &\le \frac{1}{\alpha^2 - 1} \cdot \frac{r^2}{\log p_1 \log p_2 \cdots \log p_r} (\log x)^r \\ &\le \frac{1}{\alpha^2 - 1} \cdot \frac{r^2}{\log q_1 \log q_2 \cdots \log q_r} (\log x)^r, \end{aligned} \tag{4}$$

where q_i is the ith odd prime, $q_1 = 3, q_2 = 5, \ldots$. For convenience, we denote

$$f(r) := \frac{r^2}{\log q_1 \log q_2 \cdots \log q_r}.$$

By simple calculation, we find that $f(r)$ is a decreasing function of r for $r \ge 3$. The maximal value of $f(r)$ is

$$f(3) = \frac{9}{\log 3 \log 5 \log 7} < 2.62. \tag{5}$$

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If $\alpha = 2$, then Nielsen [6] showed that $\omega(n) \geq 9$ for any odd perfect n. If $\alpha \geq 3$ is an integer and n is an odd α-perfect number, then Cohen and Hagis [1] proved that $\omega(n) \geq 11$. It follows that for any integer $\alpha \geq 2$,

$$f(r) \leq f(9) = \frac{81}{\log 3 \log 5 \cdots \log 29} < 0.043.$$

Therefore

$$I \leq \frac{1}{\alpha^2 - 1} f(r)(\log x)^r \leq \frac{1}{3} \times 0.043(\log x)^r < 0.02(\log x)^r. \tag{6}$$

Lemma 3 follows from (4), (5) and (6). □

Lemma 4. *Let $x \geq 1$, $r \geq 1$ and integer $k \geq 2$. The number of odd k-perfect $n \leq x$ with $\omega(n) \leq r$ is bounded by $(k-1)(\log x)^{(r^2+8r)/9}$.*

Proof. Suppose that $\sigma(n) = kn$. Let d_1 be the smallest positive divisor of n with $1 < d_1 < n$, $(d_1, n/d_1) = 1$ and $d_1 | \sigma(d_1)$. Then d_1 is an odd primitive multiperfect number. We write $\sigma(d_1) = k_1 d_1$ for some integer k_1. Similarly, let d_2 be the smallest positive divisor of n/d_1 with $1 < d_2 < n/d_1$, $(d_2, n/d_1 d_2) = 1$ and $d_2 | \sigma(d_2)$. Then d_2 is also an odd primitive multiperfect number. Write $\sigma(d_2) = k_2 d_2$ for some integer k_2. Iterating this argument, we can find divisors d_i of n and integers k_i, $i = 2, \ldots, j$, such that

$$d_i \,\Big|\, \frac{n}{d_1 \cdots d_{i-1}}, \quad \Big(d_i, \frac{n}{d_1 \cdots d_{i-1} d_i}\Big) = 1,$$

and $\sigma(d_i) = k_i d_i$ for some integer $k_i \geq 2$. We assume that the procedure stops at the $(j+1)$th step when $n/(d_1 d_2 \cdots d_j) = 1$ or $n/(d_1 d_2 \cdots d_j)$ is primitive and

$$\frac{n}{d_1 d_2 \cdots d_j} \,\Big\nmid\, \sigma\Big(\frac{n}{d_1 d_2 \cdots d_j}\Big).$$

Denote $d_{j+1} := n/(d_1 d_2 \cdots d_j)$. Then we have

$$n = d_1 d_2 \cdots d_j d_{j+1}. \tag{7}$$

If $d_{j+1} \neq 1$, then

$$\begin{aligned} kn &= \sigma(n) \\ &= \sigma(d_1 d_2 \cdots d_{j+1}) \\ &= \sigma(d_1)\sigma(d_2) \cdots \sigma(d_{j+1}) \\ &= k_1 d_1 k_2 d_2 \cdots k_j d_j \sigma(d_{j+1}). \end{aligned}$$

Therefore

$$\sigma(d_{j+1}) = \frac{k}{k_1 k_2 \cdots k_j} d_{j+1}.$$

It follows that d_{j+1} is $k/(k_1 k_2 \cdots k_j)$-perfect and $k_1 k_2 \cdots k_j \nmid k$. Since $k_1, \ldots, k_s$ are integers,

$$k_1 k_2 \cdots k_j \leq k - 1.$$

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In view of Lemma 3, the number of such d_{j+1} not exceeding x is bounded by

$$\begin{aligned}2.62\frac{1}{(\frac{k}{k_1k_2\cdots k_j})^2-1}(\log x)^r &\le 2.62\frac{1}{(\frac{k}{k-1})^2-1}(\log x)^r\\ &= 2.62\frac{(k-1)^2}{2k-1}(\log x)^r\\ &< 1.31(k-1)(\log x)^r.\end{aligned}$$

By the minimality of $d_1,\dots,d_j$, one can see that all $d_1,\dots,d_j$ are primitive. The results of Nielsen [7] and Cohen and Hagis [1] imply that $\omega(d_i)\ge 9$, $i=1,\dots,j$. Therefore

$$r\ge\omega(n)=\omega(d_{j+1})+\sum_{i=1}^{j}\omega(d_i)\ge 1+9j.$$

It follows that

$$j\le\frac{r-1}{9}.$$

By (7) and Lemma 3, the number of k-perfect numbers $n\le x$ with $\omega(n)\le r$ is at most

$$\begin{aligned}(0.02(\log x)^r)^j(1.31(k-1)(\log x)^r) &\le (0.02(\log x)^r)^{(r-1)/9}(1.31(k-1)(\log x)^r)\\ &\le\frac{k-1}{2}(\log x)^{(r^2+8r)/9}.\end{aligned}$$

If $d_{j+1}=1$, then $j\le r/9$ and the bound is

$$(0.02(\log x)^r)^j\le(0.02)^{r/9}(\log x)^{r^2/9}\le\frac{k-1}{2}(\log x)^{r^2/9}.$$

This completes the proof of Lemma 4. □

Proof of Theorem 1. Let $x=2^{4^r}$. Applying Lemma 4 and Nielson's bound (1), we deduce that the number of odd k-perfect numbers n with $\omega(n)\le r$ is at most

$$(k-1)(\log x)^{(r^2+8r)/9}<(k-1)(4^r)^{(r^2+8r)/9}=(k-1)4^{(r^3+8r^2)/9}\le(k-1)4^{r^3}.$$

This concludes the proof. □

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