Bull. Austral. Math. Soc.
Vol. 75 (2005) [373-379]

54C30, 54C10

THE DISCONTINUITY POINT SETS OF QUASI-CONTINUOUS FUNCTIONS

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It is proved that a subset E of a hereditarily normal topological space X is a discontinuity point set of some quasi-continuous function $f : X \to \mathbb{R}$ if and only if E is a countable union of sets $E_n = \overline{A}_n \cap \overline{B}_n$ where $\overline{A}_n \cap B_n = A_n \cap \overline{B}_n = \emptyset$.

1. Introduction

We deal with an old problem of the construction of a function with a given discontinuity point set. Separately continuous functions with given discontinuity point sets were constructed in [**6, 5, 1, 8**]. A complete characterisation of discontinuity point sets for separately continuous functions defined on metrisable spaces was obtained in [**9**]. An exact problem on construction of functions with given oscillations was considered in [**7, 2, 4, 10, 11, 12**]. Note that the domain space of all functions was assumed to be metrisable or "near" metrisable in all above papers.

In our note we characterise discontinuity point sets of quasi-continuous functions defined on a hereditarily normal space. We prove in Theorem 4.2 that a subset E of a hereditarily normal topological space X is a discontinuity point set of some quasi-continuous function $f : X \to \mathbb{R}$ if and only if E is a σ-junction set (that is, E is a countable union of sets $E_n = \overline{A}_n \cap \overline{B}_n$ where $\overline{A}_n \cap B_n = A_n \cap \overline{B}_n = \emptyset$).

In [**12**, Theorem 2.11.1] it was proved that a function $\varphi : X \to [0, +\infty]$ is the oscillation of some quasi-continuous function $f : X \to \mathbb{R}$ if and only if the sets $\varphi^{-1}\big([\varepsilon, +\infty]\big)$ are closed nowhere dense, for each $\varepsilon > 0$. But it is easy to see that for any meager F_σ-set $E \subseteq X$ there is a function $\varphi : X \to [0, +\infty]$ such that $E = \varphi^{-1}\big(\{0\}\big)$ and the sets $\varphi^{-1}\big([\varepsilon, +\infty]\big)$ are closed nowhere dense, for each $\varepsilon > 0$. So, a subset E of a metrisable space X is a discontinuity point set of some quasi-continuous function $f : X \to \mathbb{R}$ if and only if E is a meager F_σ-set.

But this fact may to be implied also from our the main result (Theorem 4.2). Indeed, Theorem 2.4 yields that in the metrisable case the σ-junction set is exactly the meager F_σ-set.

Moreover, in Theorem 4.3 we obtain that if X is a perfectly normal hereditary quasi-separable (or simple, hereditary separable) Frechet–Uryson space then a subset E of X

Received 16th October, 2006

https://doi.org/10.1017/S0004972700039307 Published online by Cambridge University Press

is a discontinuity point set of some quasi-continuous function $f : X \to \mathbb{R}$ if and only if E is a meager F_σ-set.

2. Junction sets

A subset E of a topological space X we call *a junction set* if there are subsets A and B of X such that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ and $E = \overline{A} \cap \overline{B}$.

Proposition 2.1. *Let E be a junction set of a topological space X. Then E is closed nowhere dense in X.*

Proof: Let A and B be subsets of X such that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ and $E = \overline{A} \cap \overline{B}$. Then obviously E is closed. Thus for nowhere density of E it is enough to show that $\operatorname{int} E = \emptyset$. Since $\operatorname{int} \overline{A} \cap B \subseteq \overline{A} \cap B = \emptyset$ then $\operatorname{int} \overline{A} \cap \overline{B} = \emptyset$. Hence $\operatorname{int} E = \operatorname{int} \overline{A} \cap \operatorname{int} \overline{B}$ $\subseteq \operatorname{int} \overline{A} \cap \overline{B} = \emptyset$. ∎

Under some additional assumptions on X (say, X is metrisable) we shall prove that the junction sets are exactly the closed nowhere dense sets.

Recall some definitions. A family $(A_i)_{i \in I}$ of subsets of a topological space X is said to be *discrete* (*locally finite*) if any point of X has a neighbourhood which intersects with at most one (a finite number) of A_i's. A system $\mathcal{A} \subseteq 2^X$ is calls *σ-locally finite* if $\mathcal{A}$ is a countable union of locally finite systems. A system $\mathcal{A}$ is said to be a *net* if for any point $x \in X$ and neighbourhood U of x there is $A \in \mathcal{A}$ with $x \in A \subseteq U$. A subset $S \subseteq X$ we call *strongly discrete* if there exists an discrete family of open sets U_s, $s \in S$, such that $s \in U_s$ for each $s \in S$. It is easy to see that any closed discrete subset of a paracompact space is strongly discrete. A subset $S \subseteq X$ will be called *strongly σ-discrete* if S is a countable union of strongly discrete sets. A subset $E \subseteq X$ we shall call *strongly $\overline{\sigma}$-discrete* if it contains a dense strongly σ-discrete subset. A topological space X will be called *hereditarily quasi-separable* if every subset of X is strongly $\overline{\sigma}$-discrete in X.

Proposition 2.2. *Let X be a paracompact space with a σ-locally finite net $\mathcal{A}$. Then X is hereditary quasi-separable.*

Proof: Suppose $E \subseteq X$. We set $\mathcal{A}_E = \{A \in \mathcal{A} : A \cap E = \emptyset\}$ and for each $A \in \mathcal{A}_E$ choose an $s_A \in A \cap E$. Since $\mathcal{A}$ is a net then the set $S = \{s_A : A \in \mathcal{A}_E\}$ is dense in E. By σ-locally finiteness of $\mathcal{A}$ we obtain that S is a countable union of closed discrete sets. Hence S is strongly σ-discrete, because X is paracompact. Thus, E is strongly $\overline{\sigma}$-discrete. ∎

Recall that a topological space X is said to be a *Frechet–Uryson space* if for any $A \subseteq X$ and $x \in \overline{A}$ there exists a sequence of points $x_n \in A$ such that $x_n \to x$. An $E \subseteq X$ is called a *$\overline{G}_\delta$-set* if there is a sequence of open sets $G_n \supseteq E$ with $E = \bigcap_{n=1}^{\infty} \overline{G}_n$

Theorem 2.3. *Let X be a Frechet–Uryson space and E be a closed nowhere dense strongly $\overline{\sigma}$-discrete $\overline{G}_\delta$-subset of X. Then E is a junction set in X.*

PROOF: Let S be a dense strongly σ-discrete subset of E and S_n be strongly discrete sets with $S = \bigcup_{n=1}^{\infty} S_n$. Without lost of generality we may assume that S_n are disjoint. Fix a decreasing sequence of open subsets $G_n \supseteq E$ with $E = \bigcap_{n=1}^{\infty} \overline{G}_n$. For arbitrary $n \in \mathbb{N}$ pick an open discrete family $(U(s) : s \in S)$ so that $s \in U(s) \subseteq G_n$ for $s \in S_n$. We construct a family $(t_k(s) : k \in \mathbb{N}, s \in S)$ so that for arbitrary $k, k', k'' \in \mathbb{N}$ and $s, s', s'' \in S$ we have

(1) $t_k(s) \in U \setminus E$;

(2) $t_{k'}(s') \neq t_{k''}(s'')$ if $(k', s') \neq (k'', s'')$;

(3) $t_k(s) \to s$ as $k \to \infty$.

Suppose that $t_k(s)$ are already constructed for some $m \in \mathbb{N}$ and each $s \in \bigcup_{n<m} S_n$ and $k \in \mathbb{N}$ so that (1)–(3) hold. Let us construct $t_k(s)$ for $s \in S_m$ and $k \in \mathbb{N}$. Put $T_n = \{t_k(s) : k \in \mathbb{N}, s \in S_n\}$ for $n < m$. Since the family $(U(s) : s \in S_n)$ is discrete and $S_n \cap S_m = \emptyset$ for $n < m$ then (1) and (3) imply that $S_m \cap \overline{T}_n = \emptyset$ for $n < m$. Now fix $s \in S_m$. Since X is a Frechet–Uryson space and

$$s \in \overline{U(s) \setminus \Bigl(E \cup \bigcup_{n<m} T_n\Bigr)}$$

then there is a sequence $t_k(s) \in U(s) \setminus (E \cup \bigcup_{n<m} T_n)$ such that $t_k(s) \to s$. Obviously, (1)–(3) are valid.

Now set

$$\begin{aligned} A_n &= \{t_{2k-1}(s) : k \in \mathbb{N}, s \in S_n\}, \\ B_n &= \{t_{2k}(s) : k \in \mathbb{N}, s \in S_n\}, \\ A &= \bigcup_{n=1}^{\infty} A_n \quad \text{and} \\ B &= \bigcup_{n=1}^{\infty} B_n. \end{aligned}$$

Since $(U(s) : s \in S_n)$ are discrete then $\overline{A}_n = A_n \cap S_n$ and $\overline{B}_n = B_n \cap S_n$. Taking into account that $U(s) \subseteq G_n$ for $s \in S_n$ we obtain that $A_n, B_n \subseteq G_n$. Then for each $m \in \mathbb{N}$ one has

$$\overline{A} = \bigcup_{n>m} \overline{A}_n \cup \overline{\bigcup_{n\geqslant m} A_n} \subseteq \bigcup_{n<m} (A_n \cup S_n) \cup \overline{G}_m \subseteq A \cup \overline{G}_n.$$

Recall that $\bigcap_{m=1}^{\infty} \overline{G}_m = E$. Hence $\overline{A} \subseteq A \cup E$. Besides,

$$E = \overline{S} = \overline{\bigcup_{n=1}^{\infty} S_n} \subseteq \overline{\bigcup_{n=1}^{\infty} \overline{A}_n} = \overline{A}.$$

Hence $\overline{A} = A \cup E$. Analogously, $\overline{B} = B \cup E$. Thus, $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ and $E = \overline{A} \cap \overline{B}$. Therefore E is a junction set. ∎

By the above results we deduce the following.

Theorem 2.4. *Suppose that for a Frechet–Uryson space X at least, one of the following conditions holds*

(i) *X is a hereditarily separable perfectly normal;*

(ii) *X is a hereditarily quasi-separable perfectly normal;*

(iii) *X is a regular space with a countable net;*

(iv) *X is a paracompact with a σ-locally finite net;*

(v) *X is metrisable.*

Then a subset E of X is a junction set in X if and only if E is closed nowhere dense.

Remark that the assumption X be a Frechet–Uryson space is essential in the above two theorems. Indeed, consider the subspace $X = \mathbb{N} \cup \{\varphi\}$ of the Čech-Stone compactification $\beta\mathbb{N}$ of the countable discrete space $\mathbb{N}$ where $\varphi \in \beta\mathbb{N} \setminus \mathbb{N}$. Then X has conditions (i)–(iv) of Theorem 2.4. But nevertheless, $E = \{\varphi\}$ is a closed nowhere dense set in X which is not junction.

3. A proof that the discontinuity point set of a quasi-continuous function is a σ-junction set

We recall that a function $f : X \to Y$ is called *quasi-continuous* if $f^{-1}(G) \subseteq \overline{\operatorname{int} f^{-1}(G)}$ for each open $G \subseteq Y$. A subset E of a topological space X we call a *σ-junction set* if there is a sequence of junction sets E_n with $E = \bigcup\limits_{n=1}^{\infty} E_n$.

Theorem 3.1. *Let X be a topological space, Y be a separable metrisable space and $f : X \to Y$ be a quasi-continuous function. Then the discontinuity point set $D(f)$ of the function f is σ-junction set.*

Proof: Let $\mathcal{V}$ be a countable base of Y. Consider a countable system

$$\{(V,W) \in \mathcal{V} \times \mathcal{V} : \overline{V} \subseteq W\} = \{(V_n, W_n) : n \in \mathbb{N}\}.$$

Set $A_n = \operatorname{int} f^{-1}(V_n)$ and $B_n = \operatorname{int} f^{-1}(Y \setminus \overline{W}_n)$. Since $f(A_n) \subseteq V_n$ and $f(B_n) \subseteq Y \setminus W_n \subseteq Y \setminus V_n$ then $A_n \cap B_n = \emptyset$. Taking into account that A_n and B_n are open one obtains $\overline{A}_n \cap B_n = A_n \cap \overline{B}_n = \emptyset$. Thus, $E_n = \overline{A}_n \cap \overline{B}_n$ are junction sets. Besides, $\overline{f(A_n)} \cap \overline{f(B_n)} = \emptyset$ since $\overline{V}_n \subseteq W_n$. Therefore $E_n \subseteq D(f)$. It is left to prove that $D(f) \subseteq \bigcup\limits_{n=1}^{\infty} E_n$.

Given $x_0 \in D(f)$, then there exists a neighbourhood V of $f(x_0)$ with $x_0 \in \overline{f^{-1}(Y \setminus V)}$. Since $\mathcal{V}$ is a base of Y then there is $n \in \mathbb{N}$ such that $f(x_0) \in V_n$ and

$\overline{W}_n \subseteq V$. But f is quasi-continuous. Then $f^{-1}(V_n) \subseteq \overline{A}_n$ and

$$f^{-1}(Y \setminus V) \subseteq f^{-1}(Y \setminus W_n) \subseteq \overline{A}_n.$$

Thus,

$$x_0 \in f^{-1}(V_n) \cap \overline{f^{-1}(Y \setminus V)} \subseteq \overline{A}_n \cap \overline{B}_n = E_n$$

and we obtain $D(f) \subseteq \bigcup_{n=1}^{\infty} E_n$. ∎

4. A construction of a quasi-continuous function with a given discontinuity point set

Recall that a function $f : X \to \overline{\mathbb{R}} = [-\infty, +\infty]$ is said to be *upper* (*lower*) *semi-continuous* if the set $f^{-1}\big([-\infty, y)\big)$ (respectively, $f^{-1}\big((y, +\infty]\big)$) is open for each $y \in \mathbb{R}$. By an *upper* (*lower*) *Baire function* of f we understand the function f^* (respectively, f_*) defined by

$$f^*(x) = \inf_{x \in \operatorname{int} U} \sup f(U) \quad \text{and} \quad f_*(x) = \sup_{x \in \operatorname{int} U} \inf f(U) \quad \text{for} \quad x \in X.$$

An upper (lower) Baire function is upper (lower) semi-continuous.

Proposition 4.1. *Let X be a topological spaces and $f : X \to \overline{\mathbb{R}}$ be an lower (upper) semi-continuous function. Then f^* (respectively, f_*) is quasi-continuous.*

Proof: Let f be, say, lower semi-continuous. Since f^* is upper semi-continuous then it remains to prove that f^* is lower quasi-continuous (that is, for given $x_0 \in X$, $y_0 < f^*(x_0)$ and a neighbourhood U_0 of x_0 there is an open nonempty set $U \subseteq U_0$ such that $f^*(x) > y_0$ when $x \in U$). Suppose $x_0 \in X$, $y_0 < f^*(x_0)$ and let U_0 be an open neighbourhood of x_0. Since $y_0 < f^*(x_0) \leqslant \sup f(U_0)$ then there is an $x_1 \in U_0$ with $f(x_1) > y_0$. But f is lower semi-continuous. Then there exists an open neighbourhood $U \subseteq U_0$ of x_1 such that $f(x) > y_0$ on U. Finally, $f^*(x) \geqslant f(x) > y_0$ on U. ∎

Theorem 4.2. *Let X be a hereditarily normal space and $E \subseteq X$. Then E is a discontinuity point set of some quasi-continuous function $f : X \to \mathbb{R}$ if and only if E is a σ-junction set.*

Proof: Let E_n be junction sets such that $E = \bigcup_{n=1}^{\infty} E_n$. Choose sets A_n and B_n such that $\overline{A}_n \cap B_n = A_n \cap \overline{B}_n = \emptyset$ and $E_n = \overline{A}_n \cap \overline{B}_n$. Fix $n \in \mathbb{N}$. Since $X \setminus E_n$ is normal, $A_n \cap E_n = B_n \cap E_n = \emptyset$ and $(\overline{A}_n \cap \overline{B}_n) \setminus E_n = \emptyset$ then by the Uryson lemma [3] there exists a continuous function $\widetilde{f}_n : X \setminus E_n \to [0, 1]$ such that $\widetilde{f}_n(x) = 0$ on A_n and $\widetilde{f}_n(x) = 1$ on B_n. Define $f_n(x) = 0$ for $x \in E_n$ and $f_n(x) = \widetilde{f}_n(x)$ for $x \in X \setminus E_n$. Since $\overline{f(A_n)} \cap \overline{f(B_n)} = \emptyset$ and $E_n = \overline{A}_n \cap \overline{B}_n$ then $E_n \subseteq D(f_n)$. But $f|_{X \setminus E_n}$ is continuous. Thus $E_n = D(f_n)$. Besides, the points $x \in E_n = D(f_n)$ are the minimum points of f_n. Therefore f_n is lower semi-continuous.

Define $g = \sum_{n=1}^{\infty} (1/4^n) f_n$ and $f = g^*$. Since g is lower semi-continuous then f is quasi-continuous by Proposition 4.1. Let us prove that $D(f) = E$. First, if all f_n are continuous at a given point $x_0 \in X$ then so is g, and hence f. Thus, $E = \bigcup_{n=1}^{\infty} D(f_n) \supseteq D(f)$.

Given $x_0 \in E$, we show that $x_0 \in D(f)$. Let n be the least integer such that $x_0 \in E_n$. We set $\varepsilon = 1/4^n$, $u = \sum_{k<n} (1/4^k) f_k$ and $v = \sum_{k>n} (1/4^k) f_k$. Since $x_0 \notin E_k = D(f_k)$ for $k < n$ then u is continuous at x_0. Therefore there is an open neighbourhood U of x_0 such that $|u(x) - u(x_0)| < \varepsilon/9$ on for $x \in U$. Put $G = U \cap \operatorname{int} f_n^{-1}\big([0, 1/9)\big)$ and $H = U \cap \operatorname{int} f_n^{-1}\big((8/9, 1]\big)$. Since f_n is continuous on $X \setminus E_n \supseteq A_n, B_n$ and $f_n(x) = 0$ on A_n and $f_n(x) = 1$ on B_n then $A_n \cap U \subseteq G$ and $B_n \cap U \subseteq H$. Hence $x_0 \in \overline{G} \cap \overline{H}$. Besides,

$$u(x) \leqslant \sum_{k>n} \frac{1}{4^k} = \frac{1/4^{n+1}}{1 - (1/4)} = \frac{1}{3 \cdot 4^n} = \frac{\varepsilon}{3}$$

for $x \in X$. Thus for $x \in G$ one has

$$g(x) = u(x) + \varepsilon f_n(x) + v(x) < u(x_0) + \frac{\varepsilon}{9} + \frac{\varepsilon}{9} + \frac{\varepsilon}{3} = u(x_0) + \frac{5\varepsilon}{9}.$$

For $y \in H$ we obtain the converse inequality

$$g(y) \geqslant u(y) + \varepsilon f_n(y) > u(x_0) - \frac{\varepsilon}{9} + \frac{8\varepsilon}{9} = u(x_0) + \frac{7\varepsilon}{9}.$$

But since $f = g^*$ and the sets G and H are open then for any $x \in G$ and $y \in H$

$$f(x) \leqslant u(x_0) + \frac{5\varepsilon}{9} < u(x_0) + \frac{7\varepsilon}{9} \leqslant f(y).$$

From $x_0 \in \overline{G} \cap \overline{H}$ we finally obtain that f is discontinuous at x_0. ∎

Theorems 2.4 and 4.2 together imply the following.

Theorem 4.3. *Let for a Frechet–Uryson space X at least, one of the following conditions holds*

(i) *X is a hereditarily separable perfectly normal;*

(ii) *X is a hereditarily quasi-separable perfectly normal;*

(iii) *X is a regular space with a countable net;*

(iv) *X is a paracompact with a σ-locally finite net;*

(v) *X is metrisable.*

Then a subset E of X is the discontinuity point sets of some quasi-continuous function $f : X \to \mathbb{R}$ if and only if E is a meager F_σ-set.

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