1

Localisation

Contents

A localisation of a category is obtained by formally inverting a specific class of morphisms. Forming localisations is one of the standard techniques in algebra; it is used throughout this book. The calculus of fractions helps to describe the morphisms of a localised category.

1.1 Localisation

We introduce the concept of localisation for categories. A localisation is obtained by formally inverting a specific class of morphisms.

Localisation of Categories

Let $\mathcal{C}$ be a category and let $S \subseteq \operatorname{Mor}\mathcal{C}$ be a class of morphisms in $\mathcal{C}$. The *localisation* of $\mathcal{C}$ with respect to S is a category $\mathcal{C}[S^{-1}]$ together with a functor $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ satisfying the following.

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(L1) For every $\sigma \in S$, the morphism $Q\sigma$ is invertible.

(L2) For every functor $F\colon \mathcal{C} \to \mathcal{D}$ such that $F\sigma$ is invertible for all $\sigma \in S$, there exists a unique functor $\bar{F}\colon \mathcal{C}[S^{-1}] \to \mathcal{D}$ such that $F = \bar{F} \circ Q$.

The localisation solves a universal problem and is therefore unique, up to a unique isomorphism. We sketch the construction of $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$. At this stage, we ignore set-theoretic issues, that is, the morphisms between two objects of $\mathcal{C}[S^{-1}]$ need not form a set. However, later on we pay attention and formulate criteria such that $\mathcal{C}[S^{-1}]$ is locally small. We put $\operatorname{Ob}\mathcal{C}[S^{-1}] = \operatorname{Ob}\mathcal{C}$. To define the morphisms of $\mathcal{C}[S^{-1}]$, consider the quiver with class of vertices $\operatorname{Ob}\mathcal{C}$ and class of arrows the disjoint union $(\operatorname{Mor}\mathcal{C}) \sqcup S^-$, where

$$S^- = \{\sigma^-\colon Y \to X \mid S \ni \sigma\colon X \to Y\}.$$

Let $\mathcal{P}$ be the category of paths in this quiver, that is, finite sequences of composable arrows, together with the obvious composition given by concatenation and denoted by $\circ_{\mathcal{P}}$. We define $\operatorname{Mor}\mathcal{C}[S^{-1}]$ as the quotient of $\mathcal{P}$ modulo the following relations:

(1) $\beta \circ_{\mathcal{P}} \alpha = \beta \circ \alpha$ for all composable morphisms $\alpha, \beta \in \operatorname{Mor}\mathcal{C}$,

(2) $\mathrm{id}_{\mathcal{P}} X = \mathrm{id}_{\mathcal{C}} X$ for all $X \in \operatorname{Ob}\mathcal{C}$,

(3) $\sigma^- \circ_{\mathcal{P}} \sigma = \mathrm{id}_{\mathcal{P}} X$ and $\sigma \circ_{\mathcal{P}} \sigma^- = \mathrm{id}_{\mathcal{P}} Y$ for all $\sigma\colon X \to Y$ in S.

The composition of morphisms in $\mathcal{P}$ induces the composition in $\mathcal{C}[S^{-1}]$. The functor Q is the identity on objects and on $\operatorname{Mor}\mathcal{C}$ the composite

$$\operatorname{Mor}\mathcal{C} \xrightarrow{\text{inc}} (\operatorname{Mor}\mathcal{C}) \sqcup S^- \xrightarrow{\text{inc}} \mathcal{P} \xrightarrow{\text{can}} \operatorname{Mor}\mathcal{C}[S^{-1}].$$

The following is a more precise formulation of the properties of the canonical functor $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$.

Lemma 1.1.1. *For any category $\mathcal{D}$, the functor*

$$\mathcal{H}om(\mathcal{C}[S^{-1}], \mathcal{D}) \longrightarrow \mathcal{H}om(\mathcal{C}, \mathcal{D}), \quad F \mapsto F \circ Q,$$

is fully faithful and identifies $\mathcal{H}om(\mathcal{C}[S^{-1}], \mathcal{D})$ with the full subcategory of functors in $\mathcal{H}om(\mathcal{C}, \mathcal{D})$ that make all morphisms in S invertible. □

Local Objects

Let $\mathcal{C}$ be a category and $S \subseteq \operatorname{Mor}\mathcal{C}$. An object Y in $\mathcal{C}$ is called *S-local* (or *S-closed*, or *S-orthogonal*) if the map $\operatorname{Hom}_{\mathcal{C}}(\sigma, Y)$ is bijective for all $\sigma \in S$. We denote by $S^{\perp}$ the full subcategory of S-local objects in $\mathcal{C}$.

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Lemma 1.1.2. *An object Y in $\mathcal{C}$ is S-local if and only if the canonical map*

$$p_{X,Y} \colon \operatorname{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \operatorname{Hom}_{\mathcal{C}[S^{-1}]}(X,Y)$$

is bijective for all $X \in \mathcal{C}$.

Proof If Y is S-local, then $\operatorname{Hom}_{\mathcal{C}}(-,Y) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ induces a functor $\operatorname{Hom}_{\mathcal{C}}(-,Y) \colon \mathcal{C}[S^{-1}]^{\mathrm{op}} \to \mathrm{Set}$. Yoneda's lemma yields a morphism

$$\operatorname{Hom}_{\mathcal{C}[S^{-1}]}(-,Y) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(-,Y)$$

corresponding to id_Y, and it is straightforward to check that this is an inverse for the canonical morphism $\operatorname{Hom}_{\mathcal{C}}(-,Y) \to \operatorname{Hom}_{\mathcal{C}[S^{-1}]}(-,Y)$.

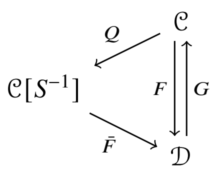
Now assume that $p_{X,Y}$ is bijective for all $X \in \mathcal{C}$. Then $\operatorname{Hom}_{\mathcal{C}}(\sigma,Y)$ is bijective for all $\sigma \in S$ since $\operatorname{Hom}_{\mathcal{C}[S^{-1}]}(\sigma,Y)$ is bijective. □

Adjoint Functors

Let $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$ be a pair of functors and assume that F is left adjoint to G. We set

$$S = S(F) = \{\sigma \in \operatorname{Mor}\mathcal{C} \mid F\sigma \text{ is invertible}\}$$

and obtain the following diagram



with $F = \bar{F} \circ Q$.

Proposition 1.1.3. *The following statements are equivalent.*

(1) *The functor G is fully faithful.*
(2) *The counit $FG(X) \to X$ is invertible for every object $X \in \mathcal{D}$.*
(3) *The functor F induces an equivalence $\bar{F} \colon \mathcal{C}[S^{-1}] \xrightarrow{\sim} \mathcal{D}$.*

Moreover, in that case G induces an equivalence $\mathcal{D} \xrightarrow{\sim} S^{\perp}$ with quasi-inverse $S^{\perp} \hookrightarrow \mathcal{C} \xrightarrow{F} \mathcal{D}$.

Proof We denote by $\eta \colon \mathrm{id}_{\mathcal{C}} \to GF$ the unit and by $\varepsilon \colon FG \to \mathrm{id}_{\mathcal{D}}$ the counit of the adjunction. Note that the composite $F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F$ equals id_F, and $G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G$ equals id_G; this characterises the fact that (F,G) is an adjoint pair.

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(1) ⇔ (2): The counit $\varepsilon_X : FG(X) \to X$ induces for all $Y \in \mathcal{D}$ a natural map

$$\operatorname{Hom}_{\mathcal{D}}(X,Y) \to \operatorname{Hom}_{\mathcal{C}}(GX,GY) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}}(FG(X),Y),$$

which is a bijection if and only if ε_X is an isomorphism, by Yoneda's lemma.

(2) ⇒ (3): We claim that QG is a quasi-inverse of $\bar{F}$. Clearly, $\bar{F}(QG) = FG \cong \mathrm{id}_{\mathcal{D}}$. On the other hand, $F\eta$ is invertible, since εF is invertible. Thus $Q\eta : Q \to QGF$ is invertible, and therefore

$$(QG)\bar{F}Q = QGF \cong Q \cong \mathrm{id}_{\mathcal{C}[S^{-1}]}\, Q.$$

Then the defining property of Q implies $(QG)\bar{F} \cong \mathrm{id}_{\mathcal{C}[S^{-1}]}$.

(3) ⇒ (2): If $\bar{F}$ is an equivalence, then composition with F induces a fully faithful functor $\mathcal{H}om(\mathcal{D},\mathcal{X}) \to \mathcal{H}om(\mathcal{C},\mathcal{X})$ for any category $\mathcal{X}$, by Lemma 1.1.1. For $\mathcal{X} = \mathcal{D}$, this implies that there is $\eta' : \mathrm{id}_{\mathcal{D}} \to FG$ such that $F\eta = \eta' F$. We claim that $(\mathrm{id}_{\mathcal{D}}, FG)$ is an adjoint pair with unit η' and counit ε. Clearly, then FG is an equivalence and ε is an isomorphism.

From the fact that $F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F$ equals id_F it follows that $(\varepsilon \circ \eta')F = \varepsilon F \circ \eta' F = \mathrm{id}_F$, and therefore $\varepsilon\eta' = \mathrm{id}_{\mathrm{id}_{\mathcal{D}}}$. On the other hand, the fact that $G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G$ equals id_G implies by applying F that $FG\varepsilon \circ \eta' FG = FG\varepsilon \circ F\eta G = \mathrm{id}_{FG}$. Thus $(\mathrm{id}_{\mathcal{D}}, FG)$ is an adjoint pair.

Now suppose that the equivalent conditions hold. In order to show that G induces an equivalence $\mathcal{D} \xrightarrow{\sim} S^{\perp}$, we need to show that the essential image of G equals $S^{\perp}$. The inclusion $\operatorname{Im} G \subseteq S^{\perp}$ is clear. If $X \in S^{\perp}$, then $\operatorname{Hom}_{\mathcal{C}}(\eta_X, X)$ is bijective since $\eta_X \in S$. This gives an inverse of η_X, so $X \cong GF(X)$. □

Example 1.1.4. Let $\mathcal{C}$ be an additive category and consider the category $\operatorname{mod} \mathcal{C}$ of functors $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ that fit into an exact sequence

$$\operatorname{Hom}_{\mathcal{C}}(-,X) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(-,Y) \longrightarrow F \longrightarrow 0.$$

Then the *Yoneda functor*

$$\mathcal{C} \longrightarrow \operatorname{mod} \mathcal{C}, \qquad X \mapsto h_X := \operatorname{Hom}_{\mathcal{C}}(-,X)$$

admits a left adjoint if and only if every morphism in $\mathcal{C}$ admits a cokernel. The left adjoint sends $F = \operatorname{Coker} h_\phi$ in $\operatorname{mod} \mathcal{C}$ (given by a morphism ϕ in $\mathcal{C}$) to $\operatorname{Coker} \phi$.

Proof Suppose that $\mathcal{C}$ has cokernels. For $C \in \mathcal{C}$ we have

$$\begin{aligned} \operatorname{Hom}(\operatorname{Coker} h_\phi, h_C) &\cong \operatorname{Ker} \operatorname{Hom}(h_\phi, h_C) \\ &\cong \operatorname{Ker} \operatorname{Hom}_{\mathcal{C}}(\phi, C) \\ &\cong \operatorname{Hom}_{\mathcal{C}}(\operatorname{Coker} \phi, C). \end{aligned}$$

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This follows from Yoneda's lemma and yields the adjointness. The converse follows from the fact that a left adjoint preserves cokernels. □

We introduce the following terminology. A diagram of additive functors

$$\mathcal{C}' \underset{E_\rho}{\overset{E}{\rightleftarrows}} \mathcal{C} \underset{F_\rho}{\overset{F}{\rightleftarrows}} \mathcal{C}''$$

is called a *localisation sequence* if

(LS1) (E, E_ρ) and (F, F_ρ) are adjoint pairs,
(LS2) E and F_ρ are fully faithful,
(LS3) $\operatorname{Im} E = \operatorname{Ker} F$ (equivalently, $EE_\rho(X) \xrightarrow{\sim} X$ if and only if $F(X) = 0$).

The dual notion is called a *colocalisation sequence* and is given by a diagram of additive functors

$$\mathcal{C}' \underset{E}{\overset{E_\lambda}{\leftrightarrows}} \mathcal{C} \underset{F}{\overset{F_\lambda}{\leftrightarrows}} \mathcal{C}''$$

satisfying the dual properties.

The above Example 1.1.4 gives rise to a localisation sequence

$$\operatorname{Ker} F \rightleftarrows \operatorname{mod} \mathcal{C} \underset{F_\rho}{\overset{F}{\rightleftarrows}} \mathcal{C}$$

provided that $\mathcal{C}$ is abelian. In that case the functor F is exact and the right adjoint of the inclusion $\operatorname{Ker} F \to \operatorname{mod} \mathcal{C}$ sends an object X to the kernel of the unit $X \to F_\rho F(X)$.

Localisation Functors

Suppose that the canonical functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$ corresponding to a class of morphisms $S \subseteq \operatorname{Mor} \mathcal{C}$ admits a right adjoint. Then the above Proposition 1.1.3 suggests we think of localisation as an endofunctor $\mathcal{C} \to \mathcal{C}$. The following definition makes this idea precise. Moreover, we see that both ways of thinking about localisation are equivalent.

A functor $L\colon \mathcal{C} \to \mathcal{C}$ is called a *localisation functor* if there exists a morphism $\eta\colon \mathrm{id}_\mathcal{C} \to L$ such that $L\eta\colon L \to L^2$ is an isomorphism and $L\eta = \eta L$. Note that we only require the existence of η; the actual morphism is not part of the definition of L. However, we will see that η is determined by L, up to a unique isomorphism $L \to L$.

Proposition 1.1.5. *Let $L\colon \mathcal{C} \to \mathcal{C}$ be a functor and $\eta\colon \mathrm{id}_\mathcal{C} \to L$ a morphism. Then the following are equivalent.*

(1) *$L\eta\colon L \to L^2$ is an isomorphism and $L\eta = \eta L$.*

(2) *There exists a functor $F\colon \mathcal{C} \to \mathcal{D}$ and a fully faithful right adjoint $G\colon \mathcal{D} \to \mathcal{C}$ such that $L = G \circ F$ and $\eta\colon \operatorname{id}_{\mathcal{C}} \to G \circ F$ is the unit of the adjunction.*

Proof (1) $\Rightarrow$ (2): Let $\mathcal{D}$ denote the essential image of L, that is, the full subcategory of $\mathcal{C}$ consisting of objects isomorphic to LX for some $X \in \mathcal{C}$. Note that $X \in \mathcal{D}$ if and only if η_X is invertible. In this case let $\theta_X\colon LX \to X$ denote the inverse of η_X. Define $F\colon \mathcal{C} \to \mathcal{D}$ by $FX = LX$ and let $G\colon \mathcal{D} \to \mathcal{C}$ be the inclusion. We claim that F and G form an adjoint pair. To this end, one checks that the maps

$$\operatorname{Hom}_{\mathcal{D}}(FX, Y) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(X, GY), \quad \alpha \mapsto G\alpha \circ \eta_X,$$

and

$$\operatorname{Hom}_{\mathcal{C}}(X, GY) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(FX, Y), \quad \beta \mapsto \theta_Y \circ F\beta,$$

are mutually inverse bijections. Consider a pair of morphisms $\alpha\colon FX \to Y$ and $\beta\colon X \to GY$. This yields a pair of commutative squares

$$\begin{array}{ccc} FX & \xrightarrow{\alpha} & Y \\ \downarrow{\scriptstyle \eta_{FX}} & & \downarrow{\scriptstyle \eta_Y} \\ GF(FX) & \xrightarrow{GF(\alpha)} & GF(Y) \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\beta} & GY \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_{GY}} \\ GF(X) & \xrightarrow{GF(\beta)} & GF(GY) \end{array}$$

giving the desired identities

$$\alpha = \theta_Y \circ \eta_Y \circ \alpha = \theta_Y \circ GF(\alpha) \circ \eta_{FX} = \theta_Y \circ FG(\alpha) \circ F\eta_X$$

and

$$\beta = \theta_{GY} \circ \eta_{GY} \circ \beta = \theta_{GY} \circ GF(\beta) \circ \eta_X = G\theta_Y \circ GF(\beta) \circ \eta_X.$$

(2) $\Rightarrow$ (1): Let $\varepsilon\colon FG \to \operatorname{id}_{\mathcal{D}}$ denote the counit. Then it is well known that the composites

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F \quad \text{and} \quad G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G$$

are identity morphisms. We know from Proposition 1.1.3 that ε is invertible because G is fully faithful. Therefore $L\eta = GF\eta$ is invertible. Moreover, we have

$$L\eta = GF\eta = (G\varepsilon F)^{-1} = \eta GF = \eta L. \qquad \square$$

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Localisation of Adjoints

Localising a pair of adjoint functors yields an adjoint pair of functors between the localised categories.

Lemma 1.1.6. *Let (F, G) be an adjoint pair of functors $\mathcal{C} \rightleftarrows \mathcal{D}$. If $S \subseteq \operatorname{Mor} \mathcal{C}$ and $T \subseteq \operatorname{Mor} \mathcal{D}$ are classes of morphisms such that $F(S) \subseteq T$ and $G(T) \subseteq S$, then (F, G) induces an adjoint pair of functors $(\bar{F}, \bar{G})$ such that the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{C} & \underset{G}{\overset{F}{\rightleftarrows}} & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{C}[S^{-1}] & \underset{\bar{G}}{\overset{\bar{F}}{\rightleftarrows}} & \mathcal{D}[T^{-1}] \end{array}$$

Proof The functors F and G induce a pair of functors $\bar{F} \colon \mathcal{C}[S^{-1}] \to \mathcal{D}[T^{-1}]$ and $\bar{G} \colon \mathcal{D}[T^{-1}] \to \mathcal{C}[S^{-1}]$. We have by definition a natural isomorphism

$$\alpha \colon \operatorname{Hom}_{\mathcal{D}}(F-, -) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(-, G-)$$

of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Set}$. These functors invert morphisms in S and T. Thus α induces a natural isomorphism

$$\operatorname{Hom}_{\mathcal{D}[T^{-1}]}(\bar{F}-, -) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}[S^{-1}]}(-, \bar{G}-)$$

of functors $\mathcal{C}^{\mathrm{op}}[S^{-1}] \times \mathcal{D}[T^{-1}] \to \mathrm{Set}$. It follows that $(\bar{F}, \bar{G})$ is an adjoint pair. □

There is a useful consequence which is obtained by setting $T = \varnothing$.

Lemma 1.1.7. *Consider a composite $\mathcal{C} \twoheadrightarrow \mathcal{C}[S^{-1}] \to \mathcal{D}$ of functors and suppose there exists a right adjoint. Then $\mathcal{C}[S^{-1}] \to \mathcal{D}$ admits a right adjoint.* □

Localisation and Coproducts

Let $\mathcal{C}$ be a category and $S \subseteq \operatorname{Mor} \mathcal{C}$. We provide a criterion for the canonical functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$ to preserve coproducts.

Lemma 1.1.8. *Let $\mathcal{C}$ be a category which admits coproducts and let $S \subseteq \operatorname{Mor} \mathcal{C}$ be a class of morphisms. If $\coprod_i \sigma_i$ belongs to S for every family $(\sigma_i)_{i \in I}$ in S, then the category $\mathcal{C}[S^{-1}]$ admits coproducts and the canonical functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$ preserves coproducts.*

Proof Let $(X_i)_{i\in I}$ be a family of objects in $\mathcal{C}[S^{-1}]$. Then the coproduct is obtained by applying the left adjoint of the diagonal functor $\Delta\colon \mathcal{C} \to \prod_{i\in I} \mathcal{C}$. The assumption on S means that we can apply Lemma 1.1.6. Thus the diagonal functor $\Delta\colon \mathcal{C}[S^{-1}] \to \prod_{i\in I} \mathcal{C}[S^{-1}]$ admits a left adjoint which provides the coproduct $\coprod_{i\in I} X_i$ in $\mathcal{C}[S^{-1}]$. □

1.2 Calculus of Fractions

We introduce the calculus of fractions; this helps to describe explicitly the morphisms of a localised category.

Calculus of Fractions

Let $\mathcal{C}$ be a category and $S \subseteq \operatorname{Mor}\mathcal{C}$. There is an explicit description of the localisation $\mathcal{C}[S^{-1}]$ provided that the class S admits a *calculus of left fractions*, that is, the following conditions are satisfied.

(LF1) The identity morphism of each object is in S. The composite of two morphisms in S is again in S.

(LF2) Each pair of morphisms $X' \xleftarrow{\sigma} X \to Y$ with $\sigma \in S$ can be completed to a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\tau} \\ X' & \longrightarrow & Y' \end{array}$$

such that $\tau \in S$.

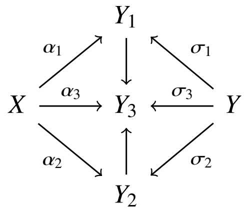
(LF3) Let $\alpha, \beta\colon X \to Y$ be morphisms in $\mathcal{C}$. If there is $\sigma\colon X' \to X$ in S such that $\alpha\sigma = \beta\sigma$, then there is $\tau\colon Y \to Y'$ in S such that $\tau\alpha = \tau\beta$.

The class S admits a *calculus of right fractions* if it admits a calculus of left fractions in the opposite category $\mathcal{C}^{\mathrm{op}}$.

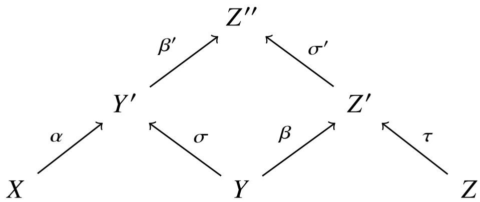
Now assume that S admits a calculus of left fractions. Then one obtains a new category $S^{-1}\mathcal{C}$ as follows. The objects are those of $\mathcal{C}$. Given objects X and Y, we call a pair (α, σ) of morphisms

$$X \xrightarrow{\alpha} Y' \xleftarrow{\sigma} Y$$

in $\mathcal{C}$ with σ in S a *left fraction*. The morphisms $X \to Y$ in $S^{-1}\mathcal{C}$ are equivalence classes $[\alpha, \sigma]$ of such left fractions, where (α_1, σ_1) and (α_2, σ_2) are *equivalent* if there exists a commutative diagram



with σ_3 in S. The composite of $[\alpha, \sigma]$ and $[\beta, \tau]$ is by definition $[\beta'\alpha, \sigma'\tau]$ where σ' and β' are obtained from condition (LF2) as in the following commutative diagram.



The canonical functor $P: \mathcal{C} \to S^{-1}\mathcal{C}$ is the identity on objects and sends a morphism $\alpha: X \to Y$ to $[\alpha, \mathrm{id}_Y]$.

Lemma 1.2.1. *Let S admit a calculus of left fractions. The functor $F: S^{-1}\mathcal{C} \to \mathcal{C}[S^{-1}]$ which is the identity on objects and takes a morphism $[\alpha, \sigma]$ to $(Q\sigma)^{-1} \circ Q\alpha$ is an isomorphism.*

Proof The functor P inverts all morphisms in S and factors therefore through $Q: \mathcal{C} \to \mathcal{C}[S^{-1}]$ via a functor $G: \mathcal{C}[S^{-1}] \to S^{-1}\mathcal{C}$. It is straightforward to check that $F \circ G = \mathrm{id}$ and $G \circ F = \mathrm{id}$. □

From now on, we identify $S^{-1}\mathcal{C}$ with $\mathcal{C}[S^{-1}]$ whenever S admits a calculus of left fractions.

A category $\mathcal{I}$ is called *filtered* if it is non-empty, for each pair of objects i, i' there is an object j with morphisms $i \to j \leftarrow i'$, and for each pair of morphisms $\alpha, \alpha': i \to j$ there is a morphism $\beta: j \to k$ such that $\beta\alpha = \beta\alpha'$.

Lemma 1.2.2. *Let S admit a calculus of left fractions and fix objects X, Y in $\mathcal{C}$. The morphisms $\sigma: Y \to Y'$ in S form a filtered category, and taking σ to $\mathrm{Hom}_{\mathcal{C}}(X, Y')$ gives a bijection*

$$\operatorname*{colim}_{\sigma: Y \to Y'} \mathrm{Hom}_{\mathcal{C}}(X, Y') \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, Y).$$

This map sends a morphism α in $\mathrm{Hom}_{\mathcal{C}}(X, Y')$ to $[\alpha, \sigma]$.

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Proof Straightforward. □

Examples for classes of morphisms with a calculus of fractions arise from pairs of adjoint functors (F, G) by taking left fractions of the form

$$X \xrightarrow{\alpha} GF(Y) \xleftarrow{\eta_Y} Y.$$

Example 1.2.3. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor with a fully faithful right adjoint. Then $S = \{\sigma \in \operatorname{Mor} \mathcal{C} \mid F\sigma \text{ is invertible}\}$ admits a calculus of left fractions.

Another class of examples arises from localising a ring. A ring may be viewed as a category with one object, by viewing the elements as morphisms.

Example 1.2.4. Let A be a ring. Then a subset $S \subseteq A$ admits a calculus of right fractions if the following holds.

(1) If $s, t \in S$, then $st \in S$. Also $1_A \in S$.
(2) For $a \in A$ and $s \in S$ there are $b \in A$ and $t \in S$ such that $at = sb$.
(3) If $sa = 0$ for $a \in A$ and $s \in S$, then there is $t \in S$ such that $at = 0$.

In this case $AS^{-1} = A[S^{-1}]$ is a ring and $A \to A[S^{-1}]$ is the universal homomorphism that makes all elements in S invertible.

Calculus of Fractions for Subcategories

Let $\mathcal{C}$ be a category and $S \subseteq \operatorname{Mor} \mathcal{C}$. A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is *left cofinal* with respect to S if for every morphism $\sigma\colon X \to Y$ in S with X in $\mathcal{D}$ there is a morphism $\tau\colon Y \to Z$ with $\tau \circ \sigma$ in $S \cap \mathcal{D}$.

Lemma 1.2.5. *Let S admit a calculus of left fractions and $\mathcal{D} \subseteq \mathcal{C}$ be left cofinal with respect to S. Then $S \cap \mathcal{D}$ admits a calculus of left fractions and the induced functor $\mathcal{D}[(S \cap \mathcal{D})^{-1}] \to \mathcal{C}[S^{-1}]$ is fully faithful.*

Proof It is straightforward to check (LF1)–(LF3) for $S \cap \mathcal{D}$. Now let X, Y be objects in $\mathcal{D}$. We need to show that the induced map

$$f\colon \operatorname{Hom}_{\mathcal{D}[(S\cap\mathcal{D})^{-1}]}(X, Y) \longrightarrow \operatorname{Hom}_{\mathcal{C}[S^{-1}]}(X, Y)$$

is bijective. The map sends the equivalence class of a fraction to the equivalence class of the same fraction. If $[\alpha, \sigma]$ belongs to $\operatorname{Hom}_{\mathcal{C}[S^{-1}]}(X, Y)$ and τ is a morphism with $\tau \circ \sigma$ in $S \cap \mathcal{D}$, then $[\tau \circ \alpha, \tau \circ \sigma]$ belongs to $\operatorname{Hom}_{\mathcal{D}[(S\cap\mathcal{D})^{-1}]}(X, Y)$ and f sends it to $[\alpha, \sigma]$. Thus f is surjective. A similar argument shows that f is injective.

For an alternative proof using filtered colimits, combine Lemma 1.2.2 and Lemma 11.1.5. □

Notes

The standard reference for localisation and the calculus of fractions is the book of Gabriel and Zisman [85]. The localisation of a category generalises the concept for rings. For instance, rings of functions are localised in order to study the local properties of a geometric object. The localisation of non-commutative rings was pioneered by Ore in 1931, who introduced the 'Ore condition' [151]. For a survey about localisation in algebra and topology, see [166].