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REMARKS ON THE PURE CRITICAL EXPONENT PROBLEM

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In this paper, we use geometric and analytic methods to study the existence of positive solutions of the pure critical exponent problem with Dirichlet boundary conditions. In particular we prove that there is no solution for domains which are nearly star-shaped and we show that being conformal to a star-shaped domain does *not* characterise the domains for which the problem has no solution. We also answer some questions of Rodriguez et al.

In this paper, we discuss the existence of positive solutions of the pure critical exponent problem

$$(1)\qquad \begin{aligned} -\Delta u &= u^p \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega\,. \end{aligned}$$

Here Ω is a bounded domain in R^m with smooth boundary, $m > 2$ and p is the critical exponent $(m+2)(m-2)^{-1}$. We discuss a slight generalisation of star-shapedness namely weakly star shaped and then prove that, if Ω is C^2 close to a weakly star-shaped domain, then (1) has no positive solution. (We do not know if this is true for sign changing solutions.) In a companion paper [**4**], Kewei Zhang and I have proved a similar result for $p > (m+2)(m-2)^{-1}$ (and related results). The proofs there are quite different. As a consequence of my result above, it follows that (1) has no positive solution on Ω if Ω is C^2 close to a domain D where D is conformally equivalent to a bounded weakly star shaped domain. We then use this to show that there exists domains Ω which are not conformally equivalent to bounded weakly star shaped domain but (1) has no positive solution. This disposes of a natural conjecture for when (1) has a positive solution. In the process, we obtain a useful characterisation of which domains are conformal to bounded weakly star shaped domains. We use this to give a much simpler proof of a much stronger version of a result of Rodriguez, Comte and Lewandowski [**12**] and answer two questions there.

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1. Weakly star shaped domains

A domain $\Omega \subseteq R^n$ is weakly star shaped if there exists $x_0 \in \overline{\Omega}$ such that every half line through x_0 intersects $\overline{\Omega}$ in a connected set (possibly consisting only of $\{x_0\}$). This connected set is an "interval". This is basically the usual definition of star-shapedness except that we allow x_0 to belong to $\partial\Omega$. This is more convenient later. We refer to x_0 as a centre of Ω. The following result is possibly known but we do not know a reference.

Proposition. *Assume that Ω is a bounded C^2 domain. Then Ω is weakly star shaped with centre x_0 if and only if $\langle x - x_0, n(x)\rangle \geqslant 0$ on $\partial\Omega$ where $n(x)$ is the outward normal.*

Remark. Clearly, if $x_0 \in \partial\Omega$, it suffices to assume the condition for x not equal to x_0.

Proof: Since our conditions only involve the line segment joining x_0 to x and the component of the normal in the direction of $x - x_0$, it suffices to assume Ω is 2-dimensional.

The condition is clearly necessary because if $\langle x-x_0, n(x)\rangle < 0$, the outward normal at x makes an obtuse angle with $x - x_0$ (in the direction away from x_0) and hence the outward normal makes an acute angle with $x_0 - x$. Thus $\alpha x + (1-\alpha)x_0 \notin \overline{\Omega}$ if α is close to 1 and less than 1. (Since Ω is smooth, we must move out of Ω if we move in any direction making an acute angle with the outward normal). This contradicts the connectedness of $\overline{\Omega} \cap \{tx + (1-t)x_0 : t \geqslant 0\}$.

To prove the condition is sufficient, assume $x_1 \in \overline{\Omega}$ but $t_0 x_1 + (1-t_0)x_0 \notin \overline{\Omega}$ for some $t_0 \in (0,1)$. By approximating x_1 by points of $\operatorname{int}\Omega$ and by a limit argument, we see that we can assume $x_1 \in \operatorname{int}\Omega$. By a similar argument, the same condition holds for all points near x_1. We shall prove below that there exist points $\widehat{x}$ arbitrarily close to x_1 such that $\langle n(t\widehat{x} + (1-t)x_0), (\widehat{x} - x_0)\rangle \neq 0$ if $t \neq 1$ and $t\widehat{x} + (1-t)x_0 \in \partial\Omega$. Assuming this for the moment, it follows that we can also assume that x_1 has this property. If $sx_1 + (1-s)x_0 \notin \overline{\Omega}$ where $s \in (0,1)$, let $\alpha = \inf\{t : t > 0,\ \alpha x_1 + (1-\alpha)x_0 \in \overline{\Omega}\}$. Then $\alpha x_1 + (1-\alpha)x_0 \in \partial\Omega$ while $(\alpha - \varepsilon)x_1 + (1 - (\alpha - \varepsilon))x_0 \in \overline{\Omega}$ if ε is small and positive. We then argue much as in the first part to check that $\langle n(\alpha x_1 + (1-\alpha)x_0), (x_1 - x_0)\rangle \geqslant 0$. But by our choice of $x_1, \langle n(\alpha x_1 + (1-\alpha)x_0), (x_1 - x_0)\rangle \neq 0$. Hence $n(\alpha x_1 + (1-\alpha)x_0) \cdot (x_1 - x_0) < 0$ which contradicts our assumptions. This proves our claim except we have to prove that we can choose x_1 suitably.

In fact, we prove our claim that, for most $x_1, \langle n(tx_1 + (1-t)x_0), (x_1 - x_0)\rangle \neq 0$ whenever $tx_1 + (1-t)x_0 \in \partial\Omega$. To see this we look at the C^1 map $S(x) = \|x - x_0\|^{-1}(x - x_0)$ of $\partial\Omega\backslash\{x_0\}$ to S^1. By Sard's theorem, most values of S in S^1 are regular values and hence it suffices to prove that if $x_2 \in \partial\Omega\backslash\{x_0\}$ and $S'(x_2)$ has zero

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kernel, then $\langle n(x_2),(x_2-x_0)\rangle \neq 0$. To see this, note that, if $\langle n(x_2),(x_2-x_0)\rangle = 0$, then the tangent space to $\partial\Omega$ at x_2 is parallel to x_2-x_0 and hence the C^1 surface $\partial\Omega$ near x_2 is of the form $x_2+\tilde{\alpha}(x_2-x_0)+o(\tilde{\alpha})$ where $\tilde{\alpha}$ is small. A simple computation then shows $S'(x_2)(x_2-x_0)=0$ which proves our claim. This completes the proof. ∎

REMARKS.

1. It is easy to construct examples of domains which are weakly star shaped but not star shaped. The advantage of weakly star shaped is that it follows easily from the proposition that if Ω_n is weakly star shaped and $\Omega_n \longrightarrow \Omega_0$ in the C^1 sense, then Ω_0 is also weakly star shaped. (Thus the set of weakly star shaped domains is closed in a suitable topology.)

2. If Ω is weakly star shaped, $\overline{\Omega}$ is contractible and hence by duality theorems for manifolds (as in [**7**]), it follows that $\partial\Omega$ is connected.

3. If Ω is weakly star shaped and x is the point of $\partial\Omega$ furthest from x_0, then $n(x)$ is parallel to $x-x_0$ and hence $\langle n(x),(x-x_0)\rangle > 0$. Thus strict inequality must hold at some points of $\partial\Omega$.

4. It is easy to see that, if Ω is weakly star shaped and $x\in\Omega$, then $tx_0+(1-t)x\in\Omega$ for $0<t<1$. (This uses that $\operatorname{int}\overline{\Omega}=\Omega$) and this is in fact an equivalent condition. (The equivalence follows by taking closures.)

To complete this section, we discuss briefly which bounded domains are conformally equivalent to bounded weakly star shaped domains. We shall assume our domains are C^1, our conformal transformations are C^1 and $m \geqslant 3$. By Vaisala [**13**], any conformal transformation C on an open set in R^n can be extended to a conformal transformation on R^n. Note that conformal transformations which do not involve inversion are of no interest to use because these are compositions of orthogonal transformations, translations and the maps $x\longrightarrow rx$ where $r>0$ and hence preserve weak star-shapedness. A conformal transformation involving inversions can be uniquely written in the form $x\longrightarrow b+r\|x+a\|^{-2}\widetilde{T}(x+a)$ where $\widetilde{T}$ is an orthogonal transformation (see [**13**]). We refer to b as the point at ∞. (It is the image of ∞ under the transformation.) Hence if Ω_1 is bounded and weakly star-shaped and $T(\Omega_1)$ is bounded where T is conformal and involves an inversion then $-a\notin\overline{\Omega}$, and if x_0 is a centre of $\overline{\Omega}_1$ then T maps half lines segments from x_o to infinity to planar circular semi arcs joining $T(x_0)$ to b. Moreover, by connectedness, $T(\overline{\Omega}_1)$ intersected with the same semi arc will be connected. Note that a semi arc is one of the two arcs joining $\overline{x}_0$ and b on the circle. Since the argument is reversible we see that a bounded C^1 domain Ω is conformal to a weakly star shaped bounded domain if and only if there exists an $\overline{x}$ in $\overline{\Omega}$ and a $b\in R^m\backslash\overline{\Omega}$ such that every planar circular semi arc joining $\overline{x}$ and b intersects $\overline{\Omega}$ in a connected set. (Here we should include the case $b=\infty$ to include the possibility that $\overline{\Omega}$ is already weakly star-shaped and allow the limiting arc of the straight line through

$\overline{x}$ and b. Equivalently we could assume every planar circular arc joining b and $\overline{x}_0$ intersects $\overline{\Omega}$ in a connected set (since these are one dimensional sets). These arcs are all arcs with centres on the hyperplane of points equidistant from $\overline{x}$ and b. Of course the problem is in the correct choice of $\overline{x}$ and b. Each $x \in R^n \backslash \{\overline{x}, b\}$ lies on a unique circular arc. Let $v_{\overline{x},b}(x)$ be the unit tangent vector to this arc at x determined by differentiating at x in the direction moving from $\overline{x}$ to b. Then by the proposition and the remark immediately after it, we see that a bounded C^1 domain Ω is conformally equivalent to a bounded weakly star shaped domain if and only if there exists $\overline{x} \in \overline{\Omega}$ and $b \in R^n \backslash \overline{\Omega}$ such that $\langle v_{\overline{x},b}(x), n(x)\rangle \geqslant 0$ on $\partial\Omega \backslash \{\overline{x}\}$. (Note that a conformal transformation preserves angles). From the above we have a quite convenient geometric and analytic characterisation of which bounded C^1 domains are conformally equivalent to bounded weakly star-shaped domains. We shall also refer to $\overline{x}$ as a centre of Ω.

As a simple consequence of the analytic characterisation and a simple compactness argument, if the Ω_n are bounded domains which are conformally equivalent to bounded weakly star-shaped domains and if $\Omega_n \longrightarrow \Omega$ in the C^1 sense, then Ω is also conformally equivalent to a bounded weakly star shaped domain unless the points at infinity b_n for Ω_n approach Ω as n tends to infinity. (I do not mean to imply points at infinity are unique.) Note that if $\|b_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$, then Ω will be weakly star-shaped.

Finally note that, if Ω is bounded and open and conformally equivalent to a weakly star-shaped domain then $\overline{\Omega}$ is clearly contractible. On the other hand not every contractible domain is conformal to a bounded weakly star shaped domain. (For example, this follows since there are contractible domains for which (1) has a positive solution. We shall meet further examples later.)

2. Nearly weakly star shaped domains

In this section, we prove two results. We firstly show that in domains which are almost weakly starshaped, (1) has no positive solution. We also use this and the results of Section 1 to show that there exist bounded domains other than those conformal to bounded weakly star shaped domains for which (1) has no positive solution. This answers a natural question. Note that there are contractible domains for which (1) has a positive solution. (See **[1, 6]** or **[10]**.)

Theorem 1. *Assume that Ω is a bounded C^2 weakly star shaped domain and assume $\Omega_n \longrightarrow \Omega$ in the C^2 norm as $n \longrightarrow \infty$. Then (1) on Ω_n has no positive solution for large n.*

Remark. The proof is complicated because we do not have a priori bounds for positive solutions.

Proof: By the Pohozaev identity, we see that, if u_n is a positive solution of (1) on Ω_n,

$$\int_{\partial\Omega} \langle x - x_0, \nu^n(x)\rangle \left(\frac{\partial u_n}{\partial \nu^n}\right)^2 = 0. \tag{2}$$

Here it is convenient to change notation slightly and use $\nu^n(x)$ for the outward unit normal to a Ω_n at x. Suppose we can prove there is a $K > 0$ such that

$$\sup_{\partial\Omega_n} \left|\frac{\partial u_n}{\partial \nu^n}\right| \leqslant K \inf_{\partial\Omega_n} \left|\frac{\partial u_n}{\partial \nu^n}\right| \tag{3}$$

where K is independent of n. Then

$$\int_{T_n} \big\langle (x - x_0), \nu^n(x)\big\rangle \left|\frac{\partial u}{\partial \nu^n}\right|^2 \geqslant \mu_n H_{n-1}(\partial\Omega_n) K^2 \inf_{\partial\Omega_n} \left|\frac{\partial u_n}{\partial \nu^n}\right|^2 \tag{4}$$

where $T_n = \Big\{x \in \partial\Omega_n : \big\langle (x - x_0), \nu^n(x)\big\rangle < 0\Big\}$, $\mu_n = \inf\Big\{\big\langle (x - x_0), \nu^n(x)\big\rangle : x \in T_n\Big\}$ and H_{n-1} is $(n-1)$-dimensional Hausdorff measure. When x is at maximal distance on $\partial\Omega_n$ from x_0, $\big\langle (x - x_0), \nu^n(x)\big\rangle = \|x - x_0\|$ and hence we easily see that there is a subset W_n of $\partial\Omega_n \backslash T_n$ with $(n-1)$-dimensional Hausdorff measure bounded below by $\gamma > 0$ independent of n and $\big\langle (x - x_0), \nu^n(x)\big\rangle \geqslant \ell > 0$ independent of n on W_n. Hence

$$\int_{\partial\Omega_n - T_n} \big\langle x - x_0, \nu^n(x)\big\rangle \left|\frac{\partial u_n}{\partial \nu^n}\right|^2 \geqslant \gamma\ell \inf_{\partial\Omega_n} \left|\frac{\partial u_n}{\partial \nu^n}\right|^2. \tag{5}$$

Since $\mu_n \longrightarrow 0$ as $n \longrightarrow \infty$, (4) and (5) contradict (2) for n large. Hence it suffices to prove (3).

We first prove (3) assuming that there exist $\delta, K > 0$ and independent of n such that

$$u_n(x) \leqslant K \quad \text{if} \quad d(x, \partial\Omega_n) \leqslant \delta. \tag{6}$$

We prove (6) a little later. To deduce (3), choose a smooth surface S in Ω close to $\partial\Omega$ (within $\delta/2$ of S) such that S intersects each normal to $\partial\Omega$ and this intersection is unique. Let $\widetilde{S}$ be a small closed neighbourhood of S with smooth boundary. Now the u_n are uniformly bounded on $\widetilde{S}$ (also uniformly in n) and hence $-\Delta u_n = a_n u_n$ where a_n is uniformly bounded on $\widetilde{S}$ (and uniform in n). By applying the Harnack inequality (see [**9**, Corollary 8.21]) on $\widetilde{S}$, we see that on a slightly smaller neighbourhood V, there exists K_1 independent of n such that $\sup\limits_V u_n \leqslant K_1 \inf_S u_n$. We can now apply $W^{2,p}$ estimates (or the $C^{1,\alpha}$ estimates) in [**9**] to obtain a $C^{1,\alpha}$ independent of n for

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$B_n^{-1}u_n$ on W_n, where W_n is the region bounded by S and $\partial\Omega_n$ and $B_n = \sup_{W_n} u_n$. There are a couple of points to be noted here. Firstly we need to follow the discussion in [**9**, p.237–230 (see also p.98 there)] to check the bounds can be made independent of n. The argument is a localisation argument. Secondly, to obtain the bounds near S we use that our earlier Harnack inequality estimate implies $B_n^{-1}u_n$ is bounded on $W_n \cup V$ and hence we can use interior estimates near S. Moreover it follows that we have a positive lower bound for $-(1/B_n)\dfrac{\partial u_n}{\partial \nu^n}$ on $\partial\Omega_n$ where, as before, ν^n is the outward unit normal. To see that this is true, suppose by way of contradiction that it is false. Then there exists a subsequence of the $B_n^{-1}u_n$ converging uniformly in the C^1 norm to a solution u of $\Delta u = \widehat{a}u$ on the region W between S and $\partial\Omega$ such that u is bounded by 1, has a positive lower bound on S and has a zero normal derivative at some point of $\partial\Omega$ (and $u = 0$ on $\partial\Omega$). Here $\widehat{a}$ is bounded. There are two points here. Firstly, we extend the u_n to be zero outside Ω_n to obtain functions on a fixed domain. Secondly, if $B_n^{-1}u_n$ converge uniformly to zero on S, then, by the Harnack inequality, $B_n^{-1}u_n$ converge uniformly to zero on V and hence u vanishes identically on V. By the Harnack inequality, it follows that $u = 0$ on W. This is impossible since $B_n^{-1}u_n$ converges uniformly and $\sup_{W_n} B_n^{-1}u_n = 1$. However by the Hopf maximum principle applied to u on W, $\dfrac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$. Hence we have a contradiction. This proves that $-B_n^{-1}\dfrac{\partial u_n}{\partial \nu^n}$ has a positive lower bound on $\partial\Omega_n$ (uniformly in n). Since we also have a bound for $\left|B_n^{-1}\dfrac{\partial u_n}{\partial \nu^n}\right|$ on $\partial\Omega_n$ (uniform in n), this proves (3).

It remains to prove (6). This is a more geometric argument. By our convergence assumptions there is a $\mu > 0$ independent of n and x_1 such that each point $x_1 \in \partial\Omega_n$ is on the boundary of a ball $B_{x_1,n}$ in $R^m \backslash \Omega_n$ of radius μ (that is a touching ball). Here μ is rather small. We use an inversion in $B_{x_1,n}$. Under this inversion, Ω_n becomes a C^2 domain $\widetilde{\Omega}_{x_1,n}$ contained in $B_{x_1,n}$ and touching $\partial B_{x_1,n}$ at x_1. Moreover if $x_1^n \in \partial\Omega_n$ converges to $x_1 \in \partial\Omega$ as $n \longrightarrow \infty$, $\widetilde{\Omega}^n_{x_1,n}$ converges in the C^2 norm to $\widetilde{\Omega}_{x_1}$ as $n \longrightarrow \infty$ uniformly in x_1. ($\widetilde{\Omega}_{x_1}$ is the image of Ω under the inversion in the ball $B_{x_{1,n}} \in R^m \backslash \Omega$). Now by the conformal invariance, $v_n(y) \equiv \left(\|y - \widehat{x}_0^n\|\right)^{2-m} u_n\left((y - \widehat{x}_0^n)/\|y - \widehat{x}_0^n\|^2\right)$ is a positive solution of (1) on $\widetilde{\Omega}_{x_1,n}$ with $v_n(y) = 0$ on $\partial\widetilde{\Omega}_{x_1,n}$. (Here $\widehat{x}_0^n$ is the centre of $B_{x_1,n}$.) We can use the method of sliding planes (see as in [**8**]) to v_n on $\widetilde{\Omega}_{x_1,n}$ and obtain that v_n increases in certain directions. By repeating the argument of De Figueredo, Nussbaum and Lions [**5**, p.51–53], we see that there exists $\varepsilon, \gamma, C > 0$ independent of n such that if $x \in \Omega_{\varepsilon,n} = \left\{x \in \overline{\Omega}_n : d(x, \partial\Omega_n) < \varepsilon\right\}$ there exists a measurable set $I_{x,n}$ with $\mu(I_{x,n}) \geqslant \gamma$ (where μ denotes Lebésque measure) such that $I_{x,n} \subseteq \Omega_n \backslash \Omega_{1/2\varepsilon,n} \equiv T_n$ and $u_n(\xi) \leqslant Cu_n(S)$ if $\xi \in I_{x,n}$. (That $\widetilde{\Omega}_{x_1^n,n}$ converges in

C^2 to $\tilde{\Omega}_{x_1}$ ensures that we can do this uniformly in n). Hence we see that

(7) $$\int_{T_n} u_n^p \geqslant \gamma C^{-p} u_n(x)^p \quad \text{if} \quad x \in \Omega_{\varepsilon,n} .$$

On the other hand, by scalar multiplying (1) (for $\Omega = \Omega_n$) by ϕ_n, the positive eigenfunction of $-\Delta$ on Ω_n (for Dirichlet boundary conditions on $\partial\Omega_n$), normalised so that $\|\phi_n\|_{2,\Omega_n} = 1$ we see that $\int_{\Omega_n} (u_n^p - \lambda_1(\Omega_n)u_n)\phi_n = 0$ where $\lambda_1(\Omega_n)$ is the eigenvalue corresponding to ϕ_n. Since $\lambda_1(\Omega_n) \longrightarrow \lambda_1(\Omega)$ as $n \longrightarrow \infty$ (by the theory of domain variation as in [**3**] or [**11**]) and since $y^p - \gamma y$ is bounded below on $[0,\infty)$ whenever $\gamma > 0$, it follows easily that $\int_{\Omega_n} u_n^p \phi_n \leqslant K$ where K is independent of n. Suppose that M is a compact subset of Ω. By domain variation again, $\phi_n \longrightarrow \phi$ uniformly on M where ϕ is the first eigenfunction of $-\Delta$ on Ω. Hence there is a $K_1 > 0$ such that $\phi_n(x) \geqslant K_1$ if $x \in M$ and n is large. Hence

(8) $$\int_M u_n^p \leqslant K_2$$

where K_2 is independent of n. If we choose $M = \{x \in \Omega : d(x_1\partial\Omega) \geqslant \varepsilon/4\}$ then $T_n \subseteq M$ for large n and hence the result follows from (7) and (8). This completes the proof. ∎

REMARKS 1. We do not know if the result is true for changing sign solutions. Note that, by the Pohojaev identity argument, there are no changing sign solutions if Ω is weakly star shaped.

It follows that if $\Omega_n \longrightarrow \Omega$ in the C^2 sense and Ω is conformally equivalent to a bounded C^2 weakly-star-shaped domain, then (1) has no positive solution on Ω_n for large n. This follows because, if $\tilde{\Omega}$ is conformally equivalent to Ω where both are bounded domains, then (1) has a positive solution on Ω if and only if it has a positive solution on $\tilde{\Omega}$. This is obvious for translations, rotations and rescalings so that we need only consider inversions in a ball of radius 1 centre zero. Now if $\tilde{\Omega}$ is the image of Ω under this inversion, it is well known and easy to see that u is a solution on Ω if and only if $v(y) = \|y\|^{2-m} u\left(y/\|y\|^2\right)$ is a positive solution on $\{x/\|x\|^2 : x \in \Omega\}$ and hence our claim follows.

It is a natural question to ask whether being conformal to a weakly star shaped set characterises the domains Ω for which (1) has no positive solution. We now construct an example to show that this is not the case.

To construct such an example we shall construct a smooth family of domains $W_t, t \in [0,1]$ such that W_0 is weakly star shaped and W_1 is not conformal to a bounded weakly star shaped domain. If we can construct such a family with $T = \{t \in [0,1] : W_t$

is conformally equivalent to a bounded weakly star shaped domain$\}$ is closed, then we shall have the required example. This follows because if $\gamma = \sup T$, then $\gamma < 1, W_{\gamma+\varepsilon}$ is not conformally equivalent to a bounded weakly star shaped domain while by one of our remarks after Theorem 1, the critical exponent problem on $W_{\gamma+\varepsilon}$ has no positive solution as required (because $W_{\gamma+\varepsilon}$ is a C^2 small perturbation of W_γ).

We now consider when T is closed. If W_{t_n} is conformal to a bounded weakly star shaped domain for all n, and $t_n \longrightarrow t$ as $n \longrightarrow \infty$, then by the remarks at the end of Section 1, the only way that W_t can fail to be conformal to a bounded weakly star shaped domain is that the point at infinity b_n of W_{t_n} satisfies $b_n \longrightarrow \partial W_t$ as $n \longrightarrow \infty$. We show that this does not occur if we choose the family W_t carefully. This will complete the construction of our example.

We construct W_0 as follows. Let T_k, $k = 1, \cdots, s$ (where $s \geqslant 4$) be open balls of small radius with centres on the ball of radius 4 and so that the centres are well spaced. We define $W_0 = B_1 \cup \bigcup_{k=1}^{s} \overline{co}\big(T_k \cup \{0\}\big)$. It is easy to see that W_0 is weakly star shaped but the only point it is weakly star shaped from is zero. We can easily smooth the corners without affecting those properties. We note that W_0 can not be conformal to a weakly star shaped domain with the "point at infinity" b close to ∂W_0. To see this, note that, whatever the location of the centre a, there must be at least $s-1$ circular arcs, each in one of the $\overline{co}\big(T_k \cup \{0\}\big)'s$ joining $\|x\| = 2$ to $\|x\| = 3$. However, if the $T_k's$ are chosen small, these circular arc's must have small curvature. However, some elementary geometry shows that the only circular arc joining a and b having small curvature between a and b must have direction nearly parallel to $b-a$ in the set $\{x : \|x\| \leqslant 4\}$. Here we use that $\|b\|$ is not large. Provided the centres of the $T_k's$ are well spaced, this clearly leads to contradiction. Thus the point b "at infinity" can not be near ∂W_0. For future reference note that this argument shows that the point b "at infinity" must be large, the circular arcs must have small curvature within our set (except possibly for those in the T_k containing the centre a) and the centre a which is the intersection points of all those arcs with small curvature must be near zero.

We now define the W_t for $t > 0$ by squeezing each of the tubes $\overline{co}\big(\{0\} \cup T_k\big)$; or more precisely the part of the tubes with $1 < \|x\| < 2$. We explain this by drawing a two dimensional cross section of a tube. (We do the squeezing in a symmetric way within a tube and we squeeze each tube in the same way). We could easily give a formula for the squeezing but the diagram is more informative.

In the diagrams, we have not drawn them to scale (by widening the tube) to make them easier to understand. Since the $W_t's$ are the same as W_0 for $2 \leqslant \|x\| \leqslant 3$, the argument in the previous paragraph shows that, if W_t is conformal to a bounded weakly star shaped domain, then the point at infinity b_t is not near the boundary of W_t and this holds uniformly in t. (Remember that the part of W_t in $\{x : 2 < \|x\| < 3\}$ is

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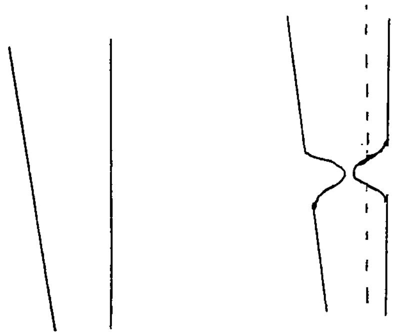

Figure 1

independent of t.) Hence, as we saw earlier, $\{t : W_t$ is conformal to a weakly star shaped bounded domain$\}$ is closed and hence we are finished if we prove that W_t becomes not conformal to a bounded star shaped domain before the two sides of $\overline{co}(T_k \cup \{0\})$ are squeezed together. To see this, note that, by the remarks at the end of the previous paragraph, the point at infinity b_t must be large, a_t must be small and the circular arcs must be nearly straight within W_t. Now consider x in W_0 with $2 < \|x\| < 3$ and x very close to the boundary. Since the circular arcs are very straight, a_t is small and b_t is large, it is easy to see that the semi-arc joining a_t to b_t through x will cross ∂W_t at three points when the squeezing is significant (one with $\|x\|$ close to 4 and two where the circular arc crosses the region being squeezed (as in the diagram). Hence the connectedness condition fails and W_t is not conformal to a weakly star shaped bounded domain. This completes the construction of the counterexample.

Finally, another example is mentioned in Section 3, while a higher dimensional version of the example in [**4**] could almost certainly be used to obtain another example (though there would be a good deal of tedious geometric arguments to justify them). The example in [**4**] (or more precisely a higher dimensional analogue) probably also provides an example where we make a small perturbation of a star-shaped domain and obtain domains not conformal to bounded weakly star shaped domains.

3. An example

Here we show that our geometric techniques can be used to give a simple proof of a result much stronger than one in Rodriguez, Comte and Lewandowski [**12**]. In fact, we answer two open questions in [**12**].

Let B be the open ball in R^m, $\varepsilon > 0$, $\rho < -1$, $\ell \in (0,1)$. We define

$$C_\varepsilon = \left\{(x', x_m) : x' \in R^{m-1},\ \|x'\| \leqslant \varepsilon(x_m - \rho)(1-\rho)^{-1}\right\}$$
$$\Omega_\varepsilon = B \backslash \{x \in C_\varepsilon : x_m \geqslant \ell\}.$$

THEOREM 2. *For $\Omega = \Omega_\varepsilon$, (1) has no solution.*

REMARK. If we replace u^p by $|u|^{p-1}u$, the result also applies to sign changing solutions.

PROOF: Except for a technical smoothness problem, it suffices to show that Ω_ε is conformal to a bounded weakly star-shaped domain. (Note that this contradicts a comment in [**12,** p.245]). We choose $a = (0, -B)$ where $B \in (0,1)$ and is close to 1 and $b = (0, \alpha)$ where α is larger than 1. Here a is the centre and b is the point at infinity. We shall specify α more closely later. Note that, unlike [**12**], we do not require ε to be small. It suffices to show that every planar circular semi arc joining a and b intersects Ω_ε in a connected set. By the symmetry of Ω_ε under rotations about the e_m axis, we see that this reduces to a two dimensional problem. Hence we can set $m = 2$. It is now elementary geometry. Any circular semi arc joining a and b is part of a circle with centre $(s, (\alpha - B)/2)$, containing a and b. This circle will intersect the unit circle T $x_1^2 + x_2^2 = 1$ in exactly two points, once on each of the two semi arcs joining a and b. (It must intersect more than once because b is outside T while a is inside T). We need only consider semi arcs in $x_1 \geqslant 0$. Suppose we can prove the circular semi arc P through the corner where a straight edge meets a curved edge does not intersect any straight edge (or other corner). If we prove this, a simple connectedness argument implies that semi arcs to the "right" of P will not intersect the straight edge of $\partial\Omega_\varepsilon$ and will intersect the curve edge exactly once while semi arcs to the "left" of P will not intersect the curved part of $\partial\Omega_\varepsilon$. Now our semi arcs to the "right" of P will be of the form $x_1 = g(x_2)$ for $\alpha \geqslant x_2 \geqslant -B$ where g decreases for $\alpha \geqslant x_2 \geqslant (\alpha - B)/2$ and g is even about $x_2 = (\alpha - B)/2$. On the other hand the boundary of C_ε in $x_1 > 0$ is an increasing function of x_2 for $x_2 > \ell$. Hence if $(\alpha - B)/2 \leqslant \ell$, we see that the semi arc can meet the part of C_ε in $x_1 > 0$ in at most one point (and P cannot meet again this part of C_ε). The claim on our semi arcs now follows easily provided $(\alpha - B)/2 \leqslant \ell$ which is true if α and B are chosen suitably $\big($for $\ell \in (0,1)\big)$.

There is one technical point in our proof. Ω_ε will be conformal to a bounded weakly star shaped domain $\widetilde{\Omega}_\varepsilon$ with corners but still with the rotational symmetry. To prove non-existence, we need to check that $\widetilde{\Omega}_\varepsilon$ is smooth enough so that the Pohozaev identity holds. This follows by the argument in [**3,** p.655–657]. This completes the proof. ∎

REMARKS.

1. We could also take $a = (-1, 0)$ and $b = (1, 0)$ where the geometry is a little simpler. However, our choice of a and b has the advantage we can use them for many smooth perturbations of Ω_ε.

2. With a little care, it can be shown that Ω_ε is still conformal to a bounded

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weakly star shaped domain if $\ell = 1$ or indeed ℓ is slightly larger than 1. However it can be shown that $\ell \leqslant 1$ is the best condition on ℓ we can allow for Ω_ε to be conformal to a weakly star shaped domain if we want a condition independent of ε and ℓ. Note also that we could replace C_ε by $\{(x', x_m) : \|x'\| \leqslant \mu\}$ and the arguments are still valid (though here we can not go above $\ell = 1$). If $\ell < 1$, we could even allow a very small "knob" at the end of the tube. In this case, the domains are quite close to those in Dancer [1] or Ding [6] and it shows the importance of the "knob" at the end of the tube in the examples there. Note that the methods in [1] can be modified to cover cases where the spherical "knob" does not lie at the centre of the outer ball. These examples seem to suggest it is very difficult to decide for which Ω (1) has a positive solution.

3. It is possible, though a lot of tedious geometrical arguments are needed, to show that, if we round off the corners in our previous example, then as we increase ℓ we obtain a different example where (1) has no solution even though Ω is not conformal to a bounded weakly star-shaped domain. (The difficulty is in establishing that the obvious choices of the centre and the point at infinity are optimal.)

4. In general, conformal equivalences seem to be very useful for domains with one spike or reentrant region but less useful for domains with several spikes (or several reentrant regions).

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