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Faltings' Finiteness Dimension of Local Cohomology Modules Over Local Cohen–Macaulay Rings

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Abstract. Let $(R,\mathfrak{m})$ denote a local Cohen–Macaulay ring and I a non-nilpotent ideal of R. The purpose of this article is to investigate Faltings' finiteness dimension $f_I(R)$ and the equidimensionalness of certain homomorphic images of R. As a consequence we deduce that $f_I(R) = \max\{1, \operatorname{ht} I\}$, and if $\operatorname{mAss}_R(R/I)$ is contained in $\operatorname{Ass}_R(R)$, then the ring $R/I + \bigcup_{n\geq1}(0:_R I^n)$ is equidimensional of dimension $\dim R - 1$. Moreover, we will obtain a lower bound for injective dimension of the local cohomology module $H_I^{\operatorname{ht} I}(R)$, in the case where $(R,\mathfrak{m})$ is a complete equidimensional local ring.

1 Introduction

Throughout this paper, let R denote a commutative Noetherian ring (with identity) and let I be an ideal of R. For an R-module L, the i^{th} local cohomology module of L with respect to I is defined as

$$H_I^i(L) = \varinjlim_{n\geq1} \operatorname{Ext}_R^i(R/I^n, L).$$

We refer the reader to [6] or [3] for more details about local cohomology.

For any finitely generated R-module M, the notion $f_I(M)$, the *finiteness dimension* of M relative to I, is defined to be the least integer i such that $H_I^i(M)$ is not finitely generated, if there exist such i's and ∞ otherwise, *i.e.*,

$$f_I(M) := \inf\{i \in \mathbb{N}_0 \mid H_I^i(M) \text{ is not finitely generated}\}.$$

Our objective in this paper is to investigate the finiteness dimension $f_I(R)$, when R is a local Cohen–Macaulay ring. More precisely, as a main result we will prove the following theorem.

Theorem 1.1 *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring and let I be a non-nilpotent ideal of R. Then $f_I(R) = \max\{1, \operatorname{ht} I\}$.*

The following proposition will play a key role in the proof of Theorem 1.1.

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Proposition 1.2 *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay local ring and let* X *and* Y *be non-empty subsets of* $\mathrm{Ass}_R(R)$ *such that* $\mathrm{Ass}_R(R) = X \cup Y$ *and* $X \cap Y = \varnothing$*. Then* $R/(I+J)$ *is an equidimensional local ring of dimension* $\dim R - 1$*, where* $I = \bigcap_{\mathfrak{p}\in X} \mathfrak{p}$ *and* $J = \bigcap_{\mathfrak{p}\in Y} \mathfrak{p}$ *.*

Recall that a Noetherian ring R, of finite Krull dimension d, is called *equidimensional* if $\dim R/\mathfrak{p} = d$ for every minimal prime ideal $\mathfrak{p}$ of R. As an another main result, we will prove the following theorem.

Theorem 1.3 *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay local ring and let* I *be a non-nilpotent ideal of* R *such that* $\mathrm{mAss}_R(R/I) \subseteq \mathrm{Ass}_R(R)$*. Then* $R/(I + \bigcup_{n\geq 1}(0 :_R I^n))$ *is an equidimensional local ring of dimension* $\dim R - 1$*.*

Hartshorne and Speiser [7] proved that if $(R, \mathfrak{m}, k)$ is a regular local ring, contains a field of characteristic $p > 0$, and $H^i_I(R)$ is supported only at the maximal ideal, then $\mathrm{Hom}_R(k, H^i_I(R)$ is a finitely generated R-module and, moreover, $H^i_I(R)$ is injective. Also, Huneke and Sharp [8] made a remarkable breakthrough. They generalized Hartshorne–Speiser's result by proving that if R is any regular ring containing a field of characteristic $p > 0$, then $\operatorname{inj\,dim} H^i_I(R) \leq \dim \operatorname{Supp} H^i_I(R)$, where $\operatorname{inj\,dim} H^i_I(R)$ denotes the injective dimension of $H^i_I(R)$ and $\dim \operatorname{Supp} H^i_I(R)$ stands for the dimension of the support of $H^i_I(R)$ in $\operatorname{Spec} R$. Finally, Lyubeznik [9] generalized the above-mentioned result of Hartshorne–Speiser by proving that if R is any regular ring containing a field of characteristic zero and $Y \subseteq \operatorname{Spec} R$ is a locally closed subscheme, then $\operatorname{inj\,dim} H^i_Y(R) \leq \dim \operatorname{Supp} H^i_Y(R)$.

As a final main result, we are able to obtain a lower bound for the injective dimension of the local cohomology module $H^{\mathrm{ht}\, I}_I(R)$, in the case where $(R, \mathfrak{m})$ is a complete equidimensional local ring.

Theorem 1.4 *Let* $(R, \mathfrak{m})$ *be a complete local equidimensional ring and let* I *be an ideal of* R*. Then* $\operatorname{inj\,dim} H^{\mathrm{ht}\, I}_I(R) \geq \dim R - \mathrm{ht}\, I$*. In particular, if* R *is a regular local ring containing a field, then* $\operatorname{inj\,dim} H^{\mathrm{ht}\, I}_I(R) = \dim R - \mathrm{ht}\, I$*.*

We will end the paper with an example, which shows that Theorem 1.4 does not hold in general.

For each R-module L, we denote by $\mathrm{Ass}\, h_R(L)$ (resp. $\mathrm{mAss}_R\, L$) the set

$$\{\mathfrak{p} \in \mathrm{Ass}_R(L) : \dim R/\mathfrak{p} = \dim L\}$$

(resp. the set of minimal primes of $\mathrm{Ass}_R\, L$). Also, the set of all zerodivisors on L is denoted by $Z_R(L)$. For any ideal $\mathfrak{b}$ of R, *the radical of* $\mathfrak{b}$, denoted by $\mathrm{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$, and we denote $\{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{b}\}$ by $V(\mathfrak{b})$. Finally, for any ideal $\mathfrak{b}$ of R, *the cohomological dimension* of an R-module M, with respect to $\mathfrak{b}$ is defined as

$$\mathrm{cd}(\mathfrak{b}, M) := \sup\{i \in \mathbb{Z} : H^i_{\mathfrak{b}}(M) \neq 0\}.$$

For any unexplained notation and terminology, we refer the reader to [3,12].

2 The Results

The following lemmas will be quite useful in the proof of the main results. Following $D := \operatorname{Hom}_R(\bullet, E_R(R/\mathfrak{m}))$ (resp. ω_R) denotes the Matlis duality functor (resp. the canonical module for R) (see [4, 3.3]).

Lemma 2.1 *Let $(R,\mathfrak{m})$ be a local Noetherian ring and let M be a finitely generated R-module. Let $\mathfrak{p}$ be a prime ideal of R such that $\dim R/\mathfrak{p} = 1$ and let $t \geq 1$ be an integer. Then $H^t_{\mathfrak{m}}(M)$ is $\mathfrak{p}$-cofinite if and only if $(H^{t-1}_{\mathfrak{p}}(M))_{\mathfrak{p}} = 0$.*

Proof See [1, Lemma 2.1]. ∎

Lemma 2.2 *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d. Then the R-module $H^d_{\mathfrak{m}}(R)$ is indecomposable.*

Proof Without loss of generality, we may assume that R is a complete Cohen–Macaulay local ring. Now, we suppose the contrary and look for a contradiction. Let $H^d_{\mathfrak{m}}(R) = A \oplus B$, where A and B are two non-zero Artinian R-modules. Then we have $\omega_R \cong D(A) \oplus D(B)$, where ω_R denotes the canonical module of R. So as the R-module ω_R is indecomposable, it follows that $D(A) = 0$ or $D(B) = 0$. Hence, $A = 0$ or $B = 0$, which is a contradiction. ∎

The following result will be useful in the proof of the main results in this section.

Theorem 2.3 *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring and let X and Y be non-empty subsets of $\operatorname{Ass}_R(R)$ such that $\operatorname{Ass}_R(R) = X \cup Y$ and $X \cap Y = \varnothing$. Set*

$$I := \bigcap_{\mathfrak{p}\in X} \mathfrak{p} \quad \text{and} \quad J := \bigcap_{\mathfrak{q}\in Y} \mathfrak{q}.$$

Then $R/(I+J)$ is an equidimensional local ring of dimension $\dim R - 1$.

Proof It follows from the hypothesis $X \cap Y = \varnothing$ that $\operatorname{ht}(I+J) \geq 1$. Now, we show that $\operatorname{ht}(I+J) = 1$. To do this, suppose the contrary is true. Then there exists a minimal prime ideal $\mathfrak{p}$ over $I+J$ such that $\operatorname{ht}\mathfrak{p} := n > 1$. Since $\operatorname{Ass}_R(R) = X \cup Y$, it follows that $I \cap J = \operatorname{nil}(R)$, and so $I \cap J$ is a nilpotent ideal of R. Therefore,

$$H^{n-1}_{I\cap J}(R) = 0 = H^n_{I\cap J}(R).$$

Now, in view of the Mayer–Vietoris sequence (see *e.g.*, [3, Theorem 3.2.3]), we obtain the isomorphism

$$H^n_{I+J}(R) \cong H^n_I(R) \oplus H^n_J(R).$$

Therefore,

$$H^n_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = H^n_{(I+J)R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \cong H^n_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}) \oplus H^n_{JR_{\mathfrak{p}}}(R_{\mathfrak{p}}).$$

Now, using Lemma 2.2, we deduce that

$$H^n_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \cong H^n_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}) \quad \text{or} \quad H^n_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \cong H^n_{JR_{\mathfrak{p}}}(R_{\mathfrak{p}}).$$

Consequently, in view of [13, Proposition 5.1], $H^n_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})$ is an $IR_{\mathfrak{p}}$ or $JR_{\mathfrak{p}}$-cofinite $R_{\mathfrak{p}}$-module. Next, as $\operatorname{ht}\mathfrak{p} > 1$, it is easy to see that there exists a prime ideal $\mathfrak{q} \in V(I)$ or $\mathfrak{q} \in V(J)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$ and $\operatorname{ht}\mathfrak{p}/\mathfrak{q} = 1$. Now, using [13, Proposition 4.1], one

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easily sees that the $R_\mathfrak{p}$-module $H^n_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})$ is $\mathfrak{q}R_\mathfrak{p}$-cofinite. Therefore, it follows from Lemma 2.1 that $H^{n-1}_{\mathfrak{q}R_\mathfrak{q}}(R_\mathfrak{q}) = 0$. On the other hand, as R is catenary, it follows that $\operatorname{ht}\mathfrak{p}/\mathfrak{q} = \operatorname{ht}\mathfrak{p} - \operatorname{ht}\mathfrak{q}$, and so $\operatorname{ht}\mathfrak{q} = \operatorname{ht}\mathfrak{p} - 1 = n - 1$. Hence, in view of Grothendieck's non-vanishing theorem we have $H^{n-1}_{\mathfrak{q}R_\mathfrak{q}}(R_\mathfrak{q}) \neq 0$, which is a contradiction. Therefore $\operatorname{ht}\mathfrak{p} = 1$, and so $\operatorname{ht}(I + J) = 1$. Now, as R is Cohen–Macaulay, it follows easily that $R/(I + J)$ is an equidimensional ring of dimension $\dim R - 1$, as required. ∎

Corollary 2.4 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring and let $x_1, \dots, x_t$ be an R-regular sequence. Let X and Y be non-empty subsets of $\operatorname{Ass}_R(R/(x_1, \dots, x_t))$ such that $\operatorname{Ass}_R(R/(x_1, \dots, x_t)) = X \cup Y$ and $X \cap Y = \varnothing$. Set*

$$I := \bigcap_{\mathfrak{p}\in X} \mathfrak{p} \quad \textit{and} \quad J := \bigcap_{\mathfrak{q}\in Y} \mathfrak{q}.$$

Then $R/(I + J)$ is an equidimensional local ring of dimension $\dim R - t - 1$.

Proof Since $R/(x_1, \dots, x_t)$ is a Cohen–Macaulay local ring, the assertion follows easily from Theorem 2.3. ∎

Lemma 2.5 *Let R be a Noetherian ring and let I be an ideal of R such that $\operatorname{cd}(I, R) = n > 0$. Then the R-module $H^n_I(R)$ is not finitely generated.*

Proof Since by the definition we have $H^n_I(R) \neq 0$, it follows that $\operatorname{Supp} H^n_I(R) \neq \varnothing$. Let $\mathfrak{p} \in \operatorname{Supp} H^n_I(R)$. Then it is easy to see that $\operatorname{cd}(IR_\mathfrak{p}, R_\mathfrak{p}) = n > 0$. So replacing of the ring R with the local ring $(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p})$, we can assume that $(R, \mathfrak{m})$ is a Noetherian local ring and I is an ideal of R such that $\operatorname{cd}(I, R) = n > 0$. Then using [3, Exercise 6.1.8] and Grothendieck's vanishing theorem, we have

$$H^n_I(R)/\mathfrak{m}H^n_I(R) \cong H^n_I(R) \otimes_R R/\mathfrak{m} \cong H^n_I(R/\mathfrak{m}) = 0.$$

Therefore, $H^n_I(R) = \mathfrak{m}H^n_I(R)$ and hence using Nakayama's lemma we can deduce that the R-module $H^n_I(R)$ is not finitely generated. ∎

We are now in a position to state and prove the first main result of this paper, which investigates the finiteness dimension $f_I(R)$ over a Cohen–Macaulay local ring.

Theorem 2.6 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring and let I be a non-nilpotent ideal of R. Then $f_I(R) = \max\{1, \operatorname{ht} I\}$.*

Proof There are two cases to consider.

Case 1. Suppose that $\operatorname{ht} I = 0$. Put

$$X := \operatorname{Ass}_R(R) \cap V(I) \quad \text{and} \quad Y := \operatorname{Ass}_R(R) \backslash V(I).$$

Let $J := \bigcap_{\mathfrak{p}\in X} \mathfrak{p}$ and $K := \bigcap_{\mathfrak{q}\in Y} \mathfrak{q}$. Since I is not nilpotent, it follows that $Y \neq \varnothing$. Also, as $\operatorname{ht} I = 0$, it follows that $X \neq \varnothing$. Moreover, it is easy to see that $\operatorname{Ass}_R(R) = X \cup Y$. Hence, in view of the proof of Theorem 2.3, we have $\operatorname{ht}(J + K) = 1$. Therefore, there exists a minimal prime ideal $\mathfrak{p}$ over $J + K$ such that $\operatorname{ht}\mathfrak{p} = 1$. Since $K \subseteq \mathfrak{p}$, there exists an ideal $\mathfrak{q} \in Y$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. As $I \subseteq J \subseteq \mathfrak{p}$, it follows that $I + \mathfrak{q} \subseteq \mathfrak{p}$. Moreover, as $I \nsubseteq \mathfrak{q}$ it follows that $\operatorname{ht}(I + \mathfrak{q}) > 0$. Therefore, $\operatorname{ht}(I + \mathfrak{q}) = \operatorname{ht}\mathfrak{p} = 1$. Thus, $\mathfrak{p}$ is a minimal prime ideal over $I + \mathfrak{q}$ and so $IR_\mathfrak{p} + \mathfrak{q}R_\mathfrak{p}$ is a $\mathfrak{p}R_\mathfrak{p}$-primary ideal. Hence, by

Grothendieck's non-vanishing theorem, we have $H^1_{IR_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) \neq 0$. Consequently, it follows from Grothendieck's vanishing theorem that $\operatorname{cd}(IR_\mathfrak{p}, R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) = 1$. Now, as $\operatorname{Supp}(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) \subseteq \operatorname{Spec} R_\mathfrak{p}$, it follows from [5, Theorem 2.2] that

$$\operatorname{cd}(IR_\mathfrak{p}, R_\mathfrak{p}) \geq \operatorname{cd}(IR_\mathfrak{p}, R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) = 1.$$

Using Grothendieck's vanishing theorem we can deduce that $\operatorname{cd}(IR_\mathfrak{p}, R_\mathfrak{p}) = 1$ and so by Lemma 2.5, the $R_\mathfrak{p}$-module $H^1_{IR_\mathfrak{p}}(R_\mathfrak{p}) \cong (H^1_I(R))_\mathfrak{p}$ is not finitely generated. In particular, the R-module $H^1_I(R)$ is not finitely generated. Now, as the R-module $H^0_I(R)$ is finitely generated, it follows that

$$f_I(R) = 1 = \max\{1, 0\} = \max\{1, \operatorname{ht} I\},$$

as required.

Case 2. Now suppose that $\operatorname{ht} I = n \geq 1$. Then we have $\operatorname{grade} I = n$, and so in view of [3, Theorem 6.2.7], $f_I(R) \geq n$. Moreover, by the definition there exists a minimal prime ideal $\mathfrak{q}$ over I such that $\operatorname{ht}\mathfrak{q} = n$. Hence, in view of Grothendieck's vanishing and non-vanishing theorems, we have

$$\operatorname{cd}(IR_\mathfrak{q}, R_\mathfrak{q}) = \operatorname{cd}(\mathfrak{q}R_\mathfrak{q}, R_\mathfrak{q}) = n.$$

Thus, by Lemma 2.5, the $R_\mathfrak{q}$-module $H^n_{IR_\mathfrak{q}}(R_\mathfrak{q}) \cong (H^n_I(R))_\mathfrak{q}$ is not finitely generated. In particular, the R-module $H^n_I(R)$ is not finitely generated. Hence, in view of the definition, we have

$$f_I(R) = n = \max\{1, n\} = \max\{1, \operatorname{ht} I\},$$

and this completes the proof. ∎

The next theorem is the second main result of this paper.

Theorem 2.7 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring and let I be a non-nilpotent ideal of R such that $\operatorname{mAss}_R(R/I) \subseteq \operatorname{Ass}_R(R)$. Then $R/(I+\Gamma_I(R))$ is an equidimensional local ring of dimension $\dim R - 1$.*

Proof Since I is not nilpotent, it is clear that $\Gamma_I(R) \subseteq Z_R(R)$, and so it follows from [12, Theorem 17.4] that $\dim R/\Gamma_I(R) = \dim R$. Moreover, as

$$\operatorname{Ass}_R\big(R/\Gamma_I(R)\big) = \operatorname{Ass}_R(R) \smallsetminus V(I),$$

it follows that I contains an $R/\Gamma_I(R)$-regular element x, and so

$$\dim R/\big(xR + \Gamma_I(R)\big) = \dim R/\Gamma_I(R) - 1.$$

Hence, $\dim R/(I + \Gamma_I(R)) \leq d - 1$.

Next, in view of the Artin–Rees lemma there exists a positive integer s such that $I^s \cap \Gamma_I(R) = 0$, and so

$$H^{n-1}_{I^s \cap \Gamma_I(R)}(R) = 0 = H^n_{I^s \cap \Gamma_I(R)}(R).$$

Hence, the Mayer–Vietoris sequence (see *e.g.,* [3, Theorem 3.2.3]) yields the isomorphism

$$H^n_{I+\Gamma_I(R)}(R) = H^n_{I^s+\Gamma_I(R)}(R) \cong H^n_{I^s}(R) \oplus H^n_{\Gamma_I(R)}(R) \cong H^n_I(R) \oplus H^n_{\Gamma_I(R)}(R).$$

Now, suppose that $\mathfrak{p}$ is a minimal prime ideal over $I+\Gamma_I(R)$ such that $\operatorname{ht}\mathfrak{p}=n>1$. Then, as $\mathfrak{p}$ is minimal over $I+\Gamma_I(R)$, we get the isomorphism

$$H^n_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})\cong H^n_{IR_\mathfrak{p}}(R_\mathfrak{p})\oplus H^n_{\Gamma_{IR_\mathfrak{p}}}(R_\mathfrak{p}).$$

Now, using Lemma 2.2, we deduce that

$$H^n_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})\cong H^n_{IR_\mathfrak{p}}(R_\mathfrak{p})\quad\text{or}\quad H^n_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})\cong H^n_{\Gamma_{IR_\mathfrak{p}}(R_\mathfrak{p})}(R_\mathfrak{p}).$$

Assume that $H^n_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})\cong H^n_{IR_\mathfrak{p}}(R_\mathfrak{p})$. Then, in view of [13, Proposition 5.1], $H^n_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})$ is an $IR_\mathfrak{p}$-cofinite $R_\mathfrak{p}$-module. Next, as $\operatorname{ht}\mathfrak{p}>1$ and $\operatorname{mAss}_R(R/I)\subseteq\operatorname{Ass}_R(R)$, it is easy to see that there exists a prime ideal $\mathfrak{q}\in V(I)$ such that $\mathfrak{q}\subseteq\mathfrak{p}$ and $\operatorname{ht}\mathfrak{p}/\mathfrak{q}=1$. Now, using [13, Proposition 4.1], it follows easily that the $R_\mathfrak{p}$-module $H^n_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})$ is $\mathfrak{q}R_\mathfrak{p}$-cofinite. Therefore, it follows from Lemma 2.1 that $H^{n-1}_{\mathfrak{q}R_\mathfrak{q}}(R_\mathfrak{q})=0$. On the other hand, as R is catenary, it follows that $\operatorname{ht}\mathfrak{p}/\mathfrak{q}=\operatorname{ht}\mathfrak{p}-\operatorname{ht}\mathfrak{q}$, and so

$$\operatorname{ht}\mathfrak{q}=\operatorname{ht}\mathfrak{p}-1=n-1.$$

Hence, in view of Grothendieck's non-vanishing theorem, we have $H^{n-1}_{\mathfrak{q}R_\mathfrak{q}}(R_\mathfrak{q})\neq 0$, which is a contradiction.

Now, assume that $H^n_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})\cong H^n_{\Gamma_{IR_\mathfrak{p}}(R_\mathfrak{p})}(R_\mathfrak{p})$. Then, again using the fact that

$$\operatorname{Ass}_R(R/\Gamma_I(R))=\operatorname{Ass}_R(R)\backslash V(I)\subseteq\operatorname{Ass}_R(R),$$

and repeating the above argument, we derive a contradiction. Therefore $\operatorname{ht}\mathfrak{p}=1$, and so $\operatorname{ht}(I+\Gamma_I(R))=1$. Now, as R is Cohen–Macaulay, it follows easily that $R/(I+\Gamma_I(R))$ is an equidimensional local ring of dimension $\dim R-1$, as required. ∎

Corollary 2.8 *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring and let A be a non-empty proper subset of* $\operatorname{Ass}_R(R)$*. Then $R/(I+\Gamma_I(R))$ is an equidimensional local ring of dimension* $\dim R-1$*, where $I=\bigcap_{\mathfrak{p}\in A}\mathfrak{p}$.*

Proof The assertion follows easily from Theorem 2.7. ∎

Proposition 2.9 *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring and let $\operatorname{Ass}_R(R)=\{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}$, $n\geq 2$. Let $A_j=\operatorname{Ass}_R(R)\backslash\{\mathfrak{p}_j\}$ and $I_j=\bigcap_{\mathfrak{p}\in A_j}\mathfrak{p}$, for all $1\leq j\leq n$. Then $0=\bigcap_{j=1}^n\Gamma_{I_j}(R)$ is the unique reduced primary decomposition of the zero ideal 0 in R, $\Gamma_{I_j}(R)$ is a $\mathfrak{p}_j$-primary ideal of R and $R/(I_j+\Gamma_{I_j}(R))$ is an equidimensional local ring of dimension* $\dim R-1$.

Proof As

$$\operatorname{Ass}_R\big(R/\Gamma_{I_j}(R)\big)=\operatorname{Ass}_R(R)\backslash V(I_j)=\{\mathfrak{p}_j\},$$

it follows that $\Gamma_{I_j}(R)$ is a $\mathfrak{p}_j$-primary ideal of R. Now, we show that $\bigcap_{j=1}^n\Gamma_{I_j}(R)=0$. To this end, we assume that $\bigcap_{j=1}^n\Gamma_{I_j}(R)\neq 0$ and derive a contradiction. Let $a\in\bigcap_{j=1}^n\Gamma_{I_j}(R)$ be such that $a\neq 0$. Then $(0:_R a)\subseteq Z_R(R)$, and so there exists $\mathfrak{p}_j\in\operatorname{Ass}_R(R)$ such that $(0:_R a)\subseteq\mathfrak{p}_j$. Next, as $a\in\Gamma_{I_j}(R)$ it follows that there exists a positive integer k such that $I_j^k\subseteq(0:_R a)\subseteq\mathfrak{p}_j$, and so $I_j\subseteq\mathfrak{p}_j$. Therefore, there exists $\mathfrak{p}_i\in A_j$ such that $\mathfrak{p}_i\subsetneqq\mathfrak{p}_j$, which is a contradiction (note that $\operatorname{Ass}_R(R)=\operatorname{mAss}_R(R)$). Now, using [12, Theorem 6.8] we see that $\mathfrak{p}_j$-primary component $\Gamma_{I_j}(R)$ of the zero ideal 0 of R is uniquely determined. That is, $0=\bigcap_{j=1}^n\Gamma_{I_j}(R)$ is the unique reduced primary decomposition of the zero ideal 0 in R. Moreover, it follows from Corollary 2.8 that the ring $R/(I_j + \Gamma_{I_j}(R))$ is equidimensional local of dimension $\dim R - 1$. ■

The following lemma is needed in the proof of Theorem 2.11.

Lemma 2.10 *Let $(R, \mathfrak{m})$ be a local ring and M an arbitrary R-module. Let x be an element of $\mathfrak{m}$ such that $x \notin \bigcup_{\mathfrak{p}\in \mathrm{Ass}_R(M)\setminus V(\mathfrak{m})} \mathfrak{p}$. Then $\Gamma_{Rx}(M) = \Gamma_{\mathfrak{m}}(M)$.*

Proof As $x \in \mathfrak{m}$, it is enough to show that $\Gamma_{Rx}(M) \subseteq \Gamma_{\mathfrak{m}}(M)$. To do this, let $w \in \Gamma_{Rx}(M)$. Then $x \in \mathrm{Rad}(0 :_R w)$. Since $\mathrm{mAss}_R\, R/(0 :_R w) \subseteq \mathrm{Ass}_R(M)$, it follows from the assumption $x \notin \bigcup_{\mathfrak{p}\in \mathrm{Ass}_R(M)\setminus V(\mathfrak{m})} \mathfrak{p}$ that $\mathrm{Rad}(0 :_R w) = \mathfrak{m}$, and so there exists $n \in \mathbb{N}$ such that $\mathfrak{m}^n w = 0$. Thus $w \in \Gamma_{\mathfrak{m}}(M)$, as required. ■

The following theorem is in preparation for the third main result of this paper, which gives us a lower bound of injective dimension of $H_I^{\mathrm{ht}\, I}(R)$. Here, $D_I(R)$ denotes the ideal transform of R with respect to I (see [3, 2.2.1]).

Theorem 2.11 *Let $(R, \mathfrak{m})$ be a complete local equidimensional ring of dimension d and I an ideal of R such that $\mathrm{ht}\, I = t$. Then $H_{\mathfrak{m}}^{d-t}(H_I^t(R)) \neq 0$. In particular,*

$$\operatorname{inj\,dim} H_I^t(R) \geq d - t.$$

Proof As R is catenary, it follows from [12, Lemma 2, p. 250] that

$$\mathrm{ht}\, J + \dim R/J = \dim R,$$

for every ideal J of R. In particular, we have $\dim R/I = d - t$. We now use induction on $d - t$. When $d = t$, the ring R/I is Artinian and so $\mathrm{Rad}(I) = \mathfrak{m}$. Hence, $H_I^t(R) = H_{\mathfrak{m}}^t(R)$ and so as $H_{\mathfrak{m}}^0(H_I^t(R)) = H_{\mathfrak{m}}^t(R)$, the assertion follows from Grothendieck's non-vanishing theorem (see [3, Theorem 6.1.4]) in this case.

Assume, inductively, that $d - t > 0$ and that the result has been proved for the ideals J with $\dim R/J = 0, 1, \ldots, d - t - 1$. Since the sets $\mathrm{Ass}_R(H_I^t(R))$ and $\mathrm{Ass}_R(H_I^{t+1}(R))$ are countable, it follows from [11, Lemma 3.2] that

$$\mathfrak{m} \nsubseteq \Big(\bigcup_{\mathfrak{p}\in \mathrm{Ass}_R(H_I^t(R))\setminus V(\mathfrak{m})} \mathfrak{p} \Big) \cup \Big(\bigcup_{\mathfrak{p}\in \mathrm{Ass}_R(H_I^{t+1}(R))\setminus V(\mathfrak{m})} \mathfrak{p} \Big) \cup \Big(\bigcup_{\mathfrak{p}\in \mathrm{Assh}_R(R/I)} \mathfrak{p} \Big).$$

Whence, there exists $x \in \mathfrak{m}$ such that

$$x \notin \Big(\bigcup_{\mathfrak{p}\in \mathrm{Ass}_R(H_I^t(R))\setminus V(\mathfrak{m})} \mathfrak{p} \Big) \cup \Big(\bigcup_{\mathfrak{p}\in \mathrm{Ass}_R(H_I^{t+1}(R))\setminus V(\mathfrak{m})} \mathfrak{p} \Big) \cup \Big(\bigcup_{\mathfrak{p}\in \mathrm{Assh}_R(R/I)} \mathfrak{p} \Big).$$

Then it follows easily from $x \notin \bigcup_{\mathfrak{p}\in \mathrm{Assh}_R(R/I)} \mathfrak{p}$ that

$$\dim R/(I + Rx) = d - t - 1,$$

and in view of Lemma 2.10, we have

$$\Gamma_{Rx}\big(H_I^t(R)\big) = \Gamma_{\mathfrak{m}}\big(H_I^t(R)\big) \quad \text{and} \quad \Gamma_{Rx}\big(H_I^{t+1}(R)\big) = \Gamma_{\mathfrak{m}}\big(H_I^{t+1}(R)\big).$$

Moreover, there is an exact sequence

$$(2.1) \qquad 0 \longrightarrow H_{Rx}^1\big(H_I^t(R)\big) \longrightarrow H_{I+Rx}^{t+1}(R) \longrightarrow H_{Rx}^0\big(H_I^{t+1}(R)\big) \longrightarrow 0,$$

(see [14, Corollary 3.5]).

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Now, if $\dim R/I = 1$, then in view of [2, Theorem 2.6] the R-module $H_I^t(R) = H_I^{d-1}(R)$ is I-cofinite. Next, it is easy to see that $\dim \operatorname{Supp} H_I^{d-1}(R) = 1$ (note that $\dim R/I = 1$). Hence, it follows from [10, Theorem 2.9] that $H_{\mathfrak{m}}^1(H_I^{d-1}(R)) \neq 0$, and so the result has been proved in this case. Therefore, we assume that $\dim R/I \geq 2$. Then

$$\dim R/(I + Rx) = d - t - 1 \geq 1,$$

and so in view of Grothendieck's vanishing theorem

$$H_{\mathfrak{m}}^{d-t-1}\big(\Gamma_{\mathfrak{m}}\big(H_I^{t+1}(R)\big)\big) = 0.$$

Hence, using the exact sequence (2.1), we obtain the exact sequence

$$H_{\mathfrak{m}}^{d-t-1}\big(H_{Rx}^1(H_I^t(R))\big) \longrightarrow H_{\mathfrak{m}}^{d-t-1}\big(H_{I+Rx}^{t+1}(R)\big) \longrightarrow 0.$$

Thus, by the inductive hypothesis, $H_{\mathfrak{m}}^{d-t-1}(H_{Rx}^1(H_I^t(R))) \neq 0$.

On the other hand, since $d - t > 0$, it yields that

$$H_{\mathfrak{m}}^{d-t}\big(H_I^t(R)\big) \cong H_{\mathfrak{m}}^{d-t}\big(H_I^t(R)/\Gamma_{\mathfrak{m}}(H_I^t(R))\big).$$

Now, let $T := H_I^t(R)/\Gamma_{\mathfrak{m}}(H_I^t(R))$. It is thus sufficient for us to show that $H_{\mathfrak{m}}^{d-t}(T) \neq 0$. To do this, in view of [3, Remark 2.2.17], there is the exact sequence

$$0 \longrightarrow T \longrightarrow D_{Rx}(T) \longrightarrow H_{Rx}^1(T) \longrightarrow 0. \tag{2.2}$$

Also, in view of [3, Theorem 2.2.16], we have $D_{Rx}(T) \cong T_x$, and so

$$D_{Rx}(T) \xrightarrow{x} D_{Rx}(T)$$

is an R-isomorphism. Therefore, for all $i \geq 0$,

$$H_{\mathfrak{m}}^i\big(D_{Rx}(T)\big) \xrightarrow{x} H_{\mathfrak{m}}^i\big(D_{Rx}(T)\big),$$

is an R-isomorphism, and hence $H_{\mathfrak{m}}^i(D_{Rx}(T)) = 0$, for all $i \geq 0$. Consequently, it follows from the exact sequence (2.2) that

$$H_{\mathfrak{m}}^{d-t}(T) \cong H_{\mathfrak{m}}^{d-t-1}\big(H_{Rx}^1(T)\big).$$

As $H_{\mathfrak{m}}^{d-t-1}(H_{Rx}^1(T)) \neq 0$, this completes the inductive step. ∎

Corollary 2.12 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d and let I be an ideal of R such that* $\operatorname{ht} I = t$. *Then $H_{\mathfrak{m}}^{d-t}(H_I^t(R)) \neq 0$. In particular,* $\operatorname{inj\,dim} H_I^t(R) \geq d - t$.

Proof Let $\widehat{R}$ denote the completion of R with respect to the $\mathfrak{m}$-adic topology. Then as $(\widehat{R}, \mathfrak{m}\widehat{R})$ is a complete local equidimensional ring of dimension d, the assertion follows from Theorem 2.11, the faithfully flatness of the homomorphism $R \to \widehat{R}$ and the fact that

$$\operatorname{ht} I = \operatorname{grade} I = \operatorname{grade} I\widehat{R} = \operatorname{ht} I\widehat{R}.$$ ∎

Lemma 2.13 *Let $(R, \mathfrak{m})$ be a regular local ring containing a field and let I be an ideal of R. Then, for any integer n with $H_I^n(R) \neq 0$,*

$$\operatorname{inj\,dim} H_I^n(R) \leq \dim \operatorname{Supp} H_I^n(R).$$

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Proof The result follows from [8, 9]. ■

Corollary 2.14 *Let $(R, \mathfrak{m})$ be a regular local ring containing a field and let I be an ideal of R such that* $\operatorname{ht} I = t$. *Then* $\operatorname{inj\,dim} H^t_I(R) = \dim R - t$.

Proof In view of Corollary 2.12 and Lemma 2.13, it is enough to show that

$$\dim \operatorname{Supp} H^t_I(R) = \dim R - t.$$

To this end, as $\operatorname{Supp} H^t_I(R) \subseteq V(I)$ and $\dim R/I = \dim R - t$, we have

$$\dim \operatorname{Supp} H^t_I(R) \leq \dim R - t.$$

On the other hand, since $\operatorname{ht} I = t$, there exists a minimal prime $\mathfrak{p}$ over I such that $\operatorname{ht} \mathfrak{p} = t$. Now, in view of [3, Theorems 4.3.2 and 6.1.4], we deduce that

$$\big(H^t_I(R)\big)_{\mathfrak{p}} \cong H^t_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}) \cong H^t_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \neq 0.$$

Thus, $\mathfrak{p} \in \operatorname{Supp} H^t_I(R)$, and so as $\dim R/\mathfrak{p} = \dim R - t$, it follows that

$$\dim \operatorname{Supp} H^t_I(R) \geq \dim R - t.$$

This completes the proof. ■

We end the paper with the following example, which shows that Corollary 2.14 does not hold in general.

Example 2.15 Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d \geq 3$, $\mathfrak{p}$ a prime ideal of R such that $\dim R/\mathfrak{p} = 1$ and $x \in \mathfrak{m} \setminus \mathfrak{p}$. Then

$$\operatorname{inj\,dim} H^{\dim R-1}_{Rx\cap\mathfrak{p}}(R) = 0 \quad \text{and} \quad \dim \operatorname{Supp} H^{\dim R-1}_{Rx\cap\mathfrak{p}}(R) = 1.$$

Proof Let $\dim R = d$. Since $\operatorname{Rad}(\mathfrak{p} + Rx) = \mathfrak{m}$, it follows from the Mayer–Vietoris sequence (see *e.g.*, [3, Theorem 3.2.3]) that

$$0 \longrightarrow H^{d-1}_{\mathfrak{p}}(R) \longrightarrow H^{d-1}_{x\mathfrak{p}}(R) \longrightarrow H^d_{\mathfrak{m}}(R) \tag{2.3}$$

is an exact sequence. Since, in view of the proof of Corollary 2.14, $\dim \operatorname{Supp} H^{d-1}_{\mathfrak{p}}(R) = 1$ and $H^d_{\mathfrak{m}}(R)$ is Artinian, it follows that $\dim \operatorname{Supp} H^{d-1}_{x\mathfrak{p}}(R) = 1$.

On the other hand, the exact sequence

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R/xR \longrightarrow 0$$

induces the exact sequence

$$H^{d-2}_{x\mathfrak{p}}(R/xR) \longrightarrow H^{d-1}_{x\mathfrak{p}}(R) \xrightarrow{x} H^{d-1}_{x\mathfrak{p}}(R) \longrightarrow H^{d-1}_{x\mathfrak{p}}(R/xR).$$

Since $\Gamma_{x\mathfrak{p}}(R/xR) = R/xR$ and $d \geq 3$, it follows that

$$H^{d-2}_{x\mathfrak{p}}(R/xR) = 0 = H^{d-1}_{x\mathfrak{p}}(R/xR).$$

Therefore, the R-homomorphism $H^{d-1}_{x\mathfrak{p}}(R) \xrightarrow{x} H^{d-1}_{x\mathfrak{p}}(R)$ is an isomorphism, and so $(H^{d-1}_{x\mathfrak{p}}(R))_x \cong H^{d-1}_{x\mathfrak{p}}(R)$.

On the other hand, from the exact sequence (2.3), we have

$$(H^{d-1}_{x\mathfrak{p}}(R))_x \cong (H^{d-1}_{\mathfrak{p}}(R))_x.$$

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Moreover, the exact sequence $0 \longrightarrow H_{\mathfrak{p}}^{d-1}(R) \longrightarrow E_R(R/\mathfrak{p}) \longrightarrow E_R(R/\mathfrak{m})$ implies that

$$\big(H_{\mathfrak{p}}^{d-1}(R)\big)_x \cong \big(E_R(R/\mathfrak{p})\big)_x \cong E_R(R/\mathfrak{p}).$$

Therefore, $H_{x\mathfrak{p}}^{d-1}(R) \cong E_R(R/\mathfrak{p})$, and so $\operatorname{inj\,dim} H_{x\mathfrak{p}}^{d-1}(R) = 0$, as required. ■

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