ON VARIETIES OF METABELIAN p-GROUPS, AND THEIR LAWS

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1. Introduction

The purpose of this article is to present some results on varieties of metabelian p-groups, nilpotent of class c, with the prime p greater than c. After some preliminary lemmas in § 3, it is established in § 4, Theorem **3**, that there is a simple basis for the laws of such a variety, and this basis is explicitly stated. This allows the description of the lattice of such varieties, and in § 5, Theorem **4**, it is shown that each such variety has a two-generator member which generates it. Theorem 7 extends this result to show that there is in each such variety a two-generator critical group which generates it; this is established by the help of Theorem 5, which states that each critical group is a two-generator group, and Theorem 6, which gives explicitly the varieties generated by the proper subgroups, by the proper quotient groups, and by the proper factor groups of such a critical group.

Some of the theorems and lemmas, with minor re-wording of their theses and proofs, also hold for the variety $\mathfrak{M}_c$ of all metabelian groups of class c; in particular, Theorem 1 can be modified to afford an independent proof of a result due to Magnus, quoted in [2], (and also [9], corollary 36.23), that the basic c-weight commutators of a reduced free group of $\mathfrak{M}_c$ are independent.

Some of the theorems do not generalise: for example, any generalisation of Theorem 5 is limited by the existence of 3-generator critical metabelian 3-groups.

2. Notation and definitions

As usual, the commutator $x^{-1}y^{-1}xy$ is written (x, y), with left-normed commutators of weight k being defined recursively by $(x_1, x_2, \cdots, x_k) = ((x_1, x_2, \cdots, x_{k-1}), x_k)$, $k > 2$. For repeated symbols in a commutator, the notation (x, ny) is used (n being a positive integer) where $(x, 1y) = (x, y)$, and recursively $(x, ny) = ((x, (n-1)y), y)$, $n > 1$; the obvious meaning is attached to the symbol $(x_1, n_2x_2, n_3x_3, \cdots, n_sx_s)$ where all the n_i are non-

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negative integers, and at least one n_i is not zero; the symbol could clearly be defined recursively. By $\gamma_k(G)$ we mean the kth term of the lower central series of G, i.e. the verbal subgroup $\gamma_k(G) = Gp\{(x_1, x_2, \cdots, x_k)| x_i \in G,$ $i = 1, \cdots, k\}$. The notation G^p means the verbal subgroup $Gp \{x^p \mid x \in G\}$, and the Frattini subgroup is denoted by $\Phi(G)$, where we know, if G is a finite p-group, that $\Phi(G) = Gp\{G^p, \gamma_2(G)\}$.

The varieties considered are subvarieties of $\mathfrak{M}_c$, the variety of all metabelian groups of class c. The phrase "$\mathfrak{V}$ *is a variety of type* $\mathfrak{M}$ (p, c)" is defined to mean that for the fixed prime $p > c$, $\mathfrak{V}$ has p-power exponent, is metabelian, and is of class c; also, that at least one group in V is not of class $c-1$, so that there is, amongst the laws of V, a law of the form $(x_1, x_2, \cdots, x_c)^{p^{\alpha_c}} = 1$, where $\alpha_c > 0$. It is to be understood that α_c is minimal for this law in $\mathfrak{V}$, and the symbol α_c is used in this sense throughout.

The variety generated by a group G is denoted by Var (G); the variety generated by its proper subgroups is denoted by Var $(S-1)G$, and by all its proper factors by Var $(QS-1)G$. A finite group G is called *critical* if $G \notin$ Var $(QS-1)G$. The properties of varieties, and the correspondences amongst varieties, free groups, verbal subgroups, reduced free groups, and sets of laws of varieties, may all be found in [8] and [9].

Calculations will be carried out, almost entirely, in metabelian groups, i.e. groups for which $\gamma_2(G)$ is Abelian. In this case, we have that $(x_1, x_2, x_3, x_4) = (x_1, x_2, x_4, x_3)$, and the Jacobi-Witt identity becomes

$$(x, y, z)(y, z, x)(z, x, y) = 1.$$

The general rules

$$(xy, z) = (x, z)(x, z, y)(y, z)$$
$$(x, yz) = (x, z)(x, y)(x, y, z)$$

are then sufficient to perform any of the calculations used. In particular, any product of commutators may be rearranged to an equivalent product of *basic* commutators, according to some ordering of the generators involved.

3. Preliminary lemmas

The first result is a lemma about possible manipulations in a commutative group; a corollary about certain laws in a variety of p-groups follows from it.

LEMMA 1. *Let G be a group with commuting elements* W_k, $k = 1, 2, \cdots n$ *and containing no elements of prime order* $\leq n$. *If the set* $\{S_\lambda\}$ *of relations hold*:

$$S_\lambda : \prod_{k=1}^{n} W_k^{\lambda^k} = 1, \qquad \qquad \lambda = 1, 2, \cdots, n$$

then $W_k = 1$ *for each k*, $k = 1, 2, \cdots n$.

PROOF. By raising the left-hand sides of the relations S_λ to powers, taking inverses, and multiplying together, we can derive new relations of the form R:

$$R : \prod_{k=1}^{n} W_k^{\xi^k} = 1.$$

Now corresponding to the λth relation of the set $\{S_\lambda\}$, we have the exponents of the W_k as the λth row of the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^n \\ 3 & 3^2 & \cdots & 3^n \\ \vdots & \vdots & & \vdots \\ n & n^2 & \cdots & n^n \end{pmatrix}.$$

Clearly the set of ξ_k in a relation of the form R will correspond to the result, in some row, after some sequence of row operations has been carried out on this matrix, entries being considered over the ring of integers, (i.e. disallowing division). But from an explicit formula [1] for the Stirling numbers of the second kind, $S_j^{(i)}$, namely

$$S_{ij} = i!\, S_j^{(i)} = \sum_{t=0}^{i} (-1)^{i-t} \binom{i}{t} t^j$$

(which yields $S_i^{(i)} = 1$, $S_j^{(i)} = 0$ for $i > j$), it is clear that ring operations alone suffice to bring the matrix, by row operations, to the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 2! & 6 & 14 & \cdots & \\ 0 & 0 & 3! & 36 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & n! \end{pmatrix}.$$

The entry in the ith row and j^{th} column is S_{ij}, and the entries in a suitable row now correspond to the ξ_k of some relation of the form R. In particular, since $W_n^{n!} = 1$ is a law, and G contains no elements of prime order $\leqq n$, it follows that $W_n = 1$. Since the rth row of the matrix corresponds to the relation

$$W_r^{r!} \prod_{k=r+1}^{n} W_k^{s_{rk}} = 1, \qquad\qquad r \leqq n,$$

an induction over $(n-r)$ completes the proof of the lemma.

We now turn to an immediate corollary, which deals with certain laws in a variety $\mathfrak{B}$ of p-groups. When dealing with laws in a variety,

certain transformations applied to the laws (e.g. thorough replacement of one variable by another, thorough replacement of a variable by 1, etc.) yield valid laws; essentially, such transformations are endomorphisms on the reduced free group of suitable rank. Such a transformation is termed "valid" in the following:

COROLLARY 1.1. *Let* $\mathfrak{B}$ *be a variety of p-groups, with the law* $\prod_{k=1}^{n} W_k = 1$, $n < p$, *where the* W_k *are distinct commuting words. If there is a set of valid transformations* $\{\phi_\lambda\}$, $\lambda = 1, 2, \cdots, n$, *such that* $\phi_\lambda(W_k) = W_k^{\lambda^k}$, *then* $W_k = 1$ *is a law in* $\mathfrak{B}$ *for each* k: $k = 1, 2, \cdots, n$.

PROOF. By the transformations ϕ_λ, we deduce the set $\{L_\lambda\}$ of laws of $\mathfrak{B}$:

$$L_\lambda : \prod_{k=1}^{n} W_k^{\lambda^k} = 1, \qquad\qquad \lambda = 1, 2, \cdots, n.$$

Interpreting these laws as relations on the generators of a reduced free group of suitable rank, Lemma 1 applies, and the corollary follows. [An example of such a set of words and transformations is as follows: Let $\mathfrak{B}$ be a subvariety of $\mathfrak{N}_c$ (the variety of all groups nilpotent of class c), with F a reduced free group on generators $x, y, z, w, \cdots$, and let $W_k = (x, (c-k-1)y, kz)^{\beta_k}$, $k = 1, 2, \cdots, c-1$. Then a set of endomorphisms with the required property is given by ϕ_λ being the substitution of z^λ for z, leaving $x, y, w, \cdots$, fixed, in every word of F.]

We now turn to varieties of type $\mathfrak{M}(p, c)$, and first consider laws which involve two variables only, in commutator words of weight c. From corollary 1.1 we derive the further corollary:

COROLLARY 1.2. *Let* $\mathfrak{B}$ *be a variety of type* $\mathfrak{M}(p, c)$. *If* $\mathfrak{B}$ *has a law of the form*

$$\prod_{i=0}^{c-2} (x, (c-1-i)y, ix)^{\beta_i} = 1, \quad \textit{(with some } \beta_i \neq 0)$$

then $\mathfrak{B}$ *also has the set of laws* $\{L_i\}$,

$$L_i : (x, (c-1-i)y, ix)^{p^\mu} = 1, \qquad i = 0, 1, \cdots c-2,$$

where $\mu > 0$, *and* p^μ *is the g.c.d. of the non-zero* β_i.

PROOF. We may regard $(x, (c-1-i)y, ix)^{\beta_i}$ as a W_k of corollary 1.1, with $k = i+1$, $n = c-1$, and the ϕ_λ being afforded by the substitution of x^λ for x. It then follows that $(x, (c-1-i)y, ix)^{\beta_i} = 1$ is a law for each i. Since $\mathfrak{B}$ is of type $\mathfrak{M}(p, c)$, clearly we may take $\beta_i = p^{\xi_i}$ for some $\xi_i \geq 0$, (or else $\beta_i = 0$, and the corresponding term is trivial in the original product). Further, if $(x, (c-1-i)y, ix) = 1$ were a law, then a result due to Gupta and Newman [4] would require that, for any $G \in \mathfrak{B}$, the exponent of $\gamma_c(G)$

divide the number $c(c-i)(c-2-i)!(c-1-i)!i!\prod_{k=1}^{i}k!\prod_{t=1}^{c-2-i}t!$. Since $p>c$, this is clearly not so, and thus $\xi_i>0$. Then for any non-zero β_i of the original law, we have the law L_i:.

$$L_i : (x,\ (c-1-i)y,\ ix)^{\beta_i} = 1,\ \beta_i = p^{\xi_i},\ \xi_i > 0.$$

In L_i, substitute xy for x: then

$$L_i' : \prod_{s=0}^{i} (x,\ (c-1-i+s)y,\cdot(i-s)x)^{\binom{i}{s}\beta_i} = 1.$$

In L_i, substitute xy for y: then

$$L_i'' : \prod_{s=0}^{c-2-i} (x,\ (c-1-i-s)y,\ (i+s)x)^{\binom{c-2-i}{s}\beta_i} = 1.$$

By what we have already proved above, *each* of these laws may be appropriately "factored"; since p divides neither $\binom{i}{s}$ nor $\binom{c-2-i}{s}$, we have for each non-zero β_i the set of laws $\{S_m\}$

$$\{S_m\} : (x,\ (c-1-m)y,\ mx)^{\beta_i} = 1,\quad m = 0, 1, 2, \cdots, c-2.$$

Since each non-zero β_i is of the form p^{ξ_i}, $\xi_i>0$, we may take the minimal ξ_i, and the corollary has been proved.

It is worth noting that the following corollary, which will be used in § 4, can be derived immediately from Gupta and Newman's result;

COROLLARY 1.3. *If $\mathfrak{B}$ is a variety of metabelian p-groups, of class c, with $p>c$, and the nth-Engel law $(x, ny)=1$ holds in $\mathfrak{B}$, with $n<c$, then $\mathfrak{B}$ is of class $n+1$.*

We now turn to laws involving more than two variables.

LEMMA 2. *In a variety $\mathfrak{B}$ of type $\mathfrak{M}(p, c)$, there is no law of the form $(x_1, x_2, s_3x_3, \cdots, s_tx_t)^{p^\mu} = 1$, with $\mu<\alpha_c$, where the commutator is of weight c in the t distinct variables $x_1, x_2, \cdots, x_t$, $t\geq 3$, and all $s_i>0$.*

This lemma can be extended to the case of two variables by a corollary:

COROLLARY 2.1. *In a variety $\mathfrak{B}$ of type $\mathfrak{M}(p, c)$, there is no law of the form $(x, (c-1)y)^{p^\mu} = 1$, with $\mu<\alpha_c$.*

The proofs follow.

PROOF OF LEMMA 2. Since the result for $t=c$ is immediate by definition of $\mathfrak{B}$, we proceed by induction on $c-t$, the inductive hypothesis being that the lemma is true for $t=c, c-1, \cdots, c-r$, with $r\leq c-4$. Then if there were a law of the above form for $t=c-r-1$, it could be written in the form L:

$$L : (x_1, x_2, \cdots, s_{c-r-2}x_{c-r-2}, ix_{c-r-1})^{p^\mu} = 1$$

where the commutator is of weight c, and $i = s_{c-r-1} > 0$. By substituting

$x_{c-r-1}x_{c-r}$ for x_{c-r-1} wherever it occurs, and using L to remove the term with $k = 0$, we have the new law L';

$$L' : \prod_{k=1}^{i} (x_1, x_2, \cdots, x_{c-r-2}, (i-k)x_{c-r-1}, kx_{c-r})^{\binom{i}{k}p\mu} = 1.$$

Lemma 1 now applies to L', W_k being the power of the commutator with k entries of x_{c-r}, n being i, and ϕ_λ being the substitution of x_{c-r}^λ for x_{c-r} throughout. Thus, in particular,

$$(x_1, x_2, \cdots, x_{c-r-2}, (i-1)x_{c-r-1}, x_{c-r})^{ip\mu} = 1,$$

and since $p > c > i$, we have a contradiction to the inductive hypothesis if $\mu < \alpha_c$, and the lemma is established.

We turn to the corollary;

PROOF OF COROLLARY 2.1. In the proposed law $(x, (c-1)y)^{p\mu} = 1$, substitute yz for y, giving the law

$$(x, (c-1)y)^{p\mu} \prod_{k=1}^{c-2} W_k (x, (c-1)z)^{p\mu} = 1$$

where

$$W_k = (x, y, (c-2-k)y, kz)^{\binom{c-2}{k}n\mu} (x, z, (c-1-k)y, (k-1)z)^{\binom{c-2}{k-1}p\mu}.$$

But the proposed law gives $\prod_{k=1}^{c-2} W_k = 1$, to which Lemma 1 may be applied, ϕ_λ being the substitution of z^λ for z; in particular,

$$W_{c-2} = (x, y, (c-2)z)^{p\mu} (x, z, y, (c-3)z)^{(c-2)p\mu} = 1.$$

In this law, interchange x and y and take inverses: this gives

$$W'_{c-2} = (x, y, (c-2)z)^{p\mu} (z, y, x, (c-3)z)^{(c-2)p\mu} = 1.$$

Since $\mathfrak{B}$ is metabelian, we have the Jacobi identity

$$(x, z, y)(z, y, x)(y, x, z) = 1,$$

from which, since $\mathfrak{B}$ is of class c,

$$(x, z, y, (c-3)z) (z, y, x, (c-3)z) (y, x, (c-2)z) = 1.$$

Taking this relation with the law $W_{c-2}W'_{c-2}=1$, we derive $(x, y, (c-2)z)^{cp\mu}=1$, and since $p > c$, we have $(x, y, (c-2)z)^{p\mu} = 1$, so that $\mu \not< \alpha_c$ by Lemma 2, and the corollary is proved.

From corollary 2.1, we have immediately:

COROLLARY 2.2. *In a variety $\mathfrak{B}$ of type $\mathfrak{M}$ (p, c), the laws $(x, (c-1)y)^{p^{\alpha_c}}=1$ and $(x_1, x_2, \cdots, x_c)^{p^{\alpha_c}} = 1$ are equivalent.*

By taking Corollary 1.2 and Corollary 2.1 together, we have the stronger result:

LEMMA 3. *For a variety $\mathfrak{B}$ of type $\mathfrak{M}(p, c)$, any law of the form $\prod_{i=0}^{c-2} (x, (c-1-i)y, ix)^{\beta_i} = 1$ can be assumed to have $\beta_i = 0 \bmod p^{\alpha_\bullet}$ for each i.*

The next lemma will provide an inductive step to help establish a theorem about basic commutators in a reduced free group of a variety $\mathfrak{B}$ of type $\mathfrak{M}(p, c)$. Using the properties of metabelian groups, any commutator law can be shown to be equivalent to a law which involves basic commutators only, according to some ordering of the generators of the corresponding free group. Lemma 4 will be used later to establish Theorem 1, which deals with basic commutators; to state Lemma 4, we need the following notation.

Let $A(i, j)$ be a product of commutators of weight c, whose entries are from the set of n variables $x_1, x_2, \cdots, x_n$, of the form

$$A(i, j) = \prod_{q \in Q_j} (x_i, x_j, N_j x_j, N_{j+1} x_{j+1}, \cdots, N_n x_n)^{\alpha_q(i, j)}, \quad n \geq i > j \geq 1.$$

Q_j is the set of all possible partitions of $c-2$ into $n-j+1$ parts $\{\beta_j, \beta_{j+1}, \cdots, \beta_n\}$ so that $\beta_j + \beta_{j+1} + \cdots + \beta_n = c-2$, $\beta_k \geq 0$ for $k = j, j+1, \cdots, n$; q is the element of Q_j with $\beta_k = N_k$, and $\alpha_q(i, j)$ is an integer.

Using this notation, we establish

LEMMA 4. *If, for a variety $\mathfrak{B}$ of type $\mathfrak{M}(p, c)$, there is an integer s, $2 \leq s < n$ such that all laws of the form*

$$L(s) : A(s+1, s)A(s+2, s) \cdots A(n, s) = 1$$

are consequences of the law $(x_1, x_2, \cdots, x_c)^{p^{\alpha_\bullet}} = 1$ in the sense that all $\alpha_q(k, s)$ in $L(s)$ are $0 \bmod p^{\alpha_\bullet}$, then the same is true for all laws of the form $L(s-1)$.

PROOF. Any particular law of the form $L(s-1)$ may be explicitly written out as

$$L(s-1) : P(1) \prod_{q \in Q_{s-1}} (x_s, x_{s-1}, N_{s-1} x_{s-1}, \cdots, N_n x_n)^{\alpha_q(s, s-1)} = 1$$

where $P(1)$ is the case $\lambda = 1$ for the product

$$P(\lambda) = \prod_{k=s+1}^{n} \prod_{q \in Q_{s-1}} (x_k, x_{s-1}, N_{s-1} x_{s-1}, \cdots N_n x_n)^{\xi_q}$$

with $\xi_q = \lambda^{N_s} \alpha_q(k, s-1)$, λ an integer, the elements q of Q_{s-1} being identified by the entries in the $(n-s+2)$-tuple $\{N_{s-1}, N_s, \cdots, N_n\}$. (The form $P(\lambda)$ is needed later.)

Substituting x_s^λ for x_s throughout $L(s-1)$, we derive the law L':

$$L' : P(\lambda) \prod_{q \in Q_{s-1}} (x_s, x_{s-1}, N_{s-1} x_{s-1}, \cdots, N_n x_n)^{\beta_q} = 1$$

where $\beta_q = \lambda^{N_s+1} \alpha_q(s, s-1)$. Then substituting x_s for x_{s-1} throughout L', we have L'':

$$L'' : \prod_{k=s+1}^{n} \prod_{q \in Q_{s-1}} (x_k, x_s, (N_{s-1}+N_s)x_s, N_{s+1}x_{s+1}, \cdots, N_n x_n)^{\xi_q} = 1.$$

Now this last law L'' may be written as a law of the form $L(s)$ by collecting terms, and considering the product over the set Q_s of partitions. Thus, for each element of Q_s of the form $\{\beta_s, N_{s+1}, N_{s+2}, \cdots, N_n\}$, and each $k > s$, we have in L'' a term of the form

$$(x_k, x_s, \beta_s x_s, N_{s+1}x_{s+1}, \cdots, N_n x_n)^{\eta}$$

where $\eta = \sum \xi_q$, $\xi_q = \lambda^{N_s} \alpha_q$ $(k, s-1)$, q is the element of Q_{s-1} of the form $\{N_{s-1}, N_s, \cdots, N_n\}$, and the sum ranges over N_s, $(N_s = 0, 1, \cdots, \beta_s)$, with q restricted by $N_{s-1}+N_s = \beta_s$. But by the inductive hypothesis, since L'' is now of the form $L(s)$ we have $\eta = 0 \bmod p^{\alpha_0}$. Taking $\lambda = 1, 2, \cdots, \beta_s+1$, we have the set E of equations:

$$E : \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & & 2^{\beta_s} \\ 1 & 3 & 3^2 & & 3^{\beta_s} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & (\beta_s+1) & (\beta_s+1)^2 & \cdots & (\beta_s+1)^{\beta_s} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{\beta_s} \end{pmatrix} = 0 \bmod p^{\alpha_0}$$

where α_i denotes $\alpha_q(k, s-1)$ for $q = \{(\beta_s-i), i, N_{s+1}, \cdots, N_n\} \in Q_{s-1}$. Since the determinant of the matrix is $\prod_{r=1}^{\beta_s} r!$, and $p > c > \beta_s$, each of the $\alpha_i = 0 \bmod p^{\alpha_0}$, and hence, by considering all the elements of Q_s, we have $\alpha_q(k, s-1) = 0 \bmod p^{\alpha_0}$ for all $k > s$ and all $q \in Q_{s-1}$. Thus, $P(1) \equiv 1$, and the law $L(s-1)$ reduces to

$$L : \prod_{q \in Q_{s-1}} (x_s, x_{s-1}, N_{s-1}x_{s-1}, \cdots, N_n x_n)^{\alpha_q(s, s-1)} = 1.$$

By substituting x_{s+1}^{λ} for x_{s+1} throughout L, we have

$$\prod_{q \in Q_{s-1}} (x_s, x_{s-1}, N_{s-1}x_{s-1}, \cdots, N_n x_n)^{\mu_q} = 1$$

with $\mu_q = \lambda^{N_{s+1}} \alpha_q(s, s-1)$. Substitution of x_{s+1} for x_s throughout this last law gives

$$\prod_{q \in Q_{s-1}} (x_{s+1}, x_{s-1}, N_{s-1}x_{s-1}, (N_s+N_{s+1})x_{s+1}, \cdots, N_n x_n)^{\mu_q} = 1,$$

and substitution of x_s for x_{s-1} throughout this then gives

$$\prod_{q \in Q_{s-1}} (x_{s+1}, x_s, N_{s-1}x_s, (N_s+N_{s+1})x_{s+1}, \cdots, N_n x_n)^{\mu_q} = 1.$$

By collecting terms, and rearranging, this last law may be put in a form $L'(s)$:

$$L'(s) : \prod_{q \in Q_s} (x_{s+1}, x_s, N_s x_s, \beta_{s+1}x_{s+1}, \cdots, N_n x_n)^{\delta_r} = 1$$

where $r = \{\beta_s, \beta_{s+1}, \cdots, \beta_n\} \in Q_s$, and r is related to the elements q of Q_{s-1} by $\beta_s = N_{s-1}$, $\beta_{s+1} = N_s + N_{s+1}$, $\beta_{s+2} = N_{s+2}, \cdots, \beta_n = N_n$. The indices δ_r are given by $\delta_r = \sum_{N_{s+1}} \lambda^{N_{s+1}} \alpha_q(s, s-1)$, and by hypothesis each $\delta_r = 0 \bmod p^{\alpha_c}$.

Putting $\lambda = 1, 2, \cdots, \beta_{s+1}+1$, a set of equations of the same form as E is obtained, and the same argument then applies, since $p > c > \beta_{s+1}$, and so each $\alpha_q(s, s-1) = 0 \bmod p^{\alpha_c}$.

Thus, all indices in $L(s-1)$ have been shown to be zero $\bmod p^{\alpha_c}$, and the lemma is proved.

So far, all the lemmas have been concerned with commutators of weight c in a variety $\mathfrak{B}$ of class c. If all the laws of $\mathfrak{B}$ are to be examined, we need a result which will allow some induction along the weight of commutators occurring in a law. The last lemma of this section provides that step:

LEMMA 5. *Let the variety $\mathfrak{B}$ of type $\mathfrak{M}(p, n+2)$ be generated by a reduced free group F. Then if $(x, (n+1)y)^{p^\alpha} = 1$ is a law in $\mathfrak{B}$, and $(x, ny)^{p^r} = 1$ is a law in $F/\gamma_{n+2}(F)$, then $(x, ny)^{p^r} = 1$ is a law in $\mathfrak{B}$.*

(Note: from the proof, the lemma is true for $n = 0$, if we interpret $(x, 0y)$ as x).

PROOF. By Corollary 2.2, we have that the pair of laws

$$(x_1, x_2, \cdots, x_{n+2})^{p^\alpha} = 1 \quad \text{and} \quad (x, (n+1)y)^{p^\alpha} = 1$$

are equivalent in $\mathfrak{B}$, and that the pair

$$(x_1, x_2, \cdots, x_{n+1})^{p^r} = 1 \quad \text{and} \quad (x, ny)^{p^r} = 1$$

are equivalent in $\mathrm{Var}\,(F/\gamma_{n+2}(F))$.

If $(x, ny)^{p^r}$ was not always trivial in F, there would have to be at least two free generators x_i, x_j of F such that $(x_i, nx_j)^{p^r} \neq 1$; but $(x_i, nx_j)^{p^r} \in \gamma_{n+2}(F)$ by hypothesis, so $(x_i, nx_j)^{p^r}$ would be a product of commutators of weight $n+2$ in the generators $x_1, x_2, x_3, \cdots$ of F, and this yields a law in F and hence in $\mathfrak{B}$. By putting $x_k = 1$ whenever k is neither i nor j, we could then deduce a two variable law in x_i and x_j. Reading x for x_i and y for x_j, we have that there would be a law in $\mathfrak{B}$ of the form L:

$$L : (x, ny)^{p^r} = \prod_{k=1}^{n+1} (x, ky, (n+1-k)x)^{\beta_k}.$$

By commuting once with y, we see that $r \geq \alpha$, and hence $(x, ny)^{p^r}$ becomes $(x, ny)^{\lambda^n p^r}$ under the substitution of y^λ for y. By first writing L in the form L':

$$L' : \prod_{k=1}^{n+1} W_k = 1, \quad W_k = (x, ky, (n+1-k)x)^{\beta_k},$$
$$k \neq n, \quad W_n = (x, ny)^{-p^r} (x, ny, x)^{\beta_n}$$

we deduce, using corollary 1.1 with ϕ_λ being substitution of y^λ for y, that $W_k = 1$, and hence by Lemma 3 that $\beta_k = 0 \bmod p^\alpha$, for each $k \neq n$. The only non-trivial part of the law remaining is $W_n = 1$, i.e.

$$(x, \, ny)^{-p^r} \, (x, \, ny, \, x)^{\beta_n} = 1.$$

By using the substitution of x^{-1} for x we deduce that $(x, \, ny)^{p^r} = 1$ is a law in $\mathfrak{B}$, and of course $\beta_n = 0 \bmod p^\alpha$.

4. Theorems on laws

In this section, we consider the laws which a variety $\mathfrak{B}$ of type $\mathfrak{M}(p, \, c)$ may have. It is found that these laws have a basis which may be stated in terms of the maximum exponents of the lower central series of groups in $\mathfrak{B}$.

THEOREM 1. *Let F_n be a reduced free group (of a variety $\mathfrak{B}$ of type $\mathfrak{M}(p, \, c)$) of finite rank n, on the free generators $g_1, g_2, \cdots, g_n$, with ordering $g_1 < g_2 < \cdots < g_n$. Then the basic commutators, of weight c, in these generators, are independent.*

PROOF. The proof depends on the fact that the free generators of F_n are such that any relation amongst them is a law in $F_n(\mathfrak{B})$ ([8], [2]).

Take the forms $A(i, j)$ as defined for the purposes of Lemma 4, with symbols $g_1, g_2, \cdots, g_n$; by the ordering $g_1 < g_2 < \cdots < g_n$, any relation amongst the basic c-weight commutators may be written in the form (R):

$$(R) : T_1 T_2 \cdots T_{n-1} = 1, \quad \text{with} \quad T_s = \prod_{\substack{k \\ n \geq k > s}} A(k, s).$$

This relation gives a law in $F_n(\mathfrak{B})$, and hence in $\mathrm{Var}\,(F_n(\mathfrak{B}))$, which is a variety of the type under consideration. By substituting 1 for the symbols $g_1, g_2, \cdots, g_{n-2}$ in this law, we have $T_{n-1} = 1$ is a law, and so $T_1 T_2 \cdots T_{n-2} = 1$ is a law. Continuing by substituting 1 for $g_1, g_2, \cdots, g_{n-3}$, and so on, we have that each of $T_s = 1$ is a law, for $s = n-1, \, n-2, \cdots 1$. But $T_{n-1} = A(n, \, n-1)$, so by Lemma 3 (using symbols x for g_n and y for g_{n-1}), the law $T_{n-1} = 1$ is a *direct* consequence of the law $(x_1, x_2, \cdots, x_c)^{p^{\alpha_c}} = 1$, in the sense that each exponent of each commutator is 0 mod p^{α_c}. Lemma 4 now provides the inductive step, and hence all the exponents of all the distinct commutators in each factor T_s of the relation (R) are 0 mod p^{α_c}, the left side of R is identically 1, and the theorem is established.

Considering the remarks preceding Lemma 4, we can restate Theorem 1 as a theorem about the laws of a variety $\mathfrak{B}$ of type $\mathfrak{M}(p, \, c)$, since $\mathfrak{B}$ is in fact generated by $F_c(\mathfrak{B})$ ([5], [2]). Thus:

THEOREM 2. *For a variety $\mathfrak{B}$ of type $\mathfrak{M}(p, \, c)$, all laws of the form*

$\prod_i W_i^{\beta_i} = 1$, where W_i are commutators of weight c, in n variables, are consequences of the single law in c variables:

$$(x_1, x_2, \cdots, x_c)^{p^{\alpha_c}} = 1.$$

We can now give the form for a basis of the laws of any given variety of the type $\mathfrak{M}(p, c)$, and name any such variety by giving its relevant exponents. This is done in Theorem 3. The existence of, and generation of, such varieties, is dealt with in § 5.

THEOREM 3. *The laws of a variety $\mathfrak{B}$ of type $\mathfrak{M}(p, c)$ can all be derived from the following basis:*
  (i) *the metabelian law* $((x_1, x_2), (x_3, x_4)) = 1$
  (ii) *the nilpotency law* $(x_1, x_2, \cdots, x_c, x_{c+1}) = 1$
  (iii) *a set of two-variable laws, c in number, of the form*

$$x^{p^{\alpha_1}} = 1, \quad (x, ky)^{p^{\alpha_{k+1}}}, \qquad k = 1, 2, \cdots, c-1$$

in which $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_c \geqq 1$.

PROOF. It is known ([5], [2]) that there must be at least one reduced free group F in $\mathfrak{B}$ such that $\mathfrak{B} = \text{Var}\,(F)$. Now $\mathfrak{B}$ certainly has laws of the form stated in the theorem, as we know by corollary 1.3, and we will take the α_i as minimal. Then $F/\gamma_c(F)$ has the law $(x, (c-2)y)^{p^{\alpha_{c-1}}} = 1$, since $F/\gamma_c(F) \in \mathfrak{B}$; by Lemma 5, α_{c-1} is minimal for $F/\gamma_c(F)$, and by corollary 2.2, this is equivalent to the law $(x_1, x_2, \cdots, x_{c-1})^{p^{\alpha_{c-1}}} = 1$ in Var $(F/\gamma_c(F))$; by Theorem 2, any law of Var $(F/\gamma_c(F))$ of the form $\prod_k W_k^{\beta_k} = 1$, where the W_k are commutators of weight $c-1$, can be derived from it. Lemma 5 now provides the inductive steps necessary to establish:
  "For all $k < c$,

$$(x, (c-k)y)^{p^{\alpha_{c-k+1}}} = 1$$

is a basis law in Var $(F/\gamma_{c-k+2}(F))$ in the sense that all laws of the form $\prod W_i^{\beta_i} = 1$, W_i commutators of weight $c-k+1$, follow from it."
  Consider now F, and $\mathfrak{B}$, and take any n-variable law of $\mathfrak{B}$. Then this law may be written in the form L:

$$L : x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} P_2 P_3 \cdots P_c = 1$$

where each P_i is a product of commutators of weight i. By substituting $x_i = 1$, for all $i \neq k$, we have $r_k = 0 \bmod p^{\alpha_1}$ for each k, and the law remaining is $P_2 P_3 \cdots P_c = 1$. But this law holds also for Var $(F/\gamma_3(F)) \subseteq \mathfrak{B}$, and hence $P_2 = 1$ is a law which follows from $(x, y)^{p^{\alpha_2}} = 1$ and the metabelian law in the strict sense of Theorem 1 and Theorem 2, (i.e. if P_2 is written in the form $\prod_k W_k^{\beta_k}$, all W_k basic commutators of weight 2, then $\beta_k = 0 \bmod p^{\alpha_2}$). Thus $P_2 = 1$ is a law in $\mathfrak{B}$. The obvious induction, using Theorem 2

at each stage, then gives that the whole law L follows from the basis explicitly stated in the theorem, and the theorem follows.

From Theorem 3, we may name any given variety $\mathfrak{V}$ of type $\mathfrak{M}(p, c)$ by the minimal indices in the basis laws, writing $\mathfrak{V} = [\alpha_1, \alpha_2, \cdots, \alpha_c]$. Clearly the varieties of type $\mathfrak{M}(p, c)$ for p, c fixed, form a lattice under the obvious rules. For fixed p, and $k < c$, we may include all varieties of type $\mathfrak{M}(p, k)$ in the lattice; by interpreting the symbols $[\alpha_1, \alpha_2, \cdots, \alpha_k]$ and $[\beta_1, \beta_2, \cdots, \beta_c]$ as equivalent when $\alpha_i = \beta_i$ for $i = 1, 2, \cdots, k$ and $\beta_i = 0$ for $i = k+1, k+2, \cdots, c$, (which is consistent in the light of corollary 1.3), the obvious rules then become

$$[\beta_1, \beta_2, \cdots, \beta_c] \vee [\beta_1', \beta_2', \cdots, \beta_c'] = [\max{(\beta_1, \beta_1')}, \cdots, \max{(\beta_c, \beta_c')}]$$

and

$$[\beta_1, \beta_2, \cdots, \beta_c] \wedge [\beta_1', \beta_2', \cdots, \beta_c'] = [\min{(\beta_1, \beta_1')}, \cdots, \min{(\beta_c, \beta_c')}].$$

5. Theorems on generation and critical groups

Theorem 4 establishes the existence of each variety, and its generation; Theorems 5 and 6 deal with critical groups, and Theorem 7 deals with the generation of each variety by a critical group in it.

THEOREM 4. *Any given variety $[\alpha_1, \alpha_2, \cdots, \alpha_c]$ of type $\mathfrak{M}(p, c)$ can be generated by a two-generator group, and hence by its free group of rank 2.*

PROOF. The following group G, of order $p^{\alpha_1 + 2\alpha_2 + \cdots + \alpha_c}$ may be constructed, as shown in the Appendix:

$$G = Gp\ \{a, b \mid a^{p^{\alpha_1}} = b^{p^{\alpha_2}} = (a, b)^{p^{\alpha_2}} = \cdots = (a, (c-1)b)^{p^{\alpha_c}} = (a, cb) = 1,$$
$$(a, b, a) = 1, \text{ metabelian}\}.$$

A simple calculation shows that $G \in [\alpha_1, \alpha_2, \cdots, \alpha_c]$ and that these indices are minimal for G and hence for Var (G). Since Theorem 3 applies to Var (G), we have Var $(G) = [\alpha_1, \alpha_2, \cdots, \alpha_c]$.

We now examine critical groups, and can immediately state:

THEOREM 5. *When $p > c > 1$, any metabelian critical p-group of class c is a two-generator group.*

PROOF. Let H be such a critical group. Then Theorem 4 applies to Var (H), so that there is a two-generator group G, with Var $(H) = $ Var (G). Then by a theorem of B. H. and H. Neumann (see [7]), H can be generated by n generators, $n \leqq 2$.

We pass immediately to

THEOREM 6. *Let H be a metabelian p-group of class c, $p > c > 1$, with Var $(H) = [\alpha_1, \alpha_2, \cdots, \alpha_c]$. If H is a two-generator group, then*

(i) $\mathrm{Var}\,(S-1)H = [\alpha_1, \beta_2, \cdots, \beta_{c-1}, \alpha_c - 1]$ where $\beta_k = \max\,(\alpha_k - 1, \alpha_{k+1})$.
If H is critical, with monolith N, then

(ii) $\mathrm{Var}\,(H/N) = [\beta_1, \beta_2, \cdots, \beta_{c-1}, \alpha_c - 1]$

(iii) $\mathrm{Var}\,(QS-1)H = \mathrm{Var}\,(S-1)H \supseteq \mathrm{Var}\,(H/N)$.

PROOF. We prove (i) first. Since H has two generators, we can designate any maximal subgroup S of H as $S = Gp\,\{g, \Phi(H)\}$, where $\Phi(H)$ is the Frattini subgroup of H, and g is some element of H. It is clear that if $(x, (i-1)y) \in \gamma_i^p \gamma_{i+1}$ $(i > 1)$ then $(x, iy) \in \gamma_{i+1}^p \gamma_{i+2}$. A simple calculation, using the forms $x = g^s f_1$, $y = g^t f_2$; $f_1, f_2 \in \Phi(H)$, shows that $(x, y) \in \gamma_2^p \gamma_3$, so we have by induction that $(x, iy) \in \gamma_{i+1}^p \gamma_{i+2}$ $(i \geq 1)$. Thus, for all proper subgroups S of H, we have laws giving $\mathrm{Var}\,(S) \subseteq [\alpha_1, \beta_2, \cdots, \beta_{c-1}, \alpha_c - 1]$. That there are proper subgroups of H such that each of these indices is minimal is established as follows:

Consider first the case where $\alpha_i = \alpha_{i+1}$ for a particular $i > 1$. Then in H there must be two elements g_1 and g_2 such that (g_1, ig_2) has order $p^{\alpha_{i+1}}$. But $S_1 = Gp\,\{(g_1, g_2), g_2\} \subset H$, since $(g_1, g_2) \in \Phi(H)$, and H is two-generator. Taking (g_1, g_2) for x, and g_2 for y, we have a contradiction to $(x, (i-1)y)^{p^{\alpha_i - 1}}$ being trivial in S_1. Consider next the case $\alpha_i > \alpha_{i+1}$ for some $i > 2$; there must be elements g_3 and g_4 in H such that $(g_3, (i-1)g_4)$ has order p^{α_i}. Then $S_2 = Gp\,\{g_3^p, g_4\} \subset H$, and using g_3^p for x, g_4 for y, we have $(x, (i-1)y)^{p^{\alpha_i - 2}} = (g_3, (i-1)g_4)^{p^{\alpha_i - 1}}$ *not* trivial in S_2. Finally, there must be an element of order p^{α_1} in H, which by itself generates a proper subgroup of H. Thus the result (i) follows.

We next consider H critical, in which case it is a two-generator group, by Theorem 5, so that (i) still applies to H.

(ii): We note first that for all x and y in H, $(x, (c-1)y)^{p^{\alpha_c - 1}}$ is central, of order p if not trivial; since N is the monolith of H, it follows that N is the corresponding verbal subgroup, and so $(x, (c-1)y)^{p^{\alpha_c - 1}} = 1$ is a law in H/N. Since it is *not* a law in H, there exist elements g_5, g_6 in H such that $(g_5, (c-1)g_6)^{p^{\alpha_c - 2}}$ is of order p^2, hence *not* in N, and so $(x, (c-1)y)^{p^{\alpha_c - 2}}$ is *not* always trivial in H/N.

Consider now the case when $\alpha_{c-1} > \alpha_c$. Then

$$\big((x, (c-2)y)^{p^{(\alpha_{c-1}-1)}}, z\big) = (x, (c-2)y, z)^{p^{(\alpha_{c-1}-1)}} = 1$$

in H. [The equivalence of the two- and three-variable laws is dealt with by corollary 2.2]. Thus, $(x, (c-2)y)^{p^{(\alpha_{c-1}-1)}}$ is always central, and of order p when not trivial, and hence lies in N, so that $(x, (c-2)y)^{p^{(\alpha_{c-1}-1)}} = 1$ is a law in H/N. Since there are elements g_7, g_8 in H such that $(g_7, (c-2)g_8)^{p^{(\alpha_{c-1}-2)}}$ is of order p^2, and hence *outside* N, the index $(\alpha_{c-1} - 1)$ is minimal for $\mathrm{Var}\,(H/N)$ when $\alpha_{c-1} > \alpha_c$.

The case when $\alpha_{c-1} = \alpha_c$ is transparent: there are elements g_9, g_{10}

in H such that $(g_9, (c-1)g_{10})^{p^{\alpha_c-1}} \neq 1$, i.e. $(g_9, (x-2)g_{10})^{p^{\alpha_c-1}}$ is *not* central, and hence lies outside N, so that $(x, (c-2)y)^{p^{(\alpha_{c-1}-1)}}$ is *not* always trivial modulo N.

Thus, β_{c-1} is the minimal index for Var (H/N); the rest of the β_k for $k > 1$ follow by induction. Whenever $\alpha_i = \alpha_{i+1}$, the centrality argument applies as above, but whenever $\alpha_i > \alpha_{i+1}$, then $(x, (i-1)y)^{p^{\alpha_i-1}}$ is central, of order p or less, and so is trivial modulo N. In that case there are elements (as g_7, g_8 above) so that $(x, (i-1)y)^{p^{\alpha_i-2}}$ is *not* a law in Var (H/N).

Finally, if $\alpha_1 > \alpha_2$, then $x^{p^{\alpha_1-1}}$ is always central, of order p if not trivial, and hence lies in N. Thus $x^{p^{\alpha_1-1}}$ is always trivial modulo N. But since there must be an element $x^{p^{\alpha_1-2}}$ of order p^2, hence not in N, we have that $x^{p^{\alpha_1-2}}$ is not always trivial modulo N, and $x^{p^{\alpha_1-1}} = 1$ is appropriate for H/N.

If $\alpha_1 = \alpha_2$, we have that if $x^{p^{\alpha_1-1}}$ is always trivial modulo N, then $x^{p^{\alpha_1-1}}$ is always central, and hence $(x^{p^{\alpha_1-1}}, y) = 1$ is a law of H, and this is equivalent to the law $\prod_{i=0}^{c-2} (x, y, ix)^{\left(p_i^{\alpha_1-1}\right)} = 1$. But (by Corollary 1.1), this would imply $(x, y)^{p^{\alpha_1-1}} = 1$ as a law of H, which is a contradiction. We conclude that $x^{p^{\beta_1}} = 1$ is the appropriate law for H/N, and so (ii) is established.

The result (iii) follows from (i) and (ii) since

$$\text{Var } (QS-1)H = \text{Var } (S-1)H \vee \text{Var } (H/N),$$

H being monolithic, and the theorem is proved.

It is worth noting that if $\alpha_1 = \beta_1$, which occurs if and only if $\alpha_1 = \alpha_2$, we are forced to Var $(QS-1)H = \text{Var } (S-1)H = \text{Var } (H/N)$. Two corollaries also follow from Theorem 6; examples of the properties they deal with are to be found in an earlier paper [3].

COROLLARY 6.1. *If Var (H) is of type $\mathfrak{M}(p, c)$, and H is critical, then at most $c-2$ varieties $\mathfrak{B}_i$ may be found such that*

$$\text{Var } (H) \supset \mathfrak{B}_1 \supset \mathfrak{B}_2 \supset \cdots \supset \mathfrak{B}_k \supset \text{Var } (QS-1)H,$$

the inclusions being proper.

COROLLARY 6.2. *If Var (H) is of type $\mathfrak{M}(p, c)$, then Var $(QS-1)H$ is maximal in Var (H) if and only if it is the unique maximal subvariety of Var (H).*

The final theorem establishes the existence of critical groups.

THEOREM 7. *For any given variety $[\alpha_1, \alpha_2, \cdots, \alpha_c]$ of the type $\mathfrak{M}(p, c)$, there exists at least one critical group H such that Var $(H) = [\alpha_1, \alpha_2, \cdots, \alpha_c]$.*

PROOF. (We note first that if $c = 1$, the cyclic group is critical, and so consider $c > 1$). For the given variety $\mathfrak{B} = [\alpha_1, \alpha_2, \cdots, \alpha_c]$, we determine

those positions where $\alpha_i > \alpha_{i+1}$, and so determine the subscripts
$(s1)$, $(s2)$, $\cdots$ (sn), that

$$\alpha_1 = \alpha_2 = \cdots = \alpha_{(s1)} = e_1$$

$$\alpha_{s1+1} = \alpha_{s1+2} = \cdots = \alpha_{s2} = e_2$$

$$\cdots$$

$$\alpha_{sk} = \alpha_{sk+1} = \cdots = \alpha_{sn} = e_n,$$

with $k = n-1$, $(sn) = c$, $e_1 > e_2 > \cdots > e_n \geqq 1$.

Let G_1 be the group which was constructed for the proof of Theorem 4:
we use the notation $A_1 = a$, $A_i = (a, (i-1)b)$, to write

$$G_1 = Gp \{a, b \mid a^{p^{\alpha_1}} = b^{p^{\alpha_2}} = A_2^{p^{\alpha_3}} = \cdots = A_c^{p^{\alpha_c}} = 1, (A_i, A_j) = 1$$
$$\text{for all } i, j, A_{c+1} = 1\}.$$

Then $\text{Var } (G_1) = [\alpha_1, \alpha_2, \cdots, \alpha_c]$. By a simple calculation it is immediate
that $A_{(s1)}^{p^{(e_1-1)}}$, $A_{(s2)}^{p^{(e_2-1)}}$, $\cdots$, $A_{(sn)}^{p^{(e_n-1)}}$ are all central. Hence, by factoring out
the product of the normal subgroups N_i, $N_i = Gp \{A_{(si)}^{p^{(e_1-1)}} A_{(si)}^{-p^{(e_i-1)}}\}$, we have
a group $G_2 \simeq Gp \{a, b \mid R\}$ where R includes all the relations in G_1, and
also the relations $A_{(s1)}^{p^{(e_1-1)}} = \cdots = A_{(sn)}^{p^{(e_n-1)}}$. It is easily verified that $\text{Var } (G_2)$
is still $\mathfrak{B}$. (In the extreme case that $\alpha_1 = \alpha_2 = \cdots = \alpha_c$, $G_2 = G_1$). Consider
now the set of quotient groups G_2/K of G_2, where K is normal in G_2. Since
this set includes both G_2 and $\{1\}$, there must be at least one group H in the
set which has the property $\text{Var } (H) = \mathfrak{B}$, and is minimal with this property.
(Thus if N is any proper normal subgroup of H, $\text{Var } (H/N)$ is a proper
subvariety of $\mathfrak{B}$.)

We now assert that H is critical.

Clearly H is a two-generator group, inheriting all the relations of G_2,
and we adopt the same notation a, b, A_i for it. But part (i) of Theorem 6
was proved using only that H was a two-generator group, hence

$$\text{Var } (S-1)H = [e_1, e_1, \cdots, e_1-1, e_2, \cdots, e_2-1, \cdots e_n, e_n-1]$$

and so $H \notin \text{Var } (S-1)H$. Suppose H is *not* critical; then by a theorem of
Weichsel [10] and Kovács and Newman [7], H belongs to the variety gener-
ated by its proper quotient groups, and from this we derive a contradiction.

Since $N_1 = Gp \{A_{(s1)}^{p^{(e_1-1)}}\}$ is still non-trivial and normal in H, we form
H/N_1 and hence

$$\text{Var } (H/N_1) = [\xi_1, \xi_2, \cdots, \xi_{(s1-1)}, \xi_{(s1)}, \cdots, \xi_{(s2)}, \cdots, \xi_{(sn)}]$$

where $\xi_{(s1)} \leqq e_1-1$, $\xi_{(s2)} \leqq e_2-1$, $\cdots$, $\xi_{(sn)} \leqq e_n-1$. Now since H belongs
to the variety formed by its proper quotient groups, and $\text{Var } (H) = \mathfrak{B}$,
there must then be *at least one* other proper normal subgroup N_2 of H such

that $\operatorname{Var}(H/N_2) = [\eta_1, \eta_2, \cdots, \eta_c]$ where $\eta_{(s1)} = e_1$. But since $\eta_1 \geq \eta_2 \geq \cdots \eta_c$, this forces $\eta_1 = \eta_2 = \cdots = \eta_{(s1)} = e_1$. But now if any $\eta_{(si)} < e_i$; then $A_{(si)}^{p^{\eta_{(si)}}} \in N_2$ and hence $A_{(s1)}^{p^{(e_1-1)}} \in N_2$, and so $\eta_{(s1)} < e_1$, a contradiction to $\eta_{(s1)} = e_1$. Thus $\eta_{(si)} = e_i$ for each i, and this forces $\operatorname{Var}(H/N_2) = \mathfrak{B}$, a contradiction to the choice of H. Thus H is critical, and the theorem is proved.

6. Appendix

For the purpose of proving Theorem 4, we needed the particular group G, the recipe for which is given herewith.

Take the c cyclic groups $C_i = Gp \{A_i \mid A_i^{p^{\alpha_i}} = 1\}$ $i = 1, 2, \cdots, c$, and form their direct product P. We now split-extend this direct product by the cyclic group $B = Gp \{b \mid b^{p^{\alpha_1}} = 1\}$ by using the mapping $A_1^b = A_1 A_2, \cdots, A_{c-1}^b = A_{c-1} A_c, A_c^b = A_c$ which defines an automorphism of the direct product. That this has the order p^{α_1} follows from $A_i^{b^k} = A_i A_{i+1}^k A_{i+2}^{\binom{k}{2}} \cdots A_c^{\binom{k}{c-i}}$ (which follows from the commuting of the A_i in P) and the facts $p > c$ and $\alpha_2 \geq \alpha_3 \geq \cdots \geq \alpha_c$. The mapping itself gives immediately $(A_i, b) = A_{i+1}$ for $i < c$, and the structure of P gives $(A_i, A_j) = 1$ for all (i, j). The resulting group is then isomorphic to G by identifying A_1 with a. By adopting the obvious normal form $b^t \prod_{i=1}^c A_i^{r_i}$ for an element of G, from which any element of $\gamma_k(G)$ is of the form $\prod_{i=k}^c A_i^{s_i}$, $k > 1$, it is clear that G is metabelian, of class c, satisfies the basis laws of $[\alpha_1, \alpha_2, \cdots, \alpha_c]$, and that these indices are minimal for the laws of G.

Acknowledgements

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