NON-RADIAL OSCILLATIONS IN RED GIANTS

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1. INTRODUCTION

The structure of red giant stars allows non-radial oscillation modes which propagate as p-modes near the surface, to propagate below the convection zone as g-modes with very high radial wave number [Dziembowski (1971, 1977), Shibahashi and Osaki (1976)]. Under some conditions the oscillations in these two propagation regions can be treated as virtually independent normal modes [Shibahashi and Osaki (1976)]. This paper examines the situation in which this approximation is not good, and discusses possible observational consequences of the interaction of the two propagation regions.

2. BASIC EQUATIONS

The linearized differential equations describing non-radial adiabatic oscillations in stars can be written in the form,

$$\frac{du}{dr} + \frac{gu}{c^2} = \frac{Yr^2}{c^2}\left(\frac{{\omega_L}^2}{\omega^2} - 1\right) \tag{1a}$$

$$\frac{dY}{dr} + \frac{{\omega_{BV}}^2 Y}{g} = \frac{u}{r^2}\left(\omega^2 - {\omega_{BV}}^2\right) \tag{1b}$$

if the Eulerian perturbation of the gravitational potential is neglected [Ando and Osaki (1975)]. The dependent variables u and Y are related to the radial displacement and Eulerian pressure perturbation by $u \equiv r^2\delta r$ and $Y \equiv \delta^E P/\rho$. The acoustic frequency is $\omega_L = c[\ell(\ell+1)]^{\frac{1}{2}}/r$, and the Brunt-Vaisala frequency is given by ${\omega_{BV}}^2 = g^2\Gamma_1 Q(\nabla_A - \nabla)/c^2$. In these expresssions, ℓ is the spherical harmonic index, $Q = -(\partial \ln \rho/\partial \ln T)p$, and other symbols have their usual meaning. The form of the solution of these equations in the WKB approximation (see for example Osaki 1977 for a similar expression) is

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$$\delta r = \frac{1}{r}\left|\frac{\omega^2 - \omega_{BV}{}^2}{\omega^2 - \omega_L{}^2}\right|^{\frac{1}{4}} \exp\left[\frac{1}{2}\int\left(\frac{g}{c^2} + \frac{\omega_{BV}{}^2}{g}\right)dr\right] \exp\left(\pm\, i\int k dr\right) \tag{2}$$

where $k^2 \equiv (\omega^2-\omega_L{}^2)(\omega^2-\omega_{BV}{}^2)/(\omega^2c^2)$. For a given mode frequency ω, the interior propagation region is characterized by $\omega_{BV},\omega_L > \omega$, and both boundaries of the region occur at radii for which $\omega_{BV}(r) = \omega$. The p-mode region has $\omega_{BV},\omega_L < \omega$. Its inner and outer boundaries are determined by $\omega_L(r) = \omega$ and $\omega_{BV}(r) = \omega$ respectively.

2.1 The General Problem

The stellar situation is quite analogous to the quantum-mechanical problem of finding the eigenfunctions for a potential consisting of two wells separated by a barrier, with the two propagation regions corresponding to the wells. Examination of the derivation of Equation (2) reveals that the main difference arises in the behavior at the turning point $\omega_L(r) = \omega$. Two non-standard WKB connection formulae more suited to that behavior have been studied, but they make no essential changes in the nature of the results discussed below. Therefore the standard connection relations have been used throughout the following discussion.

The eigenfunctions in wells (1) and (2) have the form

$$\Psi_1 = A_1 f(r) \cos\left(\int_{r_1}^{r} k dr - \tfrac{1}{4}\pi\right), \Psi_2 = A_2 f(r) \cos\left(\int_{r}^{r_4} k dr - \tfrac{1}{4}\pi\right) \tag{3}$$

in which A_1 and A_2 are constants, and r_1 and r_4 are the innermost and outermost turning points, respectively.

The equation for the eigenfrequencies ω is

$$\cot(\phi_1) \cot(\phi_2) = \tfrac{1}{4}\exp(-2\chi), \tag{4}$$

in which $$\phi_1 \equiv \int_{r_1}^{r_2} k dr, \quad \chi \equiv \int_{r_2}^{r_3} |k| dr, \quad \phi_2 \equiv \int_{r_3}^{r_4} k dr, \tag{5}$$

are all functions of ω. The relation between the constants can be written in the form

$$A_1{}^2 \sin(2\phi_1) = A_2{}^2 \sin(2\phi_2) \tag{6}$$

In the limit of an impenetrable barrier ($\chi \to \infty$), the eigenfrequencies are the resonant frequencies of the wells defined by $\cos\phi_1 = 0$ or $\cos\phi_2 = 0$. At these frequencies Equation (6) shows that the corresponding eigenfunction is entirely confined to the well for which the mode frequency is resonant. For χ large enough but finite, at least one of $\cos\phi_1$, $\cos\phi_2$ must be very close to zero if Equation (4) is satisfied. For red giant structures, ϕ_1 (hereinafter associated with the inner well) is $\gg \phi_2$, and $d\phi_1/d\omega \gg d\phi_2/d\omega$. If $d\phi_2/d\omega \simeq \eta\, d\phi_1/d\omega$ and $\eta \ll 1$, then there are $\sim\eta^{-1}$ solutions of Equation (4) associated with the inner well ($\cos\phi_1 \simeq 0$) for every solution associated with the outer well. The amplitude ratio A_2/A_1 corresponding to an eigenfrequency can be obtained by rewriting Equation (4) in the equivalent form

$$\frac{\sin 2\phi_1}{\sin 2\phi_2} = \frac{\frac{1}{4}\exp(-2\chi)}{\cos^2\phi_2 + (1/16)\sin^2\phi_2 \exp(-4\chi)} \tag{7}$$

For any solution of Equation (4) [or Equation (7)] for which ϕ_2 is within $\epsilon = \frac{1}{2}\exp(-\chi)$ of a resonant value, $A_2{}^2 > A_1{}^2$. Equivalently, if there is a solution of Equation (4) within a bandwidth $\Delta\omega/\omega = \exp(-\chi)/(d\phi_2/d\ln\omega)$ centered on the resonant frequency ($\cos\phi_2 = 0$), the corresponding eigenfunction will have a large relative amplitude in the exterior propagation region. There is always one such solution which becomes the exterior well resonant solution in the limit $\chi \to \infty$. However, if $\eta < (2\pi)^{-1}\exp(-\chi)$ there will be more than one solution of Equation (4) within the critical bandwidth; there will be many if the inequality is very strongly satisfied. All such eigenfunctions will have significant amplitude in the exterior well, and will have very similar forms in that region.

3. IMPLICATIONS FOR STARS

There are several reasons for suspecting that oscillation modes with $A_2 \gg A_1$ are more likely to be observed. In red giants the primary driving region is in the outer part of the star. Eigenfunctions belonging to a group near an exterior well resonant frequency are similar in form and will be subject to the same energy input per period, per unit (surface amplitude)^2 in the linear regime. The linear dissipation is likely to be higher for modes with large interior amplitude. Thus unstable modes near the exterior well resonance will have the largest growth rates because they have both the largest net driving and the smallest amount of kinetic energy per unit (surface amplitude)^2. This doesn't necessarily guarantee that such eigenfunctions will dominate in the non-linear regime. However, a mode which grows rapidly to high amplitude may suppress other modes. In the non-linear calculations by Christy (1966), such suppression is implied by the fact that some models could pulsate in either of two modes depending on the initial conditions. The question here is whether a whole group of near-resonant modes could survive to high amplitude and produce significant interference phenomena. In principle, non-linear effects could pull their frequencies together, or suppress all but one member of a group.

3.1 Typical Numerical Parameters

Values of the parameters η and χ have been estimated for a set of models with $M \sim M_\odot$, $L \sim 500 - 5000\ L_\odot$, and $T_e \sim 2200 - 3500$°K. On the helium shell asymptotic giant branch, η is $\sim 10^5 - 10^6$. For a low p-mode frequency $\omega \sim 10^{-7}$ and $\ell = 2$, $\exp(-\chi) \simeq 10^{-6}$. Thus the critical

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bandwidth is very narrow, and a detailed calculation is required to determine if more than one mode would fall within it. In any case, because the bandwidth is so small, the timescale for any possible beat phenomena is too long for observational detection.

For stars with $T_e \sim 3500°$ the situation is potentially more interesting. The low ℓ p-modes can propagate through much of the convection zone, and so the value of χ can be very much smaller. Critical bandwidths $\Delta\omega/\omega \sim 10^{-1}$ or 10^{-2} are possible. The higher mode frequencies as well as the higher beat frequencies are much more favorable for observation. If χ is too small, however, the interior amplitude may be significant for all the modes and they may be stabilized in the linear regime by the damping in the interior. It is likely that there is an intermediate situation in which the modes are more likely to be unstable, but for which the critical bandwidth is not too small to preclude observation of interference effects.

3.2 Summary

The conditions required for a clean separation of interior g-modes from surface p-modes may be well satisfied for extreme giant branch models. However, for somewhat hotter stars it is apparent that the interaction of the g-mode and p-mode regions may have interesting observational consequences.

REFERENCES

Ando, Hl, and Osaki, Y. 1975. Publ.Astron.Soc.Japan,27, 581
Christy, R.F., 1966. Ap.J.,144, 108.
Dziembowski, W., 1971. ActaAstron.,21, 289.
Dziembowski, W., 1977. ActaAstron.,27, 95.
Osaki, Y., 1977. Publ.Astron.Soc.Japan,29, 235.
Shibahashi, H., and Osaki, Y., 1976. Publ.Astron.Soc.Japan,28, 199.

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