Can. J. Math.Vol. **44** (5), 1992 pp. 1003–1013

A TOPOLOGICAL TRANSVERSALITY THEOREM FOR MULTI-VALUED MAPS IN LOCALLY CONVEX SPACES WITH APPLICATIONS TO NEUTRAL EQUATIONS

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ABSTRACT. The concept of essential map and topological transversality due to A. Granas is extended to multi-valued maps in locally convex spaces and it is next applied to prove the solvability of boundary value problems for certain neutral functional differential equations. In order to achieve a required compactness property, the weak topology in a Sobolev space is considered. The topological tool established in the first part of the paper allows to avoid some obstacles which are encountered when trying to use standard degree-theoretical arguments.

1. **Introduction.** The motivation of this paper is to develop a topological tool to investigate the following two-point boundary value problem of neutral functional differential equations

$$\begin{cases} \ddot{x}(t) \in f(t, x_t, \dot{x}_t, \ddot{x}_t), & \text{a.e. } t \in [0, T], \\ x_0 = \varphi \\ x(T) = b \end{cases} \tag{1.1}$$

where $r > 0$, $T > 0$, $b \in \mathbb{R}^n$, $\varphi \in W^{2,2}([-r,0];\mathbb{R}^n)$, $f\colon [0,T] \times L^\infty([-r,0];\mathbb{R}^n) \times L^\infty([-r,0];\mathbb{R}^n) \times L^2([-r,0];\mathbb{R}^n) \longrightarrow \mathbb{R}^n$ is a multi-valued map with nonempty closed convex values, and for any $x\colon [-r,T] \longrightarrow \mathbb{R}^n$, x_t, $0 \le t \le T$, denotes the map from $[-r,0]$ to $\mathbb{R}^n$ defined by $x_t(s) = x(t+s)$ for $s \in [-r,0]$. As will be shown, the problem (1.1) can be formulated as a fixed point problem for a certain *u. s. c.* map with nonempty compact convex values $F\colon \bar{U} \longrightarrow C$, where C is a closed convex bounded subset of $\big(W^{2,2}([0,T];\mathbb{R}^n), \omega\big)$ (the Sobolev space with the weak topology) and U is an open subset of C. The choice of the Sobolev space and its weak topology, motivated by the study in [1] and [7] for Cauchy initial value problems of neutral equations, enables us to consider boundary value problems of neutral equations whose right hand side functionals may not be continuous. The fact that the space $\big(W^{2,2}([0,T];\mathbb{R}^n), \omega\big)$ is non-metrizable and the bounded set U is not weakly open in the entire space makes it difficult to apply the topological degree theory in [6], [8] and [11]. This inspires us to extend the topological transversality theorem due to A. Granas to the case of convex-valued maps in a locally convex space.

The first author was supported by NSERC of Canada Grant No. 0GPIN016.

The second author was partially supported by Gordin Kaplan Postdoctoral Fellowship at the University of Alberta. This research was also partially supported by NSERC.

Received by the editors November 26, 1990; revised August 15, 1991.

AMS subject classification: 34H10, 34K10.

Key words and phrases: Topological transversality in locally convex space, multi-valued maps, neutral equations, boundary value problems.

The rest of this paper is organized as follows. In Section 2, we introduce the concept of essential maps and prove a topological transversality theorem for compact $u.s.c.$ multi-valued maps with nonempty compact convex values in a Hausdorff locally convex space. We then, in Section 3, show that the problem (1.1) can be reformulated as a fixed point problem for a multi-valued map satisfying certain monotonicity properties. The established results are then applied in Section 5 to obtain an existence result for the problem under an "*a priori* bound" condition. Finally, we present a simple example to illustrate the main result.

2. **Topological transversality.** In this section, we extend concepts of essential maps and topological transversality, due to A. Granas, to the case of convex-valued maps in locally convex spaces. For elementary properties of these maps, we refer to [6].

In what follows, E is a Hausdorff locally convex space and C is a closed convex subset of E. Given a pair of closed subsets $A \subseteq X$ of C, we dentoe by $\mathcal{K}_A(X, C)$ the class of all compact $u.s.c.$ maps $F: X \longrightarrow C$, with nonempty compact convex values, which are fixed point free on A, *i.e.*, $x \notin F(x)$ for all $x \in A$.

A map $F \in \mathcal{K}_A(X, C)$ is called *essential* if every $G \in \mathcal{K}_A(X, C)$ such that $G|_A \equiv F|_A$ has a fixed point. Two maps $F, G \in \mathcal{K}_A(X, C)$ are called *homotopic* if there exists an $u.s.c.$ compact map $H: X \times [0, 1] \longrightarrow C$, with non-empty compact convex values, such that $H_t := H(\cdot, t) \in \mathcal{K}_A(X, C)$ for all $t \in [0, 1]$, $H_0 = F$ and $H_1 = G$. We call such H a *homotopy from F to G*. Evidently, the relation "F is homotopic to G" is an equivalence relation.

LEMMA 2.1. *Let $F \in \mathcal{K}_A(X, C)$. The following statements are equivalent:*

(i) F is inessential (i.e., F is not essential);

(ii) F is homotopic to a fixed point free G in $\mathcal{K}_A(X, C)$;

(iii) F is homotopic to a fixed point free G^* in $\mathcal{K}_A(X, C)$ by a homotopy keeping $F|_A$ pointwise fixed.

PROOF. (i) $\Rightarrow$ (ii). Let $G \in \mathcal{K}_A(X, C)$ be a fixed point free map with $F|_A \equiv G|_A$. It is easily verified that $H(x, t) = (1 - t)F(x) + tG(x)$ is the required homotopy from F to G.

(ii) $\Rightarrow$ (iii). Let H be a homotopy from a fixed point free map $G = H_0$ to $F = H_1$. Let $\tilde{B} = \{(x, t) \in X \times [0, 1]; x \in H(x, t)\}$ and let $B = \{x \in X; x \in H(x, t) \text{ for some } t \in [0, 1]\}$. From the compactness of H it follows that $\tilde{B}$ is compact, therefore B is compact as the projection of $\tilde{B}$ onto the X coordinate. We may assume that $B \neq \emptyset$, otherwise F is fixed point free and we are done. Clearly, $A \cap B = \emptyset$. Since any Hausdorff locally convex space is a $T_{3\frac{1}{2}}$ space, there exists a continuous function $u: X \longrightarrow [0, 1]$ with $u(A) = 1$, $u(B) = 0$. We define $G^*(x) = H\big(x, u(x)\big)$ and $H^*(x, t) = H\big(x, (1-t) + tu(x)\big)$. It can be easily verified that G^* is fixed point free and H^* is a homotopy from F to G^* with $H^*(x, t) = F(x)$ for all $x \in A$ and all $t \in [0, 1]$.

The statement (iii) $\Rightarrow$ (i) is obvious.

As an immediate consequence we get the following

COROLLARY 2.1. *Let F, $G \in \mathcal{K}_A(X, C)$ be homotopic maps. Then F is essential if and only if G is essential.*

THEOREM 2.1. *Let U be an open subset of C, $x_0 \in U$, and let $\partial U = \partial_C U$ be the boundary of U in C. Then the constant map $\bar{U} \longrightarrow \{x_0\}$ is essential in $\mathcal{K}_{\partial U}(U, C)$.*

PROOF. We want to show that if $F \in \mathcal{K}_{\partial U}(\bar{U}, C)$ and $F|_{\partial U} = \{x_0\}$, then F has a fixed point in U. We define the extension of F to $\tilde{F}: C \longrightarrow C$ by putting $\tilde{F}(x) = F(x)$ if $x \in \bar{U}$ and $\tilde{F}(x) = \{x_0\}$ if $x \in C \setminus \bar{U}$. Then $\tilde{F}$ is $u.s.c.$ and by the Ky-Fan fixed point theorem there exists $x \in C$ such that $x \in \tilde{F}(x)$. Since no x in $C \setminus \overline{U}$ is fixed, x must be a fixed point of F.

THEOREM 2.2. *Let U be an open subset of C with $x_0 \in U$, and let $F: \bar{U} \longrightarrow C$ be an u.s.c. compact map, with nonempty compact convex values, such that $x \neq \lambda F(x) + (1-\lambda)x_0$ for all $x \in \partial_C U$ and all $0 < \lambda < 1$. Then F has a fixed point in $\bar{U}$.*

PROOF. We may assume that $F|_{\partial U}$ is fixed point free, otherwise, we are done. By the hypothesis, $H(x, t) = tF(x) + (1-t)x_0$ is a homotopy from the constant map $\{x_0\}$, which is essential by Theorem 2.1, to the map F in $\mathcal{K}_{\partial U}(U, C)$. Therefore F is essential by Corollary 2.1 and, consequently, has a fixed point.

3. **Application to neutral equations: technical lemmas.** In what follows, $r > 0$, $T > 0$, and $\varphi \in W^{2,2}([-r, 0]; \mathbb{R}^n)$ are given. We introduce the following spaces:

$$\begin{aligned} E_1 &= C([0, T]; \mathbb{R}^n) \times C([0, T]; \mathbb{R}^n) \times L^2([0, T]; \mathbb{R}^n), \\ E_2 &= L^\infty([-r, T]; \mathbb{R}^n) \times L^\infty([-r, T]; \mathbb{R}^n) \times L^2([-r, T]; \mathbb{R}^n), \\ E_3 &= L^\infty([-r, 0]; \mathbb{R}^n) \times L^\infty([-r, 0]; \mathbb{R}^n) \times L^2([-r, 0]; \mathbb{R}^n). \end{aligned}$$

We then define the extension operator $\psi: E_1 \longrightarrow E_2$ by

$$\psi(u, v, w)(t) = (\psi_1 u, \psi_2 v, \psi_3 w)(t) = \begin{cases} (\varphi, \dot{\varphi}, \ddot{\varphi})(t) & \text{if } -r \le t \le 0, \\ (u, v, w)(t) & \text{if } 0 \le t \le T \end{cases}$$

For any map $z: [-r, T] \longrightarrow \mathbb{R}^n$ and $t \in [0, T]$, z_t denotes a map from $[-r, 0]$ to $\mathbb{R}^n$ defined by $z_t(\theta) = z(t+\theta)$ for $-r \le \theta \le 0$. It is easy to see that $\psi: E_1 \longrightarrow E_2$ is continuous, that for any $t \in [0, T]$, the map from E_1 to E_3 defined by $(u, v, w)(t) = \big((\psi_1 u)_t, (\psi_2 v)_t, (\psi_3 w)_t\big)$ is continuous, and that the first two coordinates $(\psi_1 u)_t$, $(\psi_2 v)_t$ are piecewise continuous functions.

We consider the following boundary value problem

$$\text{(3.1)} \qquad \begin{cases} \ddot{x}(t) \in f(t, x_t, \dot{x}_t, \ddot{x}_t), \text{ a.e. } t \in [0, T], \\ x_0 = \varphi, x(T) = b, \end{cases}$$

where $b \in \mathbb{R}^n, f: [0, T] \times E_3 \longrightarrow \mathbb{R}^n$ is a multifunction. Define $g: [0, T] \times E_1 \longrightarrow \mathbb{R}^n$ by

$$g(t, u, v, w) = f\big(t, (\psi_1 u)_t, (\psi_2 v)_t, (\psi_3 w)_t\big).$$

Then we can rewrite (3.1) as

$$(3.2) \qquad \begin{cases} \ddot{x}(t) \in g(t, x, \dot{x}, \ddot{x}) \text{ a.e. } t \in [0, T], \\ x(0) = \varphi(0), x(T) = b. \end{cases}$$

We assume that the problem (3.2) can be imbedded in the following family of problems

$$(3.3) \qquad \begin{cases} \ddot{x}(t) \in h(\lambda, t, x, \dot{x}, \ddot{x}), \ \text{a.e } t \in [0, T], \\ x(0) = \varphi(0), x(T) = b, \end{cases}$$

where $\lambda \in [0, 1]$, $h\colon [0, 1] \times [0, T] \times E_1 \longrightarrow \mathbb{R}^n$ is a multifunction, with nonempty closed convex values, satisfying the following conditions:

(H1) $h(1, t, u, v, w) = g(t, u, v, w)$ for $(t, u, v, w) \in [0, T] \times E_1$;

(H2) $h(\cdot, t, \cdot)\colon [0, 1] \times E_1 \longrightarrow \mathbb{R}^n$ is $u.s.c.$ for a.e. $t \in [0, T]$;

(H3) $h(\lambda, \cdot, u, v, w)\colon [0, T] \longrightarrow R^n$ has measurable single-valued selections for any $(\lambda, u, v, w) \in [0, 1] \times E_1$;

(H4) for any bounded $B \subseteq E_1$ there exists $\alpha_B \in L^2\big([0, T]; [0, \infty)\big)$ such that $|h(\lambda, t, u, v, w)| \leq \alpha_B(t)$ for a.e. $t \in [0, T]$ and all $(\lambda, u, v, w) \in [0, 1] \times B$.

We define the multi-valued map $G\colon [0, 1] \times E_1 \longrightarrow L^2([0, T]; \mathbb{R}^n)$ by

$$(3.4) \qquad G(\lambda, u, v, w) = \{x \in L^2([0, T]; \mathbb{R}^n); x(t) \in h(\lambda, t, u, v, w) \text{ for a.e. } t \in [0, T]\}.$$

The above conditions on h imply that G is well defined with nonempty convex values and it sends bounded sets to bounded sets. Moreover, employing a similar argument to that in [10], we obtain the following

LEMMA 3.1. *The graph of G is closed in the product of norm topology on the domain and the weak topology on the codomain.*

PROOF. Suppose that $(\lambda_n, u_n, v_n, w_n) \longrightarrow (\lambda, u, v, w)$ in norm, $x_n \rightharpoonup x$ (weakly) and $x_n \in G(\lambda_n, u_n, v_n, w_n)$. By the Mazur theorem, for any $N = 1, 2, \ldots$,

$$x \in C_N := \overline{co\{x_{N+1}, x_{N+2}, \ldots\}}$$

and consequently, there exists a sequence $\{z_1^N, z_2^N, \ldots\} \subseteq C_N$ strongly convergent to x. Since the L^2 convergence on a bounded interval implies the pointwise almost everywhere convergence of a subsequence, we may assume without loss of generality that $z_k^N(t) \longrightarrow x(t)$ as $k \longrightarrow \infty$ for a.e. $t \in [0, T]$ and all N. Let $A \subseteq [0, T]$ be the set of those t for which the above sequence converges and for which $h(\cdot, t, \cdot)$ is $u.s.c.$ Then the Lebesgue measure of A is T. By the definition of an $u.s.c.$ map, for any $t \in A$, $\varepsilon > 0$, there exists N such that

$$x_k(t) \in D_\varepsilon := h(\lambda, t, u, v, w) + B_\varepsilon$$

for all $k > N$, where B_ε is the closed ε-ball about the origin. Since D_ε is closed and convex, $\{z_k^N\} \subseteq D_\varepsilon$ and so $x(t) \in D_\varepsilon$. This holds for all $\varepsilon > 0$ and a.e. $t \in [0, T]$, hence $x \in G(\lambda, u, v, w)$ and the conclusion follows.

In order to obtain a further closedness property of the graph of G, we consider the space $L^2 := L^2([0,T];\mathbb{R}^2)$ with the standard integral inner product, and pose the following additional condition on h:

(H5) There exists a continuous linear automorphism S of L^2 and a constant $\sigma \neq 0$ such that the map $\overline{G}(\lambda,u,v,w) = Sw + \sigma G(\lambda,u,v,w)$ is monotone with respect to $w \in L^2$ for all $(\lambda,u,v) \in [0,1] \times C([0,T];R^n) \times C([0,T];R^n)$.

Here and in what follows, a multi-valued map $F\colon H \longrightarrow H$, where $\big(H,(\cdot,\cdot)\big)$ is a Hilbert space, is called *monotone* if $(x-\bar{x}, y-\bar{y}) \geq 0$ for all $x,\bar{x} \in H$, $y \in F(x)$ and $\bar{y} \in F(\bar{x})$.

Let us note that (H5) is satisfied, for example, if either $\pm G$ is monotone in w or G is Lipschitzian in w with a Lipschitzian constant K independent of λ, u and v. Indeed, in the second case, we may take $\bar{G}(\lambda,u,v,w) = w - \frac{1}{K}G(\lambda,u,v,w)$.

LEMMA 3.2. *Let G be defined in (3.4) and satisfy (H5). Then the graph of G is closed in the following topology on* $[0,1] \times E_1$ *and L^2: norm topology on* $[0,1]$ *and the first two L^∞ components of E_1, weak topology on the last L^2 component of E_1 and on the codomain L^2.*

PROOF. We first assume that G itself is monotone. Our argument will be similar to that in [5]. Let $x_k \in G(\lambda_k,u_k,v_k,w_k)$ for $k = 1,2,\ldots,(\lambda_k,u_k,v_k) \longrightarrow (\lambda,u,v)$ in norm, $w_k \rightharpoonup w, x_k \rightharpoonup x$ weakly as $k \longrightarrow \infty$. We want to show that $x \in G(\lambda,u,v,w)$. Suppose the contrary. Since $G(\lambda,u,v,w)$ is convex and closed, the Hahn-Banach separation theorem implies the existence of $y \in L^2$ and a real α such that

$$(y,x) < \alpha < (y,z) \text{ for } z \in G(\lambda,u,v,w).$$

Let $y_m = w - t_m y$, where $t_m > 0$, $t_m \longrightarrow 0$ as $m \longrightarrow \infty$, and choose $z_k^m \in G(\lambda_k,u_k,v_k,y_m)$ for any k and m. Since any bounded set in L^2 is weakly relatively compact, there are subsequences $z_{k_p}^m \rightharpoonup z^m \in L^2$ as $p \longrightarrow \infty$. Also, by passing to a subsequence, we may assume that $z^m \rightharpoonup z \in L^2$ as $m \longrightarrow \infty$. Since $(\lambda_k,u_k,v_k,y_m) \longrightarrow (\lambda,u,v,y_m)$ in norm and $y_m \longrightarrow w$, it follows from Lemma 3.1 that $z^m \in G(\lambda,u,v,y_m)$ and $z \in G(\lambda,u,v,w)$. By the monotonicity assumption, $(x_k - z_k^m,\ w_k - w + t_m y) \geq 0$ for all k and m. Since $w_{k_p} \rightharpoonup w$ and $z_{z_p}^m \rightharpoonup z^m$ as $p \longrightarrow \infty$, we get

$$0 \leq \varliminf_{p\to\infty} \left(x_{k_p} - z_{k_p}^m, w_{k_p} - w + t_m y\right) = \varliminf_{p\to\infty} \left(x_{x_p} - z^m, t_m y\right) \text{ for all } m.$$

Next, $t_m > 0$ implies

$$0 \leq \varliminf_{p\to\infty} \left(x_{k_p} - z^m, y\right) \text{ for all } m.$$

By passing to the limit as $z^m \rightharpoonup z$ and as $x_{k_p} \rightharpoonup x$, we get

$$0 \leq \varliminf_{p\to\infty} \varliminf_{m\to\infty} \left(x_{k_p} - z^m, y\right) = \varliminf_{p\to\infty} \left(x_{k_p} - z, y\right) = (x-z,y).$$

This implies $(x,y) \geq (z,y)$ which contradicts the choice of y and α.

Let now $\overline{G}$ be monotone. The equation defining $\overline{G}$ can be rewritten as $G(\lambda,u,v,w) = \frac{1}{\sigma}\overline{G}(\lambda,u,v,w) - \frac{1}{\sigma}Sw$. Since any continuous linear operator is weakly continuous, the graph of G is closed in any one of the considered topologies if and only if the graph of $\overline{G}$ has that property, and hence the conclusion follows from the first part of the proof.

Now, let $\varepsilon > 0$ be a given sufficiently small constant. We define $\tilde{G}\colon [0,1] \times W^{2,2} \longrightarrow L^2 \times \mathbb{R}^n \times \mathbb{R}^n$ by the formula

$$\tilde{G}(\lambda, u) = \{G(\lambda, u, \dot{u}, \ddot{u}) - \varepsilon u\} \times \{\varphi(0)\} \times \{b\},$$

where $W^{2,2} := W^{2,2}([0,T]; \mathbb{R}^n)$.

LEMMA 3.3. *Under the assumptions (H1 – H5), the restriction of $\tilde{G}$ to any bounded subset of $[0,1] \times W^{2,2}$ is u. s. c. and compact in the weak topology on the domain and codomain spaces.*

PROOF. $\tilde{G}$ maps any bounded subset of $[0,1] \times W^{2,2}$ to a bounded, therefore weakly relatively compact, set in $L^2 \times \mathbb{R}^n \times \mathbb{R}^n$. The conclusion will follow if we show that the graph of $\tilde{G}$ is closed in the following topology: standard norm topology on $[0,1]$ (identical with the weak topology) and weak topology on $W^{2,2}$ and $L^2 \times \mathbb{R}^n \times \mathbb{R}^n$. Let $\lambda_k \in [0,1]$, $u_k \in W^{2,2}$, $\big(v_k, u_k(0), u_k(T)\big) \in \tilde{G}(\lambda_k, u_k)$, $k = 1, 2, \ldots$, $\lambda_k \longrightarrow \lambda$, $u_k \rightharpoonup u$, $\big(v_k, u_k(0), u_k(T)\big) \rightharpoonup (v, x, y)$ as $k \longrightarrow \infty$. We want to show that $(v, x, y) \in \tilde{G}(\lambda, u)$. Indeed, since the inclusion $W^{2,2} \subseteq C^1([0,T]; \mathbb{R}^k)$, is completely continuous, we may assume, by passing to a subsequence, that $u_k \longrightarrow u$ and $\dot{u}_k \longrightarrow \dot{u}$ in the norm topology. Evidently, $u_k(0) \longrightarrow u(0)$ and $u_k(T) \longrightarrow u(T)$, so only the convergence $\ddot{u}_n \rightharpoonup \ddot{u}$ and $v_n \rightharpoonup v$ is weak. The conclusion now follows from Lemma 3.2.

4. **Application to neutral equations: existence results.** We let $\mathcal{L}\colon W^{2,2} \longrightarrow L^2 \times \mathbb{R}^n \times \mathbb{R}^n$ be defined by $\mathcal{L}u = \big(\ddot{u} - \varepsilon u, u(0), u(T)\big)$, and $\tilde{G}\colon [0,1] \times W^{2,2} \longrightarrow L^2 \times \mathbb{R}^n \times \mathbb{R}^n$ be the multivalued map defined in the previous section. It is easily seen that $\mathcal{L}$ is an isomorphism, the problem (3.1) is equivalent to $\mathcal{L}u \in \tilde{G}(1,u)$ and, consequently, to $u \in F(u)$, where $F = \mathcal{L}^{-1} \circ \tilde{G}(1,u)\colon W^{2,2} \longrightarrow W^{2,2}$.

We assume that the following "a priori" boundedness conditions hold.

(H6) There exist $M_0, M_1 > 0$ such that, for any $u \in W^{2,2}$ satisfying either $\mathcal{L}u \in \tilde{G}(\lambda, u)$ or $\mathcal{L}u \in \lambda\tilde{G}(0,u)$, $0 \leq \lambda \leq 1$, we have $\|u\|_\infty < M_0$ and $\|\dot{u}\|_\infty < M_1$.

(H7) There exist constants $M > 0$ and $k \in [0,1)$ such that $\|h(\lambda, u, v, w)\|_2 \leq M + k\|w\|_2$ for all $(\lambda, u, v, w) \in [0,1] \times E_1$ with $\|u\|_\infty \leq M_0$ and $\|v\|_\infty \leq M_1$.

In the above statement, $\|\cdot\|_\infty$ stands for the essential supremum norm and $\|\cdot\|_2$ stands for the integral L^2 norm.

THEOREM 4.1. *Under the conditions (H1 – H7), the problem (3.1) has at least one solution $u \in W^{2,2}$ with $\|u\|_\infty \leq M_0$, $\|\dot{u}\|_\infty \leq M_1$ and $\|\ddot{u} - \varepsilon u\|_2 \leq M_2$, where $M_2 := [M + \sqrt{r}\,\varepsilon(1+k)M_0]/(1-k)$.*

PROOF. We will use the topological transversality theorem in the locally convex space $E = (W^{2,2}, \omega))$, where ω is the weak topology.

Put

$$U = \big\{u \in W^{2,2}; \|u\|_\infty < M_0, \|\dot{u}\|_\infty < M_1, \|\ddot{u} - \varepsilon u\|_2 \leq M_2\big\}.$$

Then $\mathcal{L}^{-1}\circ\tilde{G}([0,1]\times\overline{U})$ is bounded and, since the inclusion $W^{2,2}\subseteq C^1 := C^1([0,1];\mathbb{R}^n)$, is completely continuous, there are constants $N_0\geq M_0$ and $N_1\geq M_1$ such that $\|v\|_\infty\leq N_0$ and $\|\dot{v}\|_\infty\leq N_1$ for all $v\in\mathcal{L}^{-1}\circ G([0,1]\times\overline{U})$.

If $v\in\mathcal{L}^{-1}\circ\tilde{G}([0,1]\times\overline{U})$, then there exists $u\in\overline{U}$ such that

$$\ddot{v}(t)-\varepsilon v(t)\in h(\lambda,t,u,\dot{u},\ddot{u})-\varepsilon u(t).$$

From (H7) and the definition of U and noting that $\|u\|_2\leq\sqrt{r}\|u\|_\infty$ it follows that

$$\begin{aligned}\|\ddot{v}-\varepsilon v\|_2&\leq M+k\|\ddot{u}\|_2+\varepsilon\|u\|_2\\&\leq M+k\|\ddot{u}-\varepsilon u\|_2+\sqrt{r}\,\varepsilon(1+k)\|u\|_\infty\\&=[M+\sqrt{r}\,\varepsilon(1+k)M_0]+kM_2\\&\leq M_2.\end{aligned}$$

Let

$$C=\left\{u\in W^{2,2};\|u\|_\infty\leq N_0,\|\dot{u}\|_\infty\leq N_1\text{ and }\|\ddot{u}-\varepsilon u\|_2\leq M_2\right\}.$$

U is a subset of C and $\mathcal{L}^{-1}\circ\tilde{G}$ maps $[0,1]\times\overline{U}$ to C. It follows from the complete continuity of the inclusion $W^{2,2}\subset C^1$ that U is weakly open in C with $\partial_C U=\{u\in C;\|u\|_\infty=M_0,\|\dot{u}\|_\infty=M_1\}$. By Lemma 3.3, the map $\mathcal{L}^{-1}\circ\tilde{G}\colon[0,1]\times\overline{U}\longrightarrow C$ is $u.s.c.$ and compact in the norm topology of $[0,1]$ and weak topology of $W^{2,2}$ and L^2 with nonempty closed convex values. It follows from (H6) that, $\mathcal{L}^{-1}\circ\tilde{G}(1,\cdot)$ is homotopic to $\mathcal{L}^{-1}\circ\tilde{G}(0,\cdot)$ in $\mathcal{K}_{\partial_C U}(\overline{U},C)$.

On the other hand, using the same argument as that for Lemma 3.3, we can prove that $\mathcal{L}^{-1}\circ\tilde{G}(0,\cdot)\colon\overline{U}\longrightarrow C$ is an $u.s.c.$ compact map in the weak topology of $W^{2,2}$, and by assumption (H6), $x\neq\lambda\mathcal{L}^{-1}\circ\tilde{G}(0,x)$ for all $\lambda\in[0,1]$ and $x\in\partial_C U$. From the argument of Theorem 2.2, we show that $\mathcal{L}^{-1}\circ\tilde{G}(0,\cdot)$ is homotopic to the constant map $\bar{U}\longrightarrow\{y_0\}$ in $\mathcal{K}_{\partial_C U}(\bar{U},C)$, where y_0 is the unique solution of the problem $\ddot{y}_0-\varepsilon y_0=0$, $y_0(0)=\varphi(0)$ and $y_0(T)=b$.

Therefore, $\mathcal{L}^{-1}\circ\tilde{G}(1,\cdot)$ is homotopic to the constant map $\bar{U}\longrightarrow\{y_0\}$. By Theorem 2.1 and Corollary 2.1, $\mathcal{L}^{-1}\circ\tilde{G}(1,\cdot)$ is an essential map, and thus it has a fixed point in U. This completes the proof.

We now illustrate Theorem 4.1 by an example. Consider the following special case of the problem (3.1):

$$(4.1)\qquad\begin{cases}\ddot{x}(t)\in\int_{-r}^{0}p(\theta)\ddot{x}(t+\theta)\,d\theta+f\Big(t,x(t),\int_{-r}^{0}q(\theta)x(t+\theta)\,d\theta,\\\qquad\qquad x(t)-\int_{-r}^{0}p(\theta)x(t+\theta)\,d\theta,\dot{x}(t)-\int_{-r}^{0}p(\theta)\dot{x}(t+\theta)\,d\theta\Big)\\x(\theta)=\varphi(\theta),\theta\in[-r,0],\\x(T)=b\end{cases}$$

where $p,q\in L^2([-r,0];\mathbb{R}^1)$ with $\|p\|_2\sqrt{r}<\frac{1}{2}$, $f\colon[0,T]\times\mathbb{R}^4\longrightarrow\mathbb{R}^1$ is a multifunction, with nonempty closed convex values, satifying the following Carathéodory condition:
(EH1) for each $z\in\mathbb{R}^4$, the map $f(\cdot,z)\colon[0,T]\longrightarrow\mathbb{R}^1$ is measurable;

(EH2) for a.e. $t \in [0,T]$, the map $f(t,\cdot)\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^1$ is u.s.c;

(EH3) for any bounded set $W \subseteq \mathbb{R}^4$, there exists a function $\alpha_W \in L^2([0,T];\mathbb{R})$ such that $|f(t,z)| \leq \alpha_W(t)$ for all $z \in W$.

Clearly, for the above problem, the associated map $g\colon [0,T] \times E_1 \longrightarrow \mathbb{R}^1$ is defined as follows:

$$
\begin{aligned}
g(t,u,v,w) = {} & \int_{-r}^{0} p(\theta) w(t+\theta)\, d\theta \\
& + f\Big(t, u(t), \int_{-r}^{0} q(\theta) u(t+\theta)\, d\theta, u(t) - \int_{-r}^{0} p(\theta) u(t+\theta) d\theta, v(t) \\
& - \int_{-r}^{0} p(\theta) v(t+\theta)\, d\theta\Big)
\end{aligned}
$$

where we tacitly assume that

$$
(u,v,w)(t) = (\varphi, \dot{\varphi}, \ddot{\varphi})(t) \text{ for } t \in [-r,0].
$$

We now define a new multi-valued map $h\colon [0,1] \times [0,T] \times E_1 \longrightarrow \mathbb{R}^1$ as

$$
\begin{aligned}
h(\lambda, t, u, v, w) = {} & \lambda \int_{-r}^{0} p(\theta) w(t+\theta)\, d\theta \\
& + f\Big(t, u(t), \int_{-r}^{0} q(\theta) u(t+\theta) d\theta, u(t) - \lambda \int_{-r}^{0} p(\theta) u(t+\theta)\, d\theta, v(t) \\
& - \lambda \int_{-r}^{0} p(\theta) v(t+\theta)\, d\theta\Big).
\end{aligned}
$$

Evidently, (H1) is satisfied, and for each fixed $t \in [0,T]$, the map $\Psi_t\colon [0,1] \times E_1 \longrightarrow \mathbb{R}^4$ defined by

$$
\begin{aligned}
\Psi_t(\lambda, u, v, w) = {} & \Big(u(t), \int_{-r}^{0} q(\theta) u(t+\theta)\, d\theta, u(t) \\
& - \lambda \int_{-r}^{0} p(\theta) u(t+\theta)\, d\theta, v(t) - \lambda \int_{-r}^{0} p(\theta) v(t+\theta)\, d\theta\Big)
\end{aligned}
$$

is continuous. Therefore by (EH2), the composite map $f\big(t, \Psi_t(\cdot)\big)\colon [0,1] \times E_1 \longrightarrow \mathbb{R}^1$ is $u.s.c.$, so (H2) holds.

It is known (*cf.* [9]) that (EH1) and (EH2) imply that $h(\lambda, \cdot, u, v, w)$ has measurable single-valued selections for any $(\lambda, u, v, w) \in [0,1] \times E_1$, so (H3) is verified.

For any (λ, u, v, w), $(\lambda, u, v, \tilde{w}) \in [0,1] \times E_1$, we have

$$
\begin{aligned}
& \Big\| \lambda \int_{-r}^{0} p(\theta)[w(\cdot+\theta) - \tilde{w}(\cdot+\theta)]\, d\theta \Big\|_2^2 \\
& \qquad \leq \int_0^T \Big[\int_{-r}^{0} p(\theta) \big(w(t+\theta) - \tilde{w}(t+\theta)\big)\, d\theta \Big]^2 dt \\
& \qquad \leq \int_0^T \int_{-r}^{0} p^2(\theta)\, d\theta \int_{-r}^{0} [w(t+\theta) - \tilde{w}(t+\theta)]^2\, d\theta\, dt \\
& \qquad \leq r \|p\|_2^2 \|w - \tilde{w}\|_2^2.
\end{aligned}
$$

Thus (H4), (H5) and (H7) are verified.

In order to verify (H6), *i.e.*, to obtain an *a priori* bound for the following two families of problems

$$(4.2)\qquad \begin{cases} \ddot u(t) \in \lambda \int_{-r}^{0} p(\theta)\ddot u(t+\theta)\,d\theta + f\Big(t, u(t), \int_{-r}^{0} q(\theta)u(t+\theta)\,d\theta, \\ u(t) - \lambda\int_{-r}^{0} p(\theta)u(t+\theta)d\theta, \dot u(t) - \lambda \int_{-r}^{0} p(\theta)\dot u(t+\theta)\,d\theta\Big) \\ u(0) = \varphi(0) \\ u(T) = b \end{cases}$$

and

$$(4.3)\qquad \begin{cases} \ddot u(t) \in \lambda f\Big(t, u(t), \int_{-r}^{0} q(\theta)u(t+\theta)\,d\theta, u(t), \dot u(t)\Big) \\ u(0) = \varphi(0) \\ u(T) = b, \end{cases}$$

we assume the following growth conditions on f:

(EH4) there exists a constant $N > 0$ such that for $t \in [0,T]$ and $z = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4$ with $|z_1| \le \frac{|z_3|}{1-\|p\|_2\sqrt{r}}$, $|z_2| \le \frac{\|q\|_2\sqrt{r}}{1-\|p\|_2\sqrt{r}}|z_3|$ and $|z_3| \ge N$, it follows that $z_3\omega > 0$ for any $\omega \in f(t, z_1, z_2, z_3, 0)$.

(EH5) for any constant $L > 0$ there exists a function $\psi\colon [0,\infty) \longrightarrow (0,\infty)$ such that $\frac{s}{\psi(s)} \in L^{\infty}_{\mathrm{Loc}}[0,\infty)$, $\int_0^\infty \frac{s}{\psi(s)}\,ds = \infty$ and $|f(t,z)| \le \psi(|z_4|)$ for all $z \in \mathbb{R}^4$ with $|z_i| \le L$, $i = 1,2,3$.

We want to show the existence of a certain *a priori* bound for solutions of (4.2) required in (H6) under the assumptions (EH1–EH5)

We first show that if $|u(t) - \lambda\int_{-r}^{0} p(\theta)u(t+\theta)\,d\theta| \le Q$ for a constant $Q \ge \|\varphi\|_\infty$ and for all $t \in [0,T]$, then $\|u\|_\infty \le \frac{Q}{1-\|p\|_2\sqrt{r}}$. Indeed, if there exists $\tau \in [0,T]$ such that $|u(\tau)| = \max_{s\in[-r,T]} |u(s)|$, then

$$\begin{aligned} |u(\tau)| &\le Q + \left|\lambda\int_{-r}^{0} p(\theta)u(\tau+\theta)\,d\theta\right| \\ &\le Q + \|p\|_2\sqrt{r}|u(\tau)| \end{aligned}$$

from which it follows that $|u(\tau)| \le \frac{Q}{1-\|p\|_2\sqrt{r}}$, and thus $\|u\|_\infty \le \frac{Q}{1-\|p\|_2\sqrt{r}}$.

We next show that if

$$\begin{aligned} \|\varphi\|_\infty &\le \left|u(T) - \lambda\int_{-r}^{0} p(\theta)u(T+\theta)\,d\theta\right| \\ &= \max\left\{\left|u(t) - \lambda\int_{-r}^{0} p(\theta)u(t+\theta)\right|; 0 \le t \le T\right\}, \end{aligned}$$

then

$$\left|u(t) - \lambda\int_{-r}^{0} p(\theta)u(t+\theta)d\theta\right| \le \frac{1-\|p\|_2\sqrt{r}}{1-2\|p\|_2\sqrt{r}}|b| \text{ for } t \in [0,T].$$

Indeed, in this case, we have $\|u\|_\infty \le \frac{|u(T)-\lambda\int_{-r}^{0} p(\theta)u(T+\theta)\,d\theta|}{1-\|p\|_2\sqrt{r}}$ from which it follows that

$$\left|u(T) - \lambda\int_{-r}^{0} p(\theta)u(T+\theta)\,d\theta\right| \le |u(T)| + \|p\|_2\sqrt{r}\,\frac{|u(T) - \lambda\int_{-r}^{0} p(\theta)u(T+\theta)\,d\theta|}{1-\|p\|_2\sqrt{r}}$$

https://doi.org/10.4153/CJM-1992-061-5 Published online by Cambridge University Press

and thus

$$\left|u(T) - \lambda \int_{-r}^{0} p(\theta)u(T+\theta)\, d\theta\right| \leq \frac{1 - \|p\|_2\sqrt{r}}{1 - 2\|p\|_2\sqrt{r}}|u(T)| = \frac{1 - \|p\|_2\sqrt{r}}{1 - 2\|p\|_2\sqrt{r}}|b|.$$

We then claim that

$$\begin{aligned} &\left|u(t) - \lambda \int_{-r}^{0} p(\theta)u(t+\theta)\right| \\ &\quad\leq B := \max\left\{N, (1 + \|p\|_2\sqrt{r})\|\varphi\|_\infty, \frac{1 - \|p\|_2\sqrt{r}}{1 - 2\|p\|_2\sqrt{r}}|b|\right\} \text{ for } t \in [0, T] \end{aligned} \tag{4.4}$$

which, by the above argument, implies that

$$\|u\|_\infty \leq M_0 := \frac{1}{1 - \|p\|_2\sqrt{r}} \max\left\{N, \left(1 + \|p\|_2\sqrt{r}\right)\|\varphi\|_\infty, \frac{1 - \|p\|_2\sqrt{r}}{1 - 2\|p\|_2\sqrt{r}}|b|\right\}.$$

Indeed, from the above argument it suffices to verify (4.4) for $t \in (0, T)$. If this is not true, then we can find $t^* \in (0, T)$ such that $[u(t) - \lambda \int_{-r}^{0} p(\theta)u(t+\theta)\, d\theta]^2$ attains its maximum $D^2 \geq B^2$ at $t = t^*$. By the above established result, this implies that $|u(t)| \leq \frac{D}{1-\|p\|_2\sqrt{r}}$, and consequently, $|\int_{-r}^{0} q(\theta)u(t+\theta)\, d\theta| \leq \frac{\sqrt{r}\,\|q\|_2}{1-\|p\|_2\sqrt{r}} D$ for $t \in [0, T]$. However, at $t = t^*$, we have

$$\begin{aligned} \frac{d^2}{dt^2}\left[u(t) - \lambda \int_{-r}^{0} p(\theta)u(t+\theta)\, d\theta\right]^2 \in 2\Big[&f\Big(t^*, u(t^*), \int_{-r}^{0} q(\theta)u(t^*+\theta)\, d\theta, u(t^*) \\ &- \lambda \int_{-r}^{0} p(\theta)u(t^*+\theta), 0\Big)\Big]\left[u(t^*) - \lambda \int_{-r}^{0} p(\theta)u(t^*+\theta)\, d\theta\right]. \end{aligned}$$

By (EH4), we obtain $\frac{d^2}{dt^2}[u(t) - \lambda \int_{-r}^{0} p(\theta)u(t+\theta)\, d\theta]^2 > 0$ at $t = t^*$, which is a contradiction to the choice of t^*.

By assumption (EH5), we can find a function $\psi\colon [0, \infty) \longrightarrow (0, \infty)$ such that $\int_0^\infty \frac{s}{\psi(s)}\, ds = \infty$ and

$$\begin{aligned} \Big|f\Big(t, u(t), &\int_{-r}^{0} q(\theta)u(t+\theta)\, d\theta, u(t) - \lambda \int_{-r}^{0} p(\theta)u(t+\theta)\, d\theta, \dot{u}(t) \\ &- \lambda \int_{-r}^{0} p(\theta)\dot{u}(t+\theta)\, d\theta\Big)\Big| \leq \psi\left(\left|\dot{u}(t) - \lambda \int_{-r}^{0} p(\theta)\dot{u}(t+\theta)\, d\theta\right|\right). \end{aligned}$$

Therefore

$$\left|\frac{d^2}{dt^2}\left[u(t) - \lambda \int_{-r}^{0} p(\theta)u(t+\theta)\, d\theta\right]\right| \leq \psi\left(\left|\frac{d}{dt}\left[u(t) - \lambda \int_{-r}^{0} p(\theta)u(t+\theta)\right]\right|\right).$$

This implies the existence of a constant $\tilde{M}_1 > 0$ such that $|\dot{u}(t) - \lambda \int_{-r}^{0} p(\theta)\dot{u}(t+\theta)\, d\theta| \leq \tilde{M}_1$ for $t \in [0, T]$ (*cf.* [4]). Repeating the above argument, we obtain $\|\dot{u}\|_\infty \leq M_1 := \max\left\{\|\dot{\varphi}\|_\infty, \frac{\tilde{M}_1}{1-\|p\|_2\sqrt{r}}\right\}$.

Likewise, we can verify (H6) for the family of problems (4.3). Therefore by Theorem 4.1, the problem (4.1) has at least one solution.

To illustrate the above result, let us consider the following simple example:

EXAMPLE 4.1. Let $\varepsilon\colon \mathbb{R} \longrightarrow \mathbb{R}$ be the multifunction with intervals as values defined as follows

$$\varepsilon(x) = \begin{cases} \{n\} & \text{if } x \in (n, n+1),\, n \text{ is an integer,} \\ [n, n+1] & \text{if } x = n \text{ is an integer.} \end{cases}$$

Clearly, ε is an $u.s.c.$ convex valued function with $x \leq \varepsilon(x) \leq x+1$ for all $x \in \mathbb{R}$. We define $f\colon [0,1] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ as

$$f(t,z) = \varepsilon\Big(\alpha(t,z) + Az_3^{2k+1} + Bz_4^2\Big),$$

where $\alpha(t,z)$ is any bounded Carathéodory function, $A > 0$, B is any constant and k is any nonnegative integer. It is easy to verify that f satisfies the conditions EH1-EH5. Therefore, the problem (4.1) has a solution $u \in W^{2,2}$.

As a final remark, we point out that Theorem 4.1 can be applied to a much more general neutral equation than the one examined in this section. Also, we leave applications of the topological transversality theorem, to more general boundary value problems of neutral equations, for a further investigation.

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