J. Austral. Math. Soc. (Series A) 69 (2000), 153–161

TAMENESS AND GEODESIC CORES OF SUBGROUPS

RITA GITIK

(Received 13 December 1999; revised 15 February 2000)

Communicated by C. F. Miller

Abstract

Let N be a finitely generated normal subgroup of a finitely generated group G. We show that if the trivial subgroup is tame in the factor group G/N, then N is tame in G. We also give a short new proof of the fact that quasiconvex subgroups of negatively curved groups are tame. The proof utilizes the concept of the geodesic core of the subgroup and is related to the Dehn algorithm.

2000 *Mathematics subject classification*: primary 57M07, 57M30, 57N10, 20F32, 20F34.

1. Introduction

A 3-manifold M is called a missing boundary manifold if it can be embedded in a compact manifold $\bar{M}$ such that $\bar{M} \setminus M$ is a closed subset of the boundary of $\bar{M}$.

One of the long-standing open problems in the field of 3-manifolds is the missing boundary manifold conjecture due to Simon ([Sim]). He conjectured that if M_0 is a compact orientable irreducible 3-manifold, and M is the cover of M_0 corresponding to a finitely generated subgroup of $\pi_1(M_0)$, then M is a missing boundary manifold.

This conjecture has been verified in many special cases, (see [Kir, page 151] and [Gab] for additional information), however the general case is still open. In the special case when M has no boundary and $\pi_1(M)$ is finitely generated, M is a missing boundary manifold if and only if it is homeomorphic to the interior of a compact manifold. In this case M has finitely many ends, so it is a missing boundary manifold if and only if each of its ends is tame. (An end is tame if it is homeomorphic to a product (closed surface)$\times[0, \infty)$.) Hence the missing boundary manifold conjecture is also known as the tame ends conjecture.

Thurston showed in [Thu] that if M_0 is hyperbolic, geometrically finite and has infinite volume, then every cover of M_0 with a finitely generated fundamental group

https://doi.org/10.1017/S1446788700002147 Published online by Cambridge University Press

has tame ends. Bonahon showed in [Bon] that any hyperbolic manifold M with finitely generated $\pi_1(M)$ has tame ends, provided $\pi_1(M)$ is not a free product.

Tucker proved in [Tuc] that a non-compact orientable irreducible 3-manifold M is a missing boundary manifold if and only if for any compact submanifold C of M the group $\pi_1(M \setminus C)$ is finitely generated.

This observation made it possible to reduce the missing boundary manifold conjecture to a group-theoretic problem. The first results in this direction were obtained by Casson and Poénaru, (see [Poé]). Mihalik introduced the notion of a tame pair of groups in [Mi1]. Let M_0 be a 3-manifold with the fundamental group G, and let M be a cover of M_0 corresponding to a subgroup H of G. Then the pair (G, H) is tame if and only if M is a missing boundary manifold. This approach resulted in various group-theoretical results which implied some special cases of the tame ends conjecture, (see [Mi1, Mi2]).

However, the concept of a tame subgroup seems to be of independent interest. For example, it is not known weather the trivial subgroup is tame in any finitely generated group. Also there are no examples of a non-tame pair (G, H) with both H and G finitely generated.

In this paper we introduce a different, though equivalent, definition of a tame subgroup. Let M_0 be a compact orientable irreducible 3-manifold, let $G = \pi_1(M_0)$, and let H be a subgroup of G. Choose the presentation $G = \langle X | R \rangle$. Let K be the standard 2-complex representing G, that is, K has one vertex, K has an edge for any generator $x \in X$, and K has a 2-cell for any relator $r \in R$. Let $\text{Cayley}_2(G)$ be the universal cover of K, and let $\text{Cayley}_2(G, H)$ be the cover of K corresponding to a subgroup H of G.

Let $\tilde{M}_0$ be the universal cover of M_0 and let M be the cover of M_0 corresponding to H. Then $\text{Cayley}_2(G)$ imbeds quasi-isometrically in $\tilde{M}_0$, and $\text{Cayley}_2(G, H)$ embedded quasi-isometrically in M. Let C be a compact submanifold of M. It is easy to see that the fundamental group of $\text{Cayley}_2(G, H) \setminus C$ is finitely generated if and only if the fundamental group of $M \setminus C$ is. So Tucker's theorem implies that M is a missing boundary manifold if and only if $\pi_1(\text{Cayley}_2(G, H) \setminus C)$ is finitely generated for any finite subcomplex C of $\text{Cayley}_2(G, H)$.

This discussion motivates the following definition.

DEFINITION 1. A subgroup H of a group G is *tame* in G if for any finite subcomplex C of $\text{Cayley}_2(G, H)$ the group $\pi_1(\text{Cayley}_2(G, H) \setminus C)$ is finitely generated.

2. Preliminaries

Let M_0 be a compact orientable irreducible 3-manifold, and let $G = \pi_1(M_0)$. Choose a presentation $G = \langle X | R \rangle$. Let $X^* = \{x, x^{-1} | x \in X\}$, and for $x \in X$ define

$(x^{-1})^{-1} = x$. Recall that the Cayley graph of G, denoted Cayley(G), is an oriented graph whose set of vertices is G and the set of edges is $G \times X^*$, such that an edge (g, x) begins at the vertex g and ends at the vertex gx.

DEFINITION 2. Let H be a subgroup of G, and let $\{Hg\}$ be the set of right cosets of H in G. The *relative Cayley graph* of G with respect to H (or the *coset graph*) is an oriented graph whose vertices are the cosets $\{Hg\}$, the set of edges is $\{Hg\} \times X^*$, such that an edge (Hg, x) begins at the vertex Hg and ends at the vertex Hgx. We denote it Cayley(G, H).

A word in X is any finite sequence of elements of X^*. Denote the set of all words in X by W(X), and denote the equality of two words by $\equiv$.

DEFINITION 3. The *label* of a path

$$p = (Hg_1, x_1)(Hg_1x_1, x_2) \cdots (Hg_1x_1 \cdots x_{n-1}, x_n) \text{ in Cayley}(G, H)$$

is the function $\mathrm{Lab}(p) \equiv x_1x_2 \dots x_n \in W(X)$.

As usual, we identify the word Lab(p) with the corresponding element in G.

Note that Cayley(G) is the 1-skeleton of Cayley$_2(G)$, and Cayley(G, H) is the 1-skeleton of Cayley$_2(G, H)$. The following example illustrate the definitions.

LEMMA 1. *Let H be a subgroup of a group G, and let H_0 be a finite index subgroup of H. Then H is tame in G if and only if H_0 is.*

PROOF. Let ϕ : Cayley$(G, H_0) \to$ Cayley(G, H) be the covering map, and let C be a finite subcomplex of Cayley(G, H). As H_0 is a finite index subgroup of H, ϕ is a finite to one map. Then $\phi^{-1}(C)$ is a finite subcomplex of Cayley(G, H_0). If H_0 is tame in G, then $\pi_1(\text{Cayley}(G, H_0) \setminus \phi^{-1}(C))$ is finitely generated. As ϕ is a finite covering map from Cayley$(G, H_0) \setminus \phi^{-1}(C)$ to Cayley$(G, H) \setminus C$, it follows that $\pi_1(\text{Cayley}(G, H_0) \setminus \phi^{-1}(C))$ is a finite index subgroup of $\pi_1(\text{Cayley}(G, H) \setminus C)$, hence $\pi_1(\text{Cayley}(G, H) \setminus C)$ is finitely generated, so H is tame in G. In the other direction, if C_0 is a finite subcomplex in Cayley(G, H_0), then $C = \phi(C_0)$ is a finite subcomplex in Cayley(G, H). If H is tame in G, then $\pi_1(\text{Cayley}(G, H) \setminus C)$ is finitely generated, hence $\pi_1(\text{Cayley}(G, H_0) \setminus \phi^{-1}(C))$ is finitely generated. But Cayley$(G, H_0) \setminus C_0$ is constructed from Cayley$(G, H_0) \setminus \phi^{-1}(C)$ by adding a finite complex, hence $\pi_1(\text{Cayley}(G, H_0) \setminus C_0)$ is finitely generated, so H_0 is tame in G. $\square$

3. Normal subgroups

Let Y be a subset of a generating set of a group G, and let p be a path in Cayley(G, H). We say that p is labeled with Y if Lab(p) is a word in Y^*. We denote the inverse of a path p by $\bar{p}$.

THEOREM 1. *Let N be a finitely generated normal subgroup of a finitely generated group G. Then N is tame in G if the trivial subgroup is tame in the factor group G/N.*

PROOF. Let $\rho : G \to G/N$ be the projection map. Let $X = \{x_1, \dots, x_m\}$ be a generating set for N and let $Y = \{y_1, \dots, y_k\}$ be a subset of G such that $\{\rho(y_1), \dots, \rho(y_k)\}$ is a generating set for G/N. Choose the presentation $\langle X, Y | R, T\rangle$ for G, where R consists of the conjugation relators of the form $\{y_i x_j y_i^{-1} n_j | n_j \in N, 1 \le i \le k, 1 \le j \le m\}$. In this presentation Cayley(G, N) is isomorphic to a product of Cayley(G/N) and a wedge of m circles, one circle for each generator of N, however Cayley$_2(G, N)$ need not be a product. Let C be a finite subcomplex in Cayley$_2(G, N)$. Choose a basepoint v in Cayley$(G, N) \setminus C$ and let l be a loop in Cayley$(G, N) \setminus C$ beginning at v. Then l can be written as a product $l_1 l_2 \cdots l_n$, where each l_i is a loop beginning at v, all the loops l_{2i} are labeled with Y, and all the loops l_{2i-1} have the form $p_{2i-1} q_{2i-1} \bar{p}_{2i-1}$, where p_{2i-1} is labeled with Y and q_{2i-1} is labeled with X.

We claim that using the conjugation relators from R each loop l_{2i-1} is homotopic in Cayley$_2(G, N) \setminus C$ to a loop t_{2i-1} which begins at v and is labeled with Y. The proof is by induction on the length of the path p_{2i-1}. If its length is 0, then we can take $l_{2i-1} = t_{2i-1}$. Assume that the statement is true if p_{2i-1} is shorter than n. Consider a loop l_{2i-1} with p_{2i-1} of length n, so $p_{2i-1} = e_1 \cdots e_n$ and $q = e'_1 \cdots e'_l$, where e_i and e'_j are edges in Cayley$(G, N) \setminus C$. Consider a path $e_n e'_1 \bar{e}_n$. As Lab(e_n) Lab(e'_1) Lab$(e_n)^{-1}$ is a part of a conjugation relator from R, there exists a path s_1 in Cayley(G, H) labeled with X such that $e_n e'_1 \bar{e}_n s_1$ bounds a 2-cell in Cayley(G, H). This cell provides a homotopy of $e_n e'_1 \bar{e}_n$ to s_1. Similarly, for $1 \le j \le l$ we construct a homotopy of $e_n e'_j \bar{e}_n$ to a path s_j labeled with X. Combining these homotopies we obtain a homotopy of l_{2i-1} to a path $e_1 \cdots e_{n-1} s_1 \cdots s_l \bar{e}_{n-1} \cdots \bar{e}_1$, and we can use the inductive assumption. It remains to show that the paths s_j and the 2-cell bounded by $e_n e'_j \bar{e}_n s_j$ belong to Cayley$(G, H) \setminus C$. Indeed, by construction, the path $e_n e'_j \bar{e}_n$ is in Cayley$(G, H) \setminus C$. As C is a closed complex, it follows that the 2-cell bounded by $e_n e'_j \bar{e}_n s_j$ lies in Cayley$(G, H) \setminus C$. Also all the edges in s_j have the same common initial and terminal vertex, which belongs to the path l_{2i-1} and lies in the compliment of C. Hence all the edges of s_j lie in the compliment of C, proving the statement. Therefore l is homotopic to a product $t_1 l_2 t_3 \cdots l_n$, where each l_{2i} is a loop beginning at the basepoint v, labeled with Y, and all the loops t_{2i-1} are labeled with X. By construction, all the vertices of the paths t_{2i-1} coincide with the basepoint v, so all t_{2i-1} belong to the 0-neighbourhood of v. However, if the trivial subgroup is tame in G/N, then the paths l_{2i} can be homotoped to a bounded neighborhood of v in Cayley$(G/H) \setminus C$, so they can be homotoped to a bounded neighbourhood of v in Cayley$(G, H) \setminus C$. Therefore, if the trivial subgroup is tame in G/N any loop l in Cayley$(G, N) \setminus C$ can be homotoped to a loop in a bounded neighborhood of v, so N is tame in G. □

Recall that Mihalik proved in [Mi2] that if N is a normal subgroup of G and the groups G, N and G/N are finitely presented, then any finitely generated subgroup of infinite index in N is tame in G. Lemma 1 and Theorem 1 show that if the trivial subgroup is tame in G/N, then any finite index subgroup of N is tame in G.

4. Geodesic core of a subgroup

A geodesic in the Cayley(G) is a shortest path joining two vertices. Let ρ_H : Cayley$(G) \to$ Cayley(G, H) be the projection map: $\rho_H(g) = Hg$ and $\rho_H(g, x) = (Hg, x)$.

DEFINITION 4. (See [Git].) A *geodesic* in Cayley(G, H) is the image of a geodesic in Cayley(G) under the projection ρ_H. The *geodesic core* of Cayley(G, H) is the subgraph of Cayley(G, H) which consists of the union of all the vertices and all the edges which belong to closed geodesics in Cayley(G, H) beginning at the vertex $H \cdot 1$. We denote it Core(G, H).

Note that any path p in Cayley(G, H) which begins at $H \cdot 1$ ends at $H \cdot \text{Lab}(p)$, so a path p beginning at $H \cdot 1$ is closed, if and only if $\text{Lab}(p) \in H$.

A subgroup H of G is K-quasiconvex in G if any geodesic in the Cayley graph of G with the endpoints in H belongs to the K-neighbourhood of H. A subgroup is quasiconvex in G if it is K-quasiconvex in G for some K.

The following lemma from [Git] gives an important equivalent definition of quasiconvexity.

LEMMA 2. *A subgroup H of a group G is K-quasiconvex if and only if* Core(G, H) *belongs to the K-neighbourhood of $H \cdot 1$ in* Cayley(G, H).

PROOF. Let γ be a closed geodesic in Core(G, H) beginning at $H \cdot 1$. Then $\text{Lab}(\gamma) \in H$ and γ is the image of a geodesic $\tilde{\gamma}$ in Cayley(G) which begins at 1 with $\text{Lab}(\tilde{\gamma}) \equiv \text{Lab}(\gamma)$ under the projection map. But the projection map preserves distances from H, and it maps $H \subset$ Cayley(G) onto $H \cdot 1 \in$ Cayley(G, H), so $\gamma \subset N_K(H \cdot 1) \subset$ Cayley(G, H) if and only if $\tilde{\gamma} \subset N_K(H) \subset$ Cayley(G). □

https://doi.org/10.1017/S1446788700002147 Published online by Cambridge University Press

The following example illustrate the definitions.

EXAMPLE 1. Let $G = \langle x, y | xyx^{-1}y^{-1} \rangle$ be the standard presentation of $Z \times Z$, and let $H = \langle x \rangle$ be a subgroup of G. For any vertex x^n of H there exists a unique geodesic in Cayley(G) connecting 1 and x^n, namely the horizontal path p with Lab(p) $\equiv x^n$ so H is a 0-quasiconvex subgroup of Cayley(H). Cayley(G, H) can be described as follows: the set $\{Hy^n, n \in Z\}$ is the set of all cosets of H in G, hence we consider it as the set of vertices of Cayley(G, H). The edge (Hy^n, y) begins at the vertex Hy^n and ends at the vertex $Hy^ny = Hy^{n+1}$. The edge (Hy^n, x) begins at the vertex Hy^n and ends at the vertex $Hy^nx = Hxy^n = Hy^n$, hence this edge is a loop. The geodesic core of Cayley(G, H) consists of a single vertex $H \cdot 1$ and a single edge $(H \cdot 1, x)$, which begins and ends at $H \cdot 1$. The diameter of the geodesic core of Cayley(G, H) is 0, because it has only one vertex, verifying again that H is 0-quasiconvex in G.

On the other hand, consider a subgroup $L = \langle xy \rangle$ of G. A path p beginning at vertex 1 of Cayley(G) with Lab(p) $\equiv x^ny^n$ is a geodesic in Cayley(G), and the vertex x^n of Cayley(G) is in p, but $d(x^n, L) = n$, so L is not quasiconvex in Cayley(G). In order to describe Cayley(G, L), note that $\{Ly^n, n \in Z\}$ is the set of all cosets of L in G, so we consider it as the set of vertices of Cayley(G, L). The edge (Ly^n, y) begins at the vertex Ly^n and ends at the vertex $Ly^ny = Ly^{n+1}$. However, the edge (Ly^n, x) begins at the vertex Ly^n and ends at the vertex $Ly^nx = Lxyy^{n-1} = Ly^{n-1}$. The geodesic core of Cayley(G, L) is the whole graph Cayley(G, L) which is unbounded, demonstrating again that L is not quasiconvex in G.

Recall that a path p is an L-local geodesic if each subpath of p of length at most L is a geodesic. For example, any path in Cayley(G) is a 1-local geodesic. A geodesic triangle in Cayley(G) is a closed path $p = p_1p_2p_3$, where each p_i is a geodesic. A group G is δ-negatively curved if each side of each geodesic triangle in Cayley(G) belongs to the δ-neighbourhood of the union of two other sides. We consider only negatively curved groups that are finitely generated.

We use the following well-known fact.

LEMMA 3 ([Gro]). *Let G be a δ-negatively curved group, and let $L > 4\delta$. Then any L-local geodesic p in* Cayley(G) *belongs to the M-neighbourhood of a geodesic γ joining the endpoints of p, where M depends only on L and on δ.*

5. Quasiconvex subgroups

Up to this point we worked with some fixed presentation $\langle X | R \rangle$ of G. Now we need to redefine the presentation.

https://doi.org/10.1017/S1446788700002147 Published online by Cambridge University Press

DEFINITION 5. Let $\langle X | R \rangle$ be a presentation of G, and let $2L$ be a constant which is bigger than the length of the longest relator in R. Let R' be the set of all reduced words w in $W(X)$ which represent 1_G such that $|w| \leq 2L$. Then $\langle X | R' \rangle$ is a presentation for G, and we use it for the rest of the paper.

DEFINITION 6. An *L-local geodesic* in Cayley(G, H) is the image of an L-local geodesic in Cayley(G) under the projection ρ_H.

LEMMA 4. *Let $\langle X | R' \rangle$ be the presentation of a group G, as in Definition 5, let C be a finite subcomplex of* Cayley$_2(G, H)$, *and let v_0 be a vertex in* Cayley$_2(G, H) \setminus C$ *with $d(v_0, C) < L$. Then any closed path q in* Cayley$(G, H) \setminus C$ *beginning at v_0 is homotopic in* Cayley$_2(G, H) \setminus C$ *to a closed path p in* Cayley$(G, H) \setminus C$ *beginning at v_0 with the following property: there exists a decomposition $p = p_1 \cdots p_n$ such that p_{2i-1} is a maximal subpath of p which belongs to the L-neighbourhood of C, and p_{2i} is an L-local geodesic in* Cayley$(G, H) \setminus C$ *with both endpoints distance L away from C.*

PROOF. Let t be a maximal subpath of q which lies outside the L-neighbourhood of C and is not an L-local geodesic in Cayley(G, H). Then there exists a subpath t' of t shorter than L, which is not a geodesic, so t can be shortened by replacing t' with a shorter path t''. As $\bar{t}'t''$ is a closed path which is shorter than $2L$, the word Lab$(\bar{t}'t'')$ is a relator in R', so it bounds a 2-cell in Cayley$_2(G, H)$. As $|t''| < L$, this cell lies in the complement of C in Cayley$_2(G, H)$. As the path obtained from q by replacing t' with t'' is homotopic to q in Cayley$_2(G, H) \setminus C$, lies in the complement of C in Cayley(G, H) and is shorter than q, we obtain the required path p after finitely many repetitions of above procedure. $\square$

LEMMA 5. *Let H be a K-quasiconvex subgroup of a δ-negatively curved group G, let $L > 4\delta$, let M be as in Lemma* 3, *and let p be an L-local geodesic in* Cayley(G) *with both endpoints distance d away from H. Then p belongs to the $(K+M+d+2\delta)$-neighbourhood of H in* Cayley(G).

PROOF. Let s_1 and s_2 be geodesics in Cayley(G) of length at most d joining a vertex $h_1 \in H$ to the initial vertex of p, and the terminal vertex of p to a vertex $h_2 \in H$, respectively. Let γ' be a geodesic with the same endpoints, as p, and let γ be a geodesic joining h_1 to h_2. As H is K-quasiconvex in G, γ belongs to the K-neighbourhood of H. Consider a closed path $s_1\gamma' s_2 \bar{\gamma}$ in Cayley(G). This path is a geodesic 4-gon in a δ-negatively curved group G, hence γ' belongs to the 2δ-neighbourhood of the union of the other 3 sides. As s_1 and s_2 are shorter than d, it follows that γ' belongs to the $(K + d + 2\delta)$-neighbourhood of H. As p is an L-local geodesic, Lemma 3 implies that it belongs to the M-neighbourhood of γ', proving Lemma 5. $\square$

THEOREM 2. *A quasiconvex subgroup H of a negatively curved group G is tame in G.*

PROOF. Let H be a K-quasiconvex subgroup of a δ-negatively curved group G, and let C and v_0 be as in Lemma 4. Let q be any closed path in Cayley$(G, H) \setminus C$ beginning at v_0. Let p be a path equivalent to q, as described in Lemma 4. As C is a finite graph, it belongs to the ϵ-neighbourhood of $H \cdot 1$ in Cayley(G, H) for some constant ϵ. Then by construction, the subpaths p_{2i-1} of p belong to the $(\epsilon + L)$-neighbourhood of $H \cdot 1$, hence the endpoints of the subpaths p_{2i} belong to the $(\epsilon + L)$-neighbourhood of $H \cdot 1$. Then Lemma 5 implies that the subpaths p_{2i} belong to the $(\epsilon + L + M + K + 2\delta)$-neighbourhood of $H \cdot 1$. Therefore, q is homotopic in Cayley$_2(G, H) \setminus C$ to a path in the $(\epsilon + L + M + K + 2\delta)$-neighbourhood of $H \cdot 1$. As G is finitely generated, this neighbourhood is a finite graph, proving Theorem 2. $\square$

REMARK. Recall that a Dehn presentation for the group G is a finite presentation $\langle X | R \rangle$ such that any non-trivial reduced word $w \in W(X)$ representing 1_G contains more than half of some $r \in R$, that is, there exists a decomposition $r \equiv r_1 r_2$ with $|r_1| > |r_2|$ such that $w \equiv w_1 r_1 w_2$. A group with a Dehn presentation has the following algorithm, known as the Dehn algorithm, for a solution of the word problem. Let L be the length of the longest relator in R. Given a word $w \in W(X)$, check all its subwords which are shorter than $L + 1$. If none of these subwords is a bigger half of some relator in R, then w does not represent 1_G. Otherwise, there exists a relator $r = r_1 r_2 \in R$ with $|r_1| > |r_2|$ such that w contains r_1 as a subword. But then the word w' obtained from w by replacing r_1 by r_2^{-1} is equivalent to w in G and is shorter than w, so the algorithm terminates after finitely many steps. We would like to point out that the proof of Lemma 4 is based on a similar procedure.

Acknowledgement

The author thanks Gopal Prasad, Frank Raymond, John Stallings and Gadde Swarup for their support.

References

[Bon] F. Bonahon, 'Bouts ses variétés hyperboliques de dimension 3', *Ann. of Math.* **124** (1986), 71–158.

[Gab] D. Gabai, 'Problems in foliations and laminations', in: *AMS/IP Studies in Advanced Mathematics* vol. 2 part 2 (ed. W. Kazez) (AMS and International Press, Providence, 1997) pp. 1–35.

[Git] R. Gitik, 'On quasiconvex subgroups of negatively curved groups', *J. Pure Appl. Algebra* **119** (1997), 155–169.

[Gro] M. Gromov, 'Hyperbolic groups', in: *Essays in group theory* (ed. S. M. Gersten) MSRI publications vol. 8 (Springer, New York, 1987) pp. 75–263.

[Kir] R. Kirby, 'Problems in low-dimensional topology', in: *AMS/IP Studies in Advanced Mathematics* vol. 2 part 2 (ed. W. Kazez) (AMS and International Press, Providence, 1997) pp. 35–473.

[Mi1] M. Mihalik, 'Compactifying coverings of 3-manifolds', *Comment. Math. Helv.* **71** (1996), 362–372.

[Mi2] ———, 'Group extensions and tame pairs', *Trans. Amer. Math. Soc.* **351** (1999), 1095–1107.

[Poé] V. Poénaru, 'Almost convex groups, Lipschitz combing, and π_1^∞ for universal covering spaces of closed 3-manifolds', *J. Differential Geom.* **35** (1992), 103–130.

[Sim] J. Simon, 'Compactification of covering spaces of compact 3-manifolds', *Michigan Math. J.* **23** (1976), 245–256.

[Thu] W. Thurston, *Geometry and topology of 3-manifolds* (Princeton University, Princeton, 1978).

[Tuc] T. W. Tucker, 'Non-compact 3-manifolds and the missing boundary problem', *Topology* **13** (1974), 267–273.

A & H Consultants
Ann Arbor, MI 48104
USA
e-mail: ritagtk@math.lsa.umich.edu