2
Background on C^∞-schemes

One can think of C^∞-rings as a specific type of commutative $\mathbb{R}$-algebra, where there are not only the addition and multiplication operations, but operations corresponding to every smooth function $\mathbb{R}^n \to \mathbb{R}$. Alternatively, C^∞-rings can also be considered as certain product-preserving functors, and we will introduce both definitions in this chapter.

C^∞-rings, along with a spectrum functor, form the building blocks of C^∞-schemes. As in ordinary Algebraic Geometry, the image of a C^∞-ring under the spectrum functor gives an affine C^∞-scheme. However, while there will be many similarities to ordinary Algebraic Geometry, C^∞-algebraic geometry has several differences that may challenge the reader's intuition. These include the following.

- C^∞-rings are non-noetherian, so finitely presented C^∞-rings are not necessarily finitely generated.
- The spectrum functor uses maximal ideals with residue field $\mathbb{R}$, not prime ideals. This makes affine C^∞-schemes Hausdorff and regular.
- Affine C^∞-schemes are very general, enough so that all manifolds can be represented as affine C^∞-schemes, and study can be restricted to affine C^∞-schemes in many cases. However, manifolds with corners will not always be affine C^∞-schemes with corners in Chapter 5.
- C^∞-rings are not in 1-1 correspondence with affine C^∞-schemes, and the spectrum functor is neither full nor faithful. However, the full subcategory of *complete* C^∞-rings is in 1-1 correspondence with affine C^∞-schemes, and the spectrum functor is full and faithful on this subcategory.

Our motivating example throughout this chapter will be a manifold X. We will see that its set of smooth functions $C^\infty(X)$ is a complete C^∞-ring, and applying the spectrum functor returns the affine C^∞-scheme with underlying topological space X along with its sheaf of smooth functions. Transverse fibre products of manifolds map to fibre products of (affine) C^∞-schemes. Unlike the category of manifolds, all fibre products of (affine) C^∞-schemes exist, which is one of the motivating reasons for considering this category.

Our main reference for this chapter is the second author [49, §2–§4]. See also [42], Dubuc [23], Moerdijk and Reyes [75], and Kock [52].

2.1 Introduction to category theory

We begin with some well-known definitions from category theory. All the material of this section can be found in MacLane [64].

2.1.1 Categories and functors

Definition 2.1 A *category* $\mathcal{C}$ consists of a proper class of *objects* $\mathrm{Obj}(\mathcal{C})$, and for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ a set $\mathrm{Hom}(X, Y)$ of *morphisms* f from X to Y, written $f : X \to Y$, and for all $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$ a *composition map* $\circ : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$, written $(f, g) \mapsto g \circ f$. Composition must be *associative*, that is, if $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$ are morphisms in $\mathcal{C}$ then $(h \circ g) \circ f = h \circ (g \circ f)$. For each $X \in \mathrm{Obj}(\mathcal{C})$ there must exist an *identity morphism* $\mathrm{id}_X : X \to X$ such that $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$ for all $f : X \to Y$ in $\mathcal{C}$. A morphism $f : X \to Y$ is an *isomorphism* if there exists $f^{-1} : Y \to X$ with $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$.

If $\mathcal{C}$ is a category, the *opposite category* $\mathcal{C}^{\mathrm{op}}$ is $\mathcal{C}$ with the directions of all morphisms reversed. That is, we define $\mathrm{Obj}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Obj}(\mathcal{C})$, and for all $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$ we define $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$, and for $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$ we define $f \circ_{\mathcal{C}^{\mathrm{op}}} g = g \circ_{\mathcal{C}} f$, and $\mathrm{id}_{\mathcal{C}^{\mathrm{op}}} X = \mathrm{id}_{\mathcal{C}} X$.

We call $\mathcal{D}$ a *subcategory* of $\mathcal{C}$ if $\mathrm{Obj}(\mathcal{D}) \subseteq \mathrm{Obj}(\mathcal{C})$, and $\mathrm{Hom}_{\mathcal{D}}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathrm{Obj}(\mathcal{D})$, and compositions and identities in $\mathcal{D}$ agree with those in $\mathcal{C}$. We call $\mathcal{D}$ a *full* subcategory if also $\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all X, Y in $\mathcal{D}$.

Definition 2.2 Let $\mathcal{C}, \mathcal{D}$ be categories. A (*covariant*) *functor* $F : \mathcal{C} \to \mathcal{D}$ gives for all objects X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$, and for all morphisms $f : X \to Y$ in $\mathcal{C}$ a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$, such that $F(g \circ f) = F(g) \circ F(f)$ for all $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$, and $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{Obj}(\mathcal{C})$. A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Functors compose in the obvious way. Each category $\mathcal{C}$ has an obvious *identity functor* $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ with $\mathrm{id}_{\mathcal{C}}(X) = X$ and $\mathrm{id}_{\mathcal{C}}(f) = f$ for all

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X, f. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *full* if the maps $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$, $f \mapsto F(f)$ are surjective for all $X, Y \in \mathrm{Obj}(\mathcal{C})$, and *faithful* if the maps $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ are injective for all $X, Y \in \mathrm{Obj}(\mathcal{C})$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is called *essentially surjective* if for every object $Y \in \mathcal{D}$ there exists $X \in \mathcal{C}$ such that $Y \cong F(X)$ in $\mathcal{D}$.

Let $\mathcal{C}, \mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta : F \Rightarrow G$ gives, for all objects X in $\mathcal{C}$, a morphism $\eta(X) : F(X) \to G(X)$ such that if $f : X \to Y$ is a morphism in $\mathcal{C}$ then $\eta(Y) \circ F(f) = G(f) \circ \eta(X)$ as a morphism $F(X) \to G(Y)$ in $\mathcal{D}$. We call η a *natural isomorphism* if $\eta(X)$ is an isomorphism for all $X \in \mathrm{Obj}(\mathcal{C})$.

An *equivalence* between categories $\mathcal{C}, \mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that there exists a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\eta : G \circ F \Rightarrow \mathrm{id}_{\mathcal{C}}$ and $\zeta : F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$. That is, F is invertible up to natural isomorphism. Then we call $\mathcal{C}, \mathcal{D}$ *equivalent categories*. A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it is full, faithful, and essentially surjective.

It is a fundamental principle of category theory that equivalent categories $\mathcal{C}, \mathcal{D}$ should be thought of as being 'the same', and naturally isomorphic functors $F, G : \mathcal{C} \to \mathcal{D}$ should be thought of as being 'the same'. Note that equivalence of categories $\mathcal{C}, \mathcal{D}$ is much weaker than strict isomorphism: isomorphism classes of objects in $\mathcal{C}$ are naturally in bijection with isomorphism classes of objects in $\mathcal{D}$, but there is no relation between the sizes of the isomorphism classes, so that $\mathcal{C}$ could have many more objects than $\mathcal{D}$, for instance.

2.1.2 Limits, colimits and fibre products in categories

We shall be interested in various kinds of *limits* and *colimits* in our categories of spaces. These are objects in the category with a universal property with respect to some class of diagrams.

Definition 2.3 Let $\mathcal{C}$ be a category. A *diagram* Δ in $\mathcal{C}$ is a class of objects S_i in $\mathcal{C}$ for $i \in I$, and a class of morphisms $\rho_j : S_{b(j)} \to S_{e(j)}$ in $\mathcal{C}$ for $j \in J$, where $b, e : J \to I$. The diagram is called *small* if I, J are sets (rather than something too large to be a set), and *finite* if I, J are finite sets.

A *limit* of the diagram Δ is an object L in $\mathcal{C}$ and morphisms $\pi_i : L \to S_i$ for $i \in I$ such that $\rho_j \circ \pi_{b(j)} = \pi_{e(j)}$ for all $j \in J$, with the universal property that given $L' \in \mathcal{C}$ and $\pi'_i : L' \to S_i$ for $i \in I$ with $\rho_j \circ \pi'_{b(j)} = \pi'_{e(j)}$ for all $j \in J$, there is a unique morphism $\lambda : L' \to L$ with $\pi'_i = \pi_i \circ \lambda$ for all $i \in I$. The limit is called *small*, or *finite*, if Δ is small, or finite.

Here are some important kinds of limit.

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(i) A *terminal object* is a limit of the empty diagram.

(ii) Let X, Y be objects in $\mathcal{C}$. A *product* $X \times Y$ is a limit of the diagram with two objects X, Y and no morphisms.

(iii) Let $g : X \to Z$, $h : Y \to Z$ be morphisms in $\mathcal{C}$. A *fibre product* is a limit of a diagram $X \xrightarrow{g} Z \xleftarrow{h} Y$. The limit object W is often written $X \times_{g,Z,h} Y$ or $X \times_Z Y$. Explicitly, a fibre product is an object W and morphisms $e : W \to X$ and $f : W \to Y$ in $\mathcal{C}$, such that $g \circ e = h \circ f$, with the universal property that if $e' : W' \to X$ and $f' : W' \to Y$ are morphisms in $\mathcal{C}$ with $g \circ e' = h \circ f'$ then there is a unique morphism $b : W' \to W$ with $e' = e \circ b$ and $f' = f \circ b$. The commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle e} & & {\scriptstyle h}\downarrow \\ X & \xrightarrow{g} & Z \end{array} \tag{2.1}$$

is called a *Cartesian square*.

If Z is a terminal object then $X \times_Z Y$ is a product $X \times Y$.

A *colimit* of the diagram Δ is an object L in $\mathcal{C}$ and morphisms $\lambda_i : S_i \to L$ for $i \in I$ such that $\lambda_{b(j)} = \lambda_{e(j)} \circ \rho_j$ for all $j \in J$, which has the universal property that given $L' \in \mathcal{C}$ and $\lambda'_i : S_i \to L'$ for $i \in I$ with $\lambda'_{b(j)} = \lambda'_{e(j)} \circ \rho_j$ for all $j \in J$, there is a unique morphism $\pi : L \to L'$ with $\lambda'_i = \pi \circ \lambda_i$ for all $i \in I$.

Here are some important kinds of colimit.

(iv) An *initial object* is a colimit of the empty diagram.

(v) Let X, Y be objects in $\mathcal{C}$. A *coproduct* $X \amalg Y$ is a colimit of the diagram with two objects X, Y and no morphisms.

(vi) Let $e : W \to X$, $f : W \to Y$ be morphisms in $\mathcal{C}$. A *pushout* is a colimit of a diagram $X \xleftarrow{e} W \xrightarrow{f} Y$. The colimit object Z is often written $X \amalg_{e,W,f} Y$ or $X \amalg_W Y$. Explicitly, a pushout is an object Z and morphisms $g : X \to Z$ and $h : Y \to Z$ in $\mathcal{C}$, such that $g \circ e = h \circ f$, with the universal property that if $g' : X \to Z'$ and $h' : Y \to Z'$ are morphisms in $\mathcal{C}$ with $g' \circ e = h' \circ f$ then there is a unique morphism $b : Z \to Z'$ with $g' = b \circ g$ and $h' = b \circ h$. The diagram (2.1) is then called a *co-Cartesian square*.

If W is an initial object then $X \amalg_W Y$ is a coproduct $X \amalg Y$.

Limits and colimits may not may not exist. If a limit or colimit exists, it is unique up to canonical isomorphism in $\mathcal{C}$. We say that *all limits*, or *all small limits*, or *all finite limits exist in* $\mathcal{C}$, if limits exist for all diagrams, or all small diagrams, or all finite diagrams respectively; and similarly for colimits.

A category $\mathcal{C}$ is called *complete* if all small limits exist in $\mathcal{C}$, and *cocomplete* if all small colimits exist in $\mathcal{C}$.

Limits in $\mathcal{C}$ are equivalent to colimits (of the opposite diagram) in the opposite category $\mathcal{C}^{\mathrm{op}}$. So, for example, fibre products in $\mathcal{C}$ are pushouts in $\mathcal{C}^{\mathrm{op}}$.

A *directed colimit* is a colimit for which the diagram Δ is an *upward-directed set*, that is, Δ is a *preorder* (a category in which there is at most one morphism $S_i \to S_j$ between any two objects) in which every finite subset has an upper bound. Confusingly, directed colimits are also called *inductive limits* or *direct limits*, although they are actually colimits. So we can say that *all directed colimits exist in* $\mathcal{C}$.

The dual concept, called *codirected limit*, will not be used in this book.

By category theory general nonsense, one can prove the following.

Proposition 2.4 *Suppose a category $\mathcal{C}$ has a terminal object, and all fibre products exist in $\mathcal{C}$. Then all finite limits exist in $\mathcal{C}$.*

Example 2.5 Let $\mathcal{C}$ be a category of spaces, for instance, topological spaces $\mathbf{Top}$, manifolds $\mathbf{Man}$, schemes $\mathbf{Sch}$, or C^∞-schemes $\mathbf{C^\infty Sch}$ in §2.5. Then we have the following.

(i) The *terminal object* is the point $*$, and exists for all sensible $\mathcal{C}$.

(ii) *Products* $X \times Y$ are the usual products of manifolds, topological spaces, ...

(iii) *Fibre products* may or may not exist, depending on $\mathcal{C}$. All fibre products $W = X \times_{g,Z,h} Y$ exist in $\mathbf{Top}$, with $W = \{(x,y) \in X \times Y : g(x) = h(y)\}$, with the subspace topology as a subset of $X \times Y$. All fibre products also exist in $\mathbf{Sch}, \mathbf{C^\infty Sch}$. Fibre products $X \times_{g,Z,h} Y$ exist in $\mathbf{Man}$ if g, h are *transverse*, but not in general.

(iv) The *initial object* is the empty set $\emptyset$, and exists for all sensible $\mathcal{C}$.

(v) *Coproducts* $X \amalg Y$ are disjoint unions of the spaces X, Y. In $\mathbf{Man}$ this exists if $\dim X = \dim Y$.

(vi) General *pushouts* in categories such as $\mathbf{Man}, \mathbf{Sch}, \mathbf{C^\infty Sch}, \ldots$ tend not to exist, and have not been a focus of research.

Many important constructions in categories of spaces can be expressed as finite limits. For example, the intersection $X \cap Y$ of submanifolds $X, Y \subset Z$ is a fibre product $X \times_Z Y$. By Proposition 2.4, the existence of finite limits reduces to that of fibre products. So in our study of C^∞-schemes with corners, we will be particularly interested in existence and properties of fibre products.

Example 2.6 Let $\mathcal{C}$ be a category of (generalized) commutative algebras over a field $\mathbb{K}$, for example, commutative $\mathbb{C}$-algebras $\mathbf{Alg}_{\mathbb{C}}$, or C^∞-rings $\mathbf{C^\infty Rings}$ in §2.2 with $\mathbb{K} = \mathbb{R}$. Then we have the following.

(i) The *terminal object* is the zero algebra 0.
(ii) *Products* $B \times C$ are direct sums $B \oplus C$.
(iii) For morphisms $\beta : B \to D$, $\gamma : C \to D$, the *fibre product* $B \times_{\beta,D,\gamma} C$ is the subalgebra $\{(b,c) \in B \oplus C : \beta(b) = \gamma(c)\}$ in $B \oplus C$.
(iv) The *initial object* is the field $\mathbb{K}$.
(v) *Coproducts* $B \amalg C$ are (possibly completed) tensor products $B \otimes_{\mathbb{K}} C$.
(vi) For morphisms $\alpha : A \to B$, $\beta : A \to C$, a *pushout* is a (possibly completed) tensor product $B \otimes_{\alpha,A,\beta} C$.

2.1.3 Adjoint functors

Definition 2.7 Let $\mathcal{C}, \mathcal{D}$ be categories. An *adjunction* (F, G, φ) between $\mathcal{C}$ and $\mathcal{D}$ consists of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and bijections

$$\varphi(X,Y) : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

for all objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, which are natural in X, Y, that is, if $f : X_1 \to X_2$ and $g : Y_1 \to Y_2$ are morphisms in $\mathcal{C}, \mathcal{D}$ then the following commutes:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{D}}(F(X_2), Y_1) & \xrightarrow[g\circ -]{} & \mathrm{Hom}_{\mathcal{D}}(F(X_2), Y_2) & \xrightarrow[-\circ F(f)]{} & \mathrm{Hom}_{\mathcal{D}}(F(X_1), Y_2) \\
\downarrow{\scriptstyle \varphi(X_2,Y_1)} & & & & \downarrow{\scriptstyle \varphi(X_1,Y_2)} \\
\mathrm{Hom}_{\mathcal{C}}(X_2, G(Y_1)) & \xrightarrow{-\circ f} & \mathrm{Hom}_{\mathcal{C}}(X_1, G(Y_1)) & \xrightarrow{G(g)\circ -} & \mathrm{Hom}_{\mathcal{C}}(X_1, G(Y_2)).
\end{array}$$

Adjunctions are often written like this:

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}.$$

Then we say that F *is left adjoint to* G, and G *is right adjoint to* F. We say that $F : \mathcal{C} \to \mathcal{D}$ *has a right adjoint* if it can be completed to an adjunction (F, G, φ). We say that $G : \mathcal{D} \to \mathcal{C}$ *has a left adjoint* if it can be completed to an adjunction (F, G, φ).

Suppose $\mathcal{C}$ is a category, and $\mathcal{D} \subset \mathcal{C}$ is a full subcategory of $\mathcal{C}$. We say that $\mathcal{D}$ is a *reflective subcategory* of $\mathcal{C}$ if the inclusion $\mathrm{inc} : \mathcal{D} \hookrightarrow \mathcal{C}$ has a left adjoint. This left adjoint $R : \mathcal{C} \to \mathcal{D}$ is called a *reflection functor*.

Dually, if $\mathcal{C} \subset \mathcal{D}$ is a full subcategory, we say $\mathcal{C}$ is a *coreflective subcategory* of $\mathcal{D}$ if the inclusion $\mathrm{inc} : \mathcal{C} \hookrightarrow \mathcal{D}$ has a right adjoint. This right adjoint $C : \mathcal{D} \to \mathcal{C}$ is called a *coreflection functor*.

Here are some properties of adjoint functors.

Theorem 2.8 **(a)** *In Definition* 2.7 *there are natural transformations* $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F$*, called the* ***unit of the adjunction****, and* $\epsilon : F \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}}$*, called the* ***counit of the adjunction****, such that for* $X \in \mathcal{C}$ *and* $Y \in \mathcal{D}$ *we have*

$$\eta(X) = \varphi(X, F(X))(\mathrm{id}_{F(X)}) : X \longrightarrow G(F(X)),$$
$$\epsilon(Y) = \varphi(G(Y), Y)^{-1}(\mathrm{id}_{G(Y)}) : F(G(Y)) \longrightarrow Y.$$

(b) F*,*G *are both equivalences of categories if and only if* F*,*G *are both full and faithful, if and only if* η*,*ϵ *are both natural isomorphisms.*

(c) *If* $F : \mathcal{C} \to \mathcal{D}$ *has a right adjoint, then the right adjoint* $G : \mathcal{D} \to \mathcal{C}$ *is determined up to natural isomorphism by* F*.*

(d) *If* $G : \mathcal{D} \to \mathcal{C}$ *has a left adjoint, then the left adjoint* $F : \mathcal{C} \to \mathcal{D}$ *is determined up to natural isomorphism by* G*.*

(e) *If* $F : \mathcal{C} \to \mathcal{D}$ *has a right adjoint then it* ***preserves colimits****, that is,* F *maps a colimit in* $\mathcal{C}$ *to the corresponding colimit in* $\mathcal{D}$ *(which is guaranteed to exist in* $\mathcal{D}$*, if the initial colimit exists in* $\mathcal{C}$*).*

(f) *If* $G : \mathcal{D} \to \mathcal{C}$ *has a left adjoint then it* ***preserves limits****, that is,* G *maps a limit in* $\mathcal{D}$ *to the corresponding limit in* $\mathcal{C}$ *(which is guaranteed to exist in* $\mathcal{C}$*, if the initial limit exists in* $\mathcal{D}$*).*

(g) *Let* $\mathcal{D} \subset \mathcal{C}$ *be a reflective subcategory, with reflection functor* $R : \mathcal{C} \to \mathcal{D}$*. Suppose some class of colimits (e.g. all small colimits, or all pushouts) exists in* $\mathcal{C}$*. Then the same class of colimits exists in* $\mathcal{D}$*. We can obtain the colimit of a diagram in* $\mathcal{D}$ *by taking the colimit in* $\mathcal{C}$ *and then applying* R*.*

(h) *Let* $\mathcal{C} \subset \mathcal{D}$ *be a coreflective subcategory, with coreflection functor* $C : \mathcal{D} \to \mathcal{C}$*. Suppose some class of limits (e.g. all small limits, or all fibre products) exists in* $\mathcal{D}$*. Then the same class of limits exists in* $\mathcal{C}$*. We can obtain the limit of a diagram in* $\mathcal{C}$ *by taking the limit in* $\mathcal{D}$ *and then applying* C*.*

Remark 2.9 We will use adjoint functors in two main ways below. Firstly, by Theorem 2.8(e)–(h), we can use them to prove results on existence of (co)limits. The second is more philosophical, and we illustrate it by two examples.

(a) In §2.5 we define an adjunction

$$\mathbf{LC^\infty RS} \underset{\mathrm{Spec}}{\overset{\Gamma}{\rightleftarrows}} \mathbf{C^\infty Rings}$$

between the categories $\mathbf{C^\infty Rings}$ of C^∞-rings and $\mathbf{LC^\infty RS}$ of locally C^∞-ringed spaces. Here the definition of the global sections functor Γ is simple and obvious, but that of the spectrum functor Spec is complicated

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and apparently arbitrary. However, Theorem 2.8(c) implies that Spec is determined up to natural isomorphism by Γ and the adjoint property. This justifies the definition of Spec, showing it could not have been otherwise.

(b) In §6.2 we define an adjunction

$$\mathbf{C^\infty Sch^c_{in}} \underset{C}{\overset{\mathrm{inc}}{\rightleftarrows}} \mathbf{C^\infty Sch^c}.$$

Here $\mathbf{C^\infty Sch^c}$ is the category of C^∞-schemes with corners, $\mathbf{C^\infty Sch^c_{in}}$ is the non-full category with only interior morphisms, with inclusion $\mathrm{inc}: \mathbf{C^\infty Sch^c_{in}} \hookrightarrow \mathbf{C^\infty Sch^c}$, and $C : \mathbf{C^\infty Sch^c} \to \mathbf{C^\infty Sch^c_{in}}$ is the 'corner functor', which encodes the notions of boundary $\partial \boldsymbol{X}$ and k-corners $C_k(\boldsymbol{X})$ of a (firm) C^∞-scheme with corners $\boldsymbol{X}$ in a functorial way.

Again, the definition of inc is simple and obvious, and that of C is complicated and contrived. But the adjoint property shows that C is determined up to natural isomorphism by inc, and so justifies the definition.

2.2 C^∞-rings

Here are two equivalent definitions of C^∞-ring. The first definition describes C^∞-rings as functors while the second definition describes C^∞-rings as sets in the style of classical algebra.

Definition 2.10 Write $\mathbf{Man}$ for the category of manifolds, and $\mathbf{Euc}$ for the full subcategory of $\mathbf{Man}$ with objects the Euclidean spaces $\mathbb{R}^n$. That is, the objects of $\mathbf{Euc}$ are $\mathbb{R}^n$ for $n = 0, 1, 2, \ldots$, and the morphisms in $\mathbf{Euc}$ are smooth maps $f : \mathbb{R}^m \to \mathbb{R}^n$. Write $\mathbf{Sets}$ for the category of sets. In both $\mathbf{Euc}$ and $\mathbf{Sets}$ we have notions of (finite) products of objects (that is, $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$, and products $S \times T$ of sets S, T), and products of morphisms.

Define a *categorical C^∞-ring* to be a product-preserving functor $F : \mathbf{Euc} \to \mathbf{Sets}$. Here F should also preserve the empty product, that is, it maps $\mathbb{R}^0$ in $\mathbf{Euc}$ to the terminal object in $\mathbf{Sets}$, the point $*$. If $F, G : \mathbf{Euc} \to \mathbf{Sets}$ are categorical C^∞-rings, a *morphism* $\eta : F \to G$ is a natural transformation $\eta : F \Rightarrow G$. We write $\mathbf{CC^\infty Rings}$ for the category of categorical C^∞-rings.

Categorical C^∞-rings are an example of an *Algebraic Theory* in the sense of Adámek, Rosický, and Vitale [1], and many of the basic categorical properties of C^∞-rings follow from this.

Definition 2.11 A *C^∞-ring* is a set $\mathfrak{C}$ together with operations

$$\Phi_f : \mathfrak{C}^n = \overbrace{\mathfrak{C} \times \cdots \times \mathfrak{C}}^{n \text{ copies}} \longrightarrow \mathfrak{C}$$

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for all $n \geqslant 0$ and smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, where by convention when $n = 0$ we define $\mathfrak{C}^0$ to be the single point $\{\emptyset\}$. These operations must satisfy the following relations: suppose $m, n \geqslant 0$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $g : \mathbb{R}^m \to \mathbb{R}$ are smooth functions. Define a smooth function $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = g\big(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1 \ldots, x_n)\big),$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then for all $(c_1, \ldots, c_n) \in \mathfrak{C}^n$ we have

$$\Phi_h(c_1, \ldots, c_n) = \Phi_g\big(\Phi_{f_1}(c_1, \ldots, c_n), \ldots, \Phi_{f_m}(c_1, \ldots, c_n)\big).$$

We also require that for all $1 \leqslant j \leqslant n$, defining $\pi_j : \mathbb{R}^n \to \mathbb{R}$ by $\pi_j : (x_1, \ldots, x_n) \mapsto x_j$, we have $\Phi_{\pi_j}(c_1, \ldots, c_n) = c_j$ for all $(c_1, \ldots, c_n) \in \mathfrak{C}^n$. Usually we refer to $\mathfrak{C}$ as the C^∞-ring, leaving the C^∞-operations Φ_f implicit.

A *morphism* of C^∞-rings $\big(\mathfrak{C}, (\Phi_f)_{f:\mathbb{R}^n\to\mathbb{R}\ C^\infty}\big)$, $\big(\mathfrak{D}, (\Psi_f)_{f:\mathbb{R}^n\to\mathbb{R}\ C^\infty}\big)$ is a map $\phi : \mathfrak{C} \to \mathfrak{D}$ such that $\Psi_f\big(\phi(c_1), \ldots, \phi(c_n)\big) = \phi \circ \Phi_f(c_1, \ldots, c_n)$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$ and $c_1, \ldots, c_n \in \mathfrak{C}$. We will write $\mathbf{C^\infty Rings}$ for the category of C^∞-rings.

Each C^∞-ring has an underlying commutative $\mathbb{R}$-algebra structure. The addition map $f : \mathbb{R}^2 \to \mathbb{R}$, $f : (x, y) \mapsto x + y$ gives addition '$+$' on $\mathfrak{C}$ as an $\mathbb{R}$-algebra by $c + d = \Phi_f(c, d)$ for $c, d \in \mathfrak{C}$. The multiplication map $g : \mathbb{R}^2 \to \mathbb{R}$, $g : (x, y) \mapsto xy$ gives multiplication '$\cdot$' on $\mathfrak{C}$ by $c \cdot d = \Phi_g(c, d)$. For each $\lambda \in \mathbb{R}$ write $\lambda' : \mathbb{R} \to \mathbb{R}$, $\lambda' : x \mapsto \lambda x$, and then scalar multiplication is $\lambda c = \Phi_{\lambda'}(c)$. Let $0', 1' : \mathbb{R}^0 \to \mathbb{R}$ map $*$ to $0, 1$. Then $0 = \Phi_{0'}$ and $1 = \Phi_{1'}$ are the zero element and identity element for $\mathfrak{C}$. The projection and composition relations show this gives $\mathfrak{C}$ the structure of a commutative $\mathbb{R}$-algebra. However, an $\mathbb{R}$-algebra allows only for operations corresponding to polynomials, whereas a C^∞-ring allows for operations corresponding to all smooth functions and so has a richer structure.

Proposition 2.12 *There is an equivalence* $\mathbf{C^\infty Rings} \cong \mathbf{CC^\infty Rings}$. *This identifies* $\mathfrak{C}$ *in* $\mathbf{C^\infty Rings}$ *with* $F : \mathbf{Euc} \to \mathbf{Sets}$ *in* $\mathbf{CC^\infty Rings}$ *such that* $F\big(\mathbb{R}^n\big) = \mathfrak{C}^n$ *for* $n \geqslant 0$, *and for smooth* $f : \mathbb{R}^n \to \mathbb{R}$ *then* $F(f)$ *is identified with* Φ_f.

Proving Proposition 2.12 is straightforward and relies on F being product-preserving. We leave it as an exercise to guide the reader's intuition.

The following example motivates these definitions.

Example 2.13 **(a)** Let X be a manifold. Define a functor $F_X : \mathbf{Euc} \to \mathbf{Sets}$ by $F_X(\mathbb{R}^n) = \mathrm{Hom}_{\mathbf{Man}}(X, \mathbb{R}^n)$, and $F_X(g) = g\circ : \mathrm{Hom}_{\mathbf{Man}}(X, \mathbb{R}^m) \to \mathrm{Hom}_{\mathbf{Man}}(X, \mathbb{R}^n)$ for each morphism $g : \mathbb{R}^m \to \mathbb{R}^n$ in $\mathbf{Euc}$. Then F_X is a categorical C^∞-ring. If $f : X \to Y$ is a smooth map of manifolds, define a natural transformation $F_f : F_Y \Rightarrow F_X$ by $F_f(\mathbb{R}^n) = \circ f : \mathrm{Hom}_{\mathbf{Man}}(Y, \mathbb{R}^n) \to \mathrm{Hom}_{\mathbf{Man}}(X, \mathbb{R}^n)$. Then F_f is a morphism in $\mathbf{CC^\infty Rings}$. Define a functor $F_{\mathbf{Man}}^{\mathbf{CC^\infty Rings}} : \mathbf{Man} \to \mathbf{CC^\infty Rings}^{\mathrm{op}}$ to map $X \mapsto F_X$ and $f \mapsto F_f$.

(b) Let X be a manifold. Write $C^\infty(X)$ for the set of smooth functions $c : X \to \mathbb{R}$. For non-negative integers n and smooth $f : \mathbb{R}^n \to \mathbb{R}$, define C^∞-operations $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ by composition

$$\big(\Phi_f(c_1, \ldots, c_n)\big)(x) = f\big(c_1(x), \ldots, c_n(x)\big), \tag{2.2}$$

for all $c_1, \ldots, c_n \in C^\infty(X)$ and $x \in X$. The composition and projection relations follow directly from the definition of Φ_f, so that $C^\infty(X)$ forms a C^∞-ring. If we consider the $\mathbb{R}$-algebra structure of $C^\infty(X)$ as a C^∞-ring, this is the canonical $\mathbb{R}$-algebra structure on $C^\infty(X)$. If $f : X \to Y$ is a smooth map of manifolds, then $f^* : C^\infty(Y) \to C^\infty(X)$ mapping $c \mapsto c \circ f$ is a morphism of C^∞-rings.

Define $F_{\mathbf{Man}}^{\mathbf{C^\infty Rings}} : \mathbf{Man} \to \mathbf{C^\infty Rings}^{\mathrm{op}}$ to map $X \mapsto C^\infty(X)$ and $f \mapsto f^*$. Moerdijk and Reyes [75, Th. I.2.8] show that $F_{\mathbf{Man}}^{\mathbf{C^\infty Rings}}$ is full and faithful, and takes transverse fibre products in $\mathbf{Man}$ to fibre products in $\mathbf{C^\infty Rings}^{\mathrm{op}}$. This fact is non-trivial, as it relies on knowing that all manifolds X can be embedded as a closed subspace of $\mathbb{R}^n$ for some large n, that $C^\infty(X)$ is a *finitely generated* C^∞-ring (as defined later in Proposition 2.17), and that manifolds admit partitions of unity that behave well with respect to smooth maps.

There are many more C^∞-rings than those that come from manifolds. For example, if X is a smooth manifold of positive dimension, then the set $C^k(X)$ of k-differentiable maps $f : X \to \mathbb{R}$ is a C^∞-ring with operations Φ_f defined as in (2.2), and each of these C^∞-rings is different for all $k = 0, 1, \ldots$.

Example 2.14 Consider $X = *$ the point, so $\dim X = 0$, then $C^\infty(*) = \mathbb{R}$ and Example 2.13 shows the C^∞-operations $\Phi_f : \mathbb{R}^n \to \mathbb{R}$ given by $\Phi_f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$ make $\mathbb{R}$ into a C^∞-ring. It is the initial object in $\mathbf{C^\infty Rings}$, and the simplest non-zero example of a C^∞-ring. The zero C^∞-ring is the set $\{0\}$, where all C^∞-operations $\Phi_f : \{0\} \to \{0\}$ send $0 \mapsto 0$, and this is the final object in $\mathbf{C^\infty Rings}$.

Definition 2.15 An *ideal* I in $\mathfrak{C}$ is an ideal in $\mathfrak{C}$ when $\mathfrak{C}$ is considered as a commutative $\mathbb{R}$-algebra. We do not require it to be closed under all C^∞-operations, as this would force $I = \mathfrak{C}$.

We can make the $\mathbb{R}$-algebra quotient $\mathfrak{C}/I$ into a C^∞-ring using Hadamard's Lemma. That is, if $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, define $\Phi_f^I : (\mathfrak{C}/I)^n \to \mathfrak{C}/I$ by

$$\big(\Phi_f^I(c_1 + I, \ldots, c_n + I)\big)(x) = \Phi_f\big(c_1(x), \ldots, c_n(x)\big) + I.$$

Then Hadamard's Lemma says for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$, there exists $g_i : \mathbb{R}^{2n} \to \mathbb{R}$ for $i = 1, \ldots, n$, such that

$$f(x_1, \ldots, x_n) - f(y_1, \ldots, y_n) = \sum_{i=1}^{n} (x_i - y_i) g_i(x_1, \ldots, x_n, y_1, \ldots, y_n).$$

If $d_1, \ldots, d_n$ are alternative choices for $c_1, \ldots, c_n$, then $c_i - d_i \in I$ for each $i = 1, \ldots, n$ and

$$\Phi_f(c_1, \ldots, c_n) - \Phi_f(d_1, \ldots, d_n) = \sum_{i=1}^{n} (c_i - d_i) \Phi_f(c_1, \ldots, c_n, d_1, \ldots, d_n)$$

lies in I, so Φ_f^I is independent of the choice of representatives $c_1, \ldots, c_n$ in $\mathfrak{C}$ and is well defined.

The next definition and proposition come from Adámek et al. [1, Rem. 11.21, Props. 11.26, 11.28, 11.30 and Cor. 11.33].

Definition 2.16 If A is a set then by [1, Rem. 11.21] we can define the *free C^∞-ring $\mathfrak{F}^A$ generated by A*. We may think of $\mathfrak{F}^A$ as $C^\infty(\mathbb{R}^A)$, where $\mathbb{R}^A = \{(x_a)_{a \in A} : x_a \in \mathbb{R}\}$. Explicitly, we define $\mathfrak{F}^A$ to be the set of maps $c : \mathbb{R}^A \to \mathbb{R}$ which depend smoothly on only finitely many variables x_a, and operations Φ_f are defined as in (2.2). Regarding $x_a : \mathbb{R}^A \to \mathbb{R}$ as functions for $a \in A$, we have $x_a \in \mathfrak{F}^A$, and we call x_a the *generators* of $\mathfrak{F}^A$.

Then $\mathfrak{F}^A$ has the universal property that if $\mathfrak{C}$ is any C^∞-ring then a choice of map $\alpha : A \to \mathfrak{C}$ uniquely determines a morphism $\phi : \mathfrak{F}^A \to \mathfrak{C}$ with $\phi(x_a) = \alpha(a)$ for $a \in A$. When $A = \{1, \ldots, n\}$ we have $\mathfrak{F}^A \cong C^\infty(\mathbb{R}^n)$, as in [52, Prop. III.5.1].

Proposition 2.17 **(a)** *Every object $\mathfrak{C}$ in $\mathbf{C^\infty Rings}$ admits a surjective morphism $\phi : \mathfrak{F}^A \to \mathfrak{C}$ from some free C^∞-ring $\mathfrak{F}^A$. We call $\mathfrak{C}$* ***finitely generated*** *if this holds with A finite. The kernel of ϕ, $\ker(\phi)$, is an ideal in $\mathfrak{F}^A$ and the quotient $\mathfrak{F}^A / \ker(\phi)$ is isomorphic to $\mathfrak{C}$.*

(b) *Every object $\mathfrak{C}$ in $\mathbf{C^\infty Rings}$ fits into a* ***coequalizer diagram***

$$\mathfrak{F}^B \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathfrak{F}^A \xrightarrow{\phi} \mathfrak{C}, \tag{2.3}$$

that is, $\mathfrak{C}$ is the colimit of $\mathfrak{F}^B \rightrightarrows \mathfrak{F}^A$ in $\mathbf{C^\infty Rings}$, where ϕ is automatically surjective. We call $\mathfrak{C}$ ***finitely presented*** *if this holds with A, B finite.*

Actually as any relation $f = g$ in $\mathfrak{C}$ is equivalent to the relation $f - g = 0$, we can simplify (2.3) by taking β to map $x_b \mapsto 0$ for all $b \in B$. This means that finitely presented is equivalent to requiring $\ker(\phi)$ from Proposition 2.17(a) to

https://doi.org/10.1017/9781009400190.002 Published online by Cambridge University Press

be finitely generated as an ideal. But the analogue of this for C^∞-rings with corners in §4.5 will not hold. In addition, $C^\infty(\mathbb{R}^n)$ is not noetherian, so ideals in a finitely generated C^∞-ring may not be finitely generated. This implies that finitely presented C^∞-rings are a proper subcategory of finitely generated C^∞-rings, in contrast to ordinary Algebraic Geometry, where they are equal.

We now study limits and colimits in $\mathbf{C^\infty Rings}$. For the pushout of morphisms $\phi : \mathfrak{C} \to \mathfrak{D}$, $\psi : \mathfrak{C} \to \mathfrak{E}$ in $\mathbf{C^\infty Rings}$, we write $\mathfrak{D} \amalg_{\phi,\mathfrak{C},\psi} \mathfrak{E}$ or $\mathfrak{D} \amalg_{\mathfrak{C}} \mathfrak{E}$. In the special case $\mathfrak{C} = \mathbb{R}$ the coproduct $\mathfrak{D} \amalg_{\mathbb{R}} \mathfrak{E}$ will be written as $\mathfrak{D} \otimes_\infty \mathfrak{E}$. Recall that the coproduct of $\mathbb{R}$-algebras A, B is the tensor product $A \otimes B$; however, $\mathfrak{D} \otimes_\infty \mathfrak{E}$ is usually different from their tensor product $\mathfrak{D} \otimes \mathfrak{E}$, as discussed for $C^\infty(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^m)$ above. For example, for $m, n > 0$, then $C^\infty(\mathbb{R}^m) \otimes_\infty C^\infty(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^{m+n})$ as in [75, p. 22], which contains $C^\infty(\mathbb{R}^m) \otimes C^\infty(\mathbb{R}^n)$ but is larger than this, as it includes elements such as $\exp(fg)$ for $f \in C^\infty(\mathbb{R}^m)$ and $g \in C^\infty(\mathbb{R}^n)$.

By Moerdijk and Reyes [75, pp. 21–22] and Adámek et al. [1, Props. 1.21, 2.5 and Th. 4.5] we have the following.

Proposition 2.18 *The category* $\mathbf{C^\infty Rings}$ *of* C^∞*-rings has all small limits and all small colimits. The forgetful functor* $\Pi : \mathbf{C^\infty Rings} \to \mathbf{Sets}$ *preserves limits and directed colimits, and can be used to compute such (co)limits pointwise; however, it does not preserve general colimits such as pushouts.*

The proof of Proposition 2.18 is straightforward, firstly by proving that the small limits and directed colimits in the category of sets inherit their universal properties and a C^∞-ring structure. Proving separately that coproducts exist follows from the universal property and considering simple cases. This includes that for $m, n > 0$ then the coproduct of $C^\infty(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^m)$ is $C^\infty(\mathbb{R}^{n+m})$, and that for ideals I, J we have the coproduct

$$\big(C^\infty(\mathbb{R}^n)/I\big) \otimes_\infty \big(C^\infty(\mathbb{R}^m)/J\big) \cong \big(C^\infty(\mathbb{R}^n) \otimes_\infty C^\infty(\mathbb{R}^m)\big)/(I, J).$$

A similar result holds for all finitely generated C^∞-rings. The proof then uses that any C^∞-ring is a directed colimit of finitely generated C^∞-rings to deduce the result.

We will need *local* C^∞-rings and *localizations* of C^∞-rings to define local C^∞-ringed spaces and C^∞-schemes in §2.5.

Definition 2.19 Recall that a *local* $\mathbb{R}$*-algebra* is an $\mathbb{R}$-algebra R with a unique maximal ideal $\mathfrak{m}$. The *residue field* of R is the field (isomorphic to) $R/\mathfrak{m}$. A C^∞-ring $\mathfrak{C}$ is called *local* if, regarded as an $\mathbb{R}$-algebra, $\mathfrak{C}$ is a local $\mathbb{R}$-algebra with residue field $\mathbb{R}$. The quotient morphism gives a (necessarily unique) morphism of C^∞-rings $\pi : \mathfrak{C} \to \mathbb{R}$ with the property that $c \in \mathfrak{C}$ is invertible if and only if $\pi(c) \neq 0$. Equivalently, if such a morphism $\pi : \mathfrak{C} \to \mathbb{R}$ exists with this property, then $\mathfrak{C}$ is local with maximal ideal $\mathfrak{m}_{\mathfrak{C}} \cong \operatorname{Ker}\pi$. Write $\mathbf{C^\infty Rings_{lo}} \subset \mathbf{C^\infty Rings}$ for the full subcategory of local C^∞-rings.

Usually morphisms of local rings are required to send maximal ideals into maximal ideals. However, if $\phi : \mathfrak{C} \to \mathfrak{D}$ is any morphism of local C^∞-rings, we see that $\phi^{-1}(\mathfrak{m}_{\mathfrak{D}}) = \mathfrak{m}_{\mathfrak{C}}$ as the residue fields in both cases are $\mathbb{R}$, so there is no difference between local morphisms and morphisms for C^∞-rings.

Remark 2.20 We use the term 'local C^∞-ring' following Dubuc [23, Def. 4] and the second author [42]. They are known by different names in other references, such as *Archimedean local C^∞-rings* in [73, §3], *C^∞-local rings* in Dubuc [23, Def. 2.13], and *pointed local C^∞-rings* in [75, §I.3]. Moerdijk and Reyes [73, 74, 75] use 'local C^∞-ring' to mean a C^∞-ring which is a local $\mathbb{R}$-algebra, and require no restriction on its residue field.

The next proposition may be found in Moerdijk and Reyes [75, §I.3] and Dubuc [23, Prop. 5].

Proposition 2.21 *All finite colimits exist in* $\mathbf{C^\infty Rings_{lo}}$*, and agree with the corresponding colimits in* $\mathbf{C^\infty Rings}$.

In [25], the first author shows how to extend this theorem to small colimits, and how small limits also exist in $\mathbf{C^\infty Rings_{lo}}$ but usually do not agree with small limits in $\mathbf{C^\infty Rings}$. A key fact used in the proof is the existence of bump functions for $\mathbb{R}^n$.

Localizations of C^∞-rings were studied in [22, 23, 49, 73, 74, 75].

Definition 2.22 A *localization* $\mathfrak{C}(s^{-1} : s \in S) = \mathfrak{D}$ of a C^∞-ring $\mathfrak{C}$ at a subset $S \subset \mathfrak{C}$ is a C^∞-ring $\mathfrak{D}$ and a morphism $\pi : \mathfrak{C} \to \mathfrak{D}$ such that $\pi(s)$ is invertible in $\mathfrak{D}$ (as an $\mathbb{R}$-algebra) for all $s \in S$, which has the universal property that for any morphism of C^∞-rings $\phi : \mathfrak{C} \to \mathfrak{E}$ such that $\phi(s)$ is invertible in $\mathfrak{E}$ for all $s \in S$, there is a unique morphism $\psi : \mathfrak{D} \to \mathfrak{E}$ with $\phi = \psi \circ \pi$. We call $\pi : \mathfrak{C} \to \mathfrak{D}$ the *localization morphism* for $\mathfrak{D}$.

By adding an extra generator s^{-1} and extra relation $s \cdot s^{-1} - 1 = 0$ for each $s \in S$ to $\mathfrak{C}$, it can be shown that localizations $\mathfrak{C}(s^{-1} : s \in S)$ always exist and are unique up to unique isomorphism. When $S = \{c\}$ then $\mathfrak{C}(c^{-1}) \cong (\mathfrak{C} \otimes_\infty C^\infty(\mathbb{R}))/I$, where I is the ideal generated by $\iota_1(c) \cdot \iota_2(x) - 1$, and x is the generator of $C^\infty(\mathbb{R})$, and ι_1, ι_2 are the coproduct morphisms $\iota_1 : \mathfrak{C} \to \mathfrak{C} \otimes_\infty C^\infty(\mathbb{R})$ and $\iota_2 : C^\infty(\mathbb{R}) \to \mathfrak{C} \otimes_\infty C^\infty(\mathbb{R})$.

An example of this is that if $f \in C^\infty(\mathbb{R}^n)$ is a smooth function, and $U = f^{-1}(\mathbb{R}\setminus\{0\}) \subseteq \mathbb{R}^n$, then using partitions of unity one can show that $C^\infty(U) \cong C^\infty(\mathbb{R}^n)(f^{-1})$, as in [75, Prop. I.1.6].

Definition 2.23 A C^∞-ring morphism $x : \mathfrak{C} \to \mathbb{R}$, where $\mathbb{R}$ is regarded as a C^∞-ring as in Example 2.14, is called an $\mathbb{R}$-*point*. Note that a map $x : \mathfrak{C} \to \mathbb{R}$ is a morphism of C^∞-rings whenever it is a morphism of the underlying $\mathbb{R}$-algebras, as in [75, Prop. I.3.6]. We define $\mathfrak{C}_x$ as the localization $\mathfrak{C}_x = \mathfrak{C}(s^{-1} : s \in \mathfrak{C},\ x(s) \neq 0)$, and denote the projection morphism by $\pi_x : \mathfrak{C} \to \mathfrak{C}_x$. Importantly, [74, Lem. 1.1] shows $\mathfrak{C}_x$ is a local C^∞-ring.

We will use $\mathbb{R}$-points $x : \mathfrak{C} \to \mathbb{R}$ to define our spectrum functor in §2.5. We can describe $\mathfrak{C}_x$ explicitly as in [49, Prop. 2.14].

Proposition 2.24 *Let $x : \mathfrak{C} \to \mathbb{R}$ be an $\mathbb{R}$-point of a C^∞-ring $\mathfrak{C}$, and consider the projection morphism $\pi_x : \mathfrak{C} \to \mathfrak{C}_x$. Then $\mathfrak{C}_x \cong \mathfrak{C}/\operatorname{Ker}\pi_x$. This kernel is* $\operatorname{Ker}\pi_x = I$*, where*

$$I = \bigl\{ c \in \mathfrak{C} : \textit{there exists } d \in \mathfrak{C} \textit{ with } x(d) \neq 0 \textit{ in } \mathbb{R} \textit{ and } c \cdot d = 0 \textit{ in } \mathfrak{C} \bigr\}. \quad (2.4)$$

While this localization morphism $\pi_x : \mathfrak{C} \to \mathfrak{C}_x$ is surjective, general localizations of C^∞-rings need not have surjective localization morphisms. Here is an important example of localization of C^∞-rings.

Example 2.25 Let $C^\infty_p(\mathbb{R}^n)$ be the set of germs of smooth functions $c : \mathbb{R}^n \to \mathbb{R}$ at $p \in \mathbb{R}^n$ for $n \geqslant 0$ and $p \in \mathbb{R}^n$. We give $C^\infty_p(\mathbb{R}^n)$ a C^∞-ring structure by using (2.2) on germs of functions. There are several equivalent definitions.

(i) $C^\infty_p(\mathbb{R}^n)$ is the set of $\sim$-equivalence classes $[U, c]$ of pairs (U, c), where $p \in U \subseteq \mathbb{R}^n$ is open and $c : U \to \mathbb{R}$ is smooth, and $(U, c) \sim (U', c')$ if there exists open $p \in U'' \subseteq U \cap U'$ with $c|_{U''} = c'|_{U''}$.

(ii) $C^\infty_p(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^n)/I_p$, where $I_p \subset C^\infty(\mathbb{R}^n)$ is the ideal of functions vanishing near p.

(iii) $C^\infty_p(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^n)(f^{-1} : f \in C^\infty(\mathbb{R}^n),\ f(p) \neq 0)$.

Then $C^\infty_p(\mathbb{R}^n)$ is local, with maximal ideal $\mathfrak{m}_p = \bigl\{[U, c] \in C^\infty_p(\mathbb{R}^n) : c(x) = 0\bigr\}$.

Finally, we prove some facts about exponentials and logs in (local) C^∞-rings. These will be used in defining C^∞-rings with corners.

Proposition 2.26 **(a)** *Let $\mathfrak{C}$ be a C^∞-ring. Then the C^∞-operation $\Phi_{\exp} : \mathfrak{C} \to \mathfrak{C}$ induced by* $\exp : \mathbb{R} \to \mathbb{R}$ *is injective.*

(b) *Let $\mathfrak{C}$ be a local C^∞-ring, with morphism $\pi : \mathfrak{C} \to \mathbb{R}$. If $a \in \mathfrak{C}$ with $\pi(a) > 0$ then there exists $b \in \mathfrak{C}$ with $\Phi_{\exp}(b) = a$. This b is unique by* **(a)**.

Proof For (a), let $a \in \mathfrak{C}$ with $b = \Phi_{\exp}(a) \in \mathfrak{C}$. Then $\Phi_{\exp}(-a)$ is the inverse b^{-1} of b. The map $t \mapsto \exp(t) - \exp(-t)$ is a diffeomorphism $\mathbb{R} \to \mathbb{R}$. Let $e : \mathbb{R} \to \mathbb{R}$ be its inverse. Define smooth $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = e(x - y)$. Then $f(\exp t, \exp(-t)) = t$. Hence in the C^∞-ring $\mathfrak{C}$ we have

$$\Phi_f(b, b^{-1}) = \Phi_f(\Phi_{\exp}(a), \Phi_{\exp \circ -}(a)) = \Phi_{f\circ(\exp, \exp\circ -)}(a) = \Phi_{\mathrm{id}}(a) = a.$$

But b determines b^{-1} uniquely, so $\Phi_f(b, b^{-1}) = a$ implies that $b = \Phi_{\exp}(a)$ determines a uniquely, and $\Phi_{\exp} : \mathfrak{C} \to \mathfrak{C}$ is injective.

For (b), choose smooth $g, h : \mathbb{R} \to \mathbb{R}$ with $g(x) = \log x$ for $x \geqslant \frac{1}{2}\pi(a) > 0$, $h(\pi(a)) > 0$, and $h(x) = 0$ for $x \leqslant \frac{1}{2}\pi(a)$. Set $b = \Phi_g(a)$ and $c = \Phi_h(a)$. Then

$$c \cdot (\Phi_{\exp}(b) - a) = \Phi_h(a) \cdot (\Phi_{\exp\circ g}(a) - a) = \Phi_{h(x)\cdot(\exp\circ g(x) - x)}(a) = 0,$$

since $h(x) \cdot (x - \exp \circ g(x)) = 0$. Also $\pi(c) = h(\pi(a)) > 0$, so c is invertible. Thus $\Phi_{\exp}(b) = a$. □

2.3 Modules and cotangent modules of C^∞-rings

We discuss modules and cotangent modules for C^∞-rings, following [49, §5].

Definition 2.27 A *module* M over a C^∞-ring $\mathfrak{C}$ is a module over $\mathfrak{C}$ as a commutative $\mathbb{R}$-algebra, and morphisms of $\mathfrak{C}$-modules are the usual morphisms of $\mathbb{R}$-algebra modules. Denote $\mu_M : \mathfrak{C} \times M \to M$ the multiplication map, and write $\mu_M(c, m) = c \cdot m$ for $c \in \mathfrak{C}$ and $m \in M$. The category $\mathfrak{C}$-mod of $\mathfrak{C}$-modules is an abelian category.

If a $\mathfrak{C}$-module M fits into an exact sequence $\mathfrak{C} \otimes \mathbb{R}^n \to M \to 0$ in $\mathfrak{C}$-mod then it is *finitely generated*; if it further fits into an exact sequence $\mathfrak{C} \otimes \mathbb{R}^m \to \mathfrak{C} \otimes \mathbb{R}^n \to M \to 0$ it is *finitely presented*. This second condition is not automatic from the first as C^∞-rings are generally not noetherian.

For a morphism $\phi : \mathfrak{C} \to \mathfrak{D}$ of C^∞-rings and M in $\mathfrak{C}$-mod we have $\phi_*(M) = M \otimes_{\mathfrak{C}} \mathfrak{D}$ in $\mathfrak{D}$-mod, giving a functor $\phi_* : \mathfrak{C}\text{-mod} \to \mathfrak{D}\text{-mod}$. For N in $\mathfrak{D}$-mod there is a $\mathfrak{C}$-module $\phi^*(N) = N$ with $\mathfrak{C}$-action $\mu_{\phi^*(N)}(c, n) = \mu_N(\phi(c), n)$. This gives a functor $\phi^* : \mathfrak{D}\text{-mod} \to \mathfrak{C}\text{-mod}$.

Example 2.28 Let $\Gamma^\infty(E)$ be the collection of smooth sections e of a vector bundle $E \to X$ of a manifold X, so $\Gamma^\infty(E)$ is a vector space and a module over $C^\infty(X)$. If $\lambda : E \to F$ is a morphism of vector bundles over X, then there is a morphism of $C^\infty(X)$-modules $\lambda_* : \Gamma^\infty(E) \to \Gamma^\infty(F)$, where $\lambda_* : e \mapsto \lambda \circ e$.

For each smooth map of manifolds $f : X \to Y$ there is a morphism of

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C^∞-rings $f^* : C^\infty(Y) \to C^\infty(X)$. Each vector bundle $E \to Y$ gives a vector bundle $f^*(E) \to X$. Using $(f^*)_* : C^\infty(Y)\text{-mod} \to C^\infty(X)\text{-mod}$ from Definition 2.27, then $(f^*)_*\big(\Gamma^\infty(E)\big) = \Gamma^\infty(E) \otimes_{C^\infty(Y)} C^\infty(X)$ is isomorphic to $\Gamma^\infty\big(f^*(E)\big)$ in $C^\infty(X)$-mod.

The definition of $\mathfrak{C}$-module used only the commutative $\mathbb{R}$-algebra structure of $\mathfrak{C}$; however, the *cotangent module* $\Omega_{\mathfrak{C}}$ of $\mathfrak{C}$ uses the full C^∞-ring structure, making it smaller in some sense than the corresponding classical Algebraic Geometry version of *Kähler differentials* from [36, II 8].

Definition 2.29 Take a C^∞-ring $\mathfrak{C}$ and $M \in \mathfrak{C}$-mod, then a C^∞*-derivation* is a map $\mathrm{d} : \mathfrak{C} \to M$ that satisfies the following: for any smooth $f : \mathbb{R}^n \to \mathbb{R}$ and elements $c_1, \ldots, c_n \in \mathfrak{C}$, then

$$\mathrm{d}\Phi_f(c_1, \ldots, c_n) - \sum_{i=1}^{n} \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n) \cdot \mathrm{d}c_i = 0. \tag{2.5}$$

This implies that d is $\mathbb{R}$-linear and is a derivation of $\mathfrak{C}$ as a commutative $\mathbb{R}$-algebra, that is, $\mathrm{d}(c_1 c_2) = c_1 \cdot \mathrm{d}c_2 + c_2 \cdot \mathrm{d}c_1$ for all $c_1, c_2 \in \mathfrak{C}$.

The pair (M, d) is called a *cotangent module* for $\mathfrak{C}$ if it is universal in the sense that for any $M' \in \mathfrak{C}$-mod with C^∞-derivation $\mathrm{d}' : \mathfrak{C} \to M'$, there exists a unique morphism of $\mathfrak{C}$-modules $\lambda : M \to M'$ with $\mathrm{d}' = \lambda \circ \mathrm{d}$. Then a cotangent module is unique up to unique isomorphism. We can explicitly construct a cotangent module for $\mathfrak{C}$ by considering the free $\mathfrak{C}$-module over the symbols $\mathrm{d}c$ for all $c \in \mathfrak{C}$, and quotienting by all relations (2.5) for smooth $f : \mathbb{R}^n \to \mathbb{R}$ and elements $c_1, \ldots, c_n \in \mathfrak{C}$. We call this construction 'the' cotangent module of $\mathfrak{C}$, and write it as $\mathrm{d}_{\mathfrak{C}} : \mathfrak{C} \to \Omega_{\mathfrak{C}}$.

If we have a morphism of C^∞-rings $\mathfrak{C} \to \mathfrak{D}$ then $\Omega_{\mathfrak{D}} = \phi^*(\Omega_{\mathfrak{D}})$ can be considered as a $\mathfrak{C}$-module with C^∞-derivation $\mathrm{d}_{\mathfrak{D}} \circ \phi : \mathfrak{C} \to \Omega_{\mathfrak{D}}$. The universal property of $\Omega_{\mathfrak{C}}$ gives a unique morphism $\Omega_\phi : \Omega_{\mathfrak{C}} \to \Omega_{\mathfrak{D}}$ of $\mathfrak{C}$-modules such that $\mathrm{d}_{\mathfrak{D}} \circ \phi = \Omega_\phi \circ \mathrm{d}_{\mathfrak{C}}$. From this we have a morphism of $\mathfrak{D}$-modules $(\Omega_\phi)_* : \Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}} \mathfrak{D} \to \Omega_{\mathfrak{D}}$. If we have two morphisms of C^∞-rings $\phi : \mathfrak{C} \to \mathfrak{D}$, $\psi : \mathfrak{D} \to \mathfrak{E}$ then uniqueness implies that $\Omega_{\psi \circ \phi} = \Omega_\psi \circ \Omega_\phi : \Omega_{\mathfrak{C}} \to \Omega_{\mathfrak{E}}$.

Here is our motivating example.

Example 2.30 As in Example 2.28, if X is a manifold, its cotangent bundle T^*X is a vector bundle over X, and its global sections $\Gamma^\infty(T^*X)$ form a $C^\infty(X)$-module, with C^∞-derivation $\mathrm{d} : C^\infty(X) \to \Gamma^\infty(T^*X)$, $\mathrm{d} : c \mapsto \mathrm{d}c$ the usual exterior derivative, and equation (2.5) following from the chain rule.

One can show that $(\Gamma^\infty(T^*X), \mathrm{d})$ has the universal property in Definition 2.29, and so forms a cotangent module for $C^\infty(X)$. This is stated in [49, Ex. 5.4], and proved in greater generality in Theorem 7.7(a) below.

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If we have a smooth map of manifolds $f : X \to Y$, then $f^*(T^*Y)$, T^*X are vector bundles over X, and the derivative $\mathrm{d}f : f^*(T^*Y) \to T^*X$ is a vector bundle morphism. This induces a morphism of $C^\infty(X)$-modules $(\mathrm{d}f)_* : \Gamma^\infty(f^*(T^*Y)) \to \Gamma^\infty(T^*X)$, which is identified with $(\Omega_{f^*})_*$ from Definition 2.29 using that $\Gamma^\infty(f^*(T^*Y)) \cong \Gamma^\infty(T^*Y) \otimes_{C^\infty(Y)} C^\infty(X)$.

This example shows that Definition 2.29 abstracts the notion of sections of a cotangent bundle of a manifold to a concept that is well defined for any C^∞-ring. Here are further helpful examples that we later generalize for C^∞-rings with corners.

Example 2.31 **(a)** Let A be a set and $\mathfrak{F}^A$ the free C^∞-ring from Definition 2.16, with generators $x_a \in \mathfrak{F}^A$ for $a \in A$. Then there is a natural isomorphism

$$\Omega_{\mathfrak{F}^A} \cong \big\langle \mathrm{d}x_a : a \in A \big\rangle_{\mathbb{R}} \otimes_{\mathbb{R}} \mathfrak{F}^A.$$

(b) Suppose $\mathfrak{C}$ is defined by a coequalizer diagram (2.3) in $\mathbf{C^\infty Rings}$. Then writing $(x_a)_{a\in A}$, $(\tilde{x}_b)_{b\in B}$ for the generators of $\mathfrak{F}^A, \mathfrak{F}^B$, we have an exact sequence

$$\big\langle \mathrm{d}\tilde{x}_b : b \in B \big\rangle_{\mathbb{R}} \otimes_{\mathbb{R}} \mathfrak{C} \xrightarrow{\ \gamma\ } \big\langle \mathrm{d}x_a : a \in A \big\rangle_{\mathbb{R}} \otimes_{\mathbb{R}} \mathfrak{C} \xrightarrow{\ \delta\ } \Omega_{\mathfrak{C}} \longrightarrow 0$$

in $\mathfrak{C}$-mod, where if α, β in (2.3) map $\alpha : \tilde{x}_b \mapsto f_b\big((x_a)_{a\in A}\big)$, $\beta : \tilde{x}_b \mapsto g_b\big((x_a)_{a\in A}\big)$, for f_b, g_b depending only on finitely many x_a, then γ, δ are given by

$$\gamma(\mathrm{d}\tilde{x}_b) = \sum_{a\in A} \phi\Big(\frac{\partial f_b}{\partial x_a}\big((x_a)_{a\in A}\big) - \frac{\partial g_b}{\partial x_a}\big((x_a)_{a\in A}\big)\Big)\mathrm{d}x_a, \ \delta(\mathrm{d}x_a) = \mathrm{d}_{\mathfrak{C}} \circ \phi(x_a).$$

Hence as in [49, Prop. 5.6], if $\mathfrak{C}$ is finitely generated (or finitely presented) in the sense of Proposition 2.17, then $\Omega_{\mathfrak{C}}$ is finitely generated (or finitely presented).

Cotangent modules behave well under localization, as in the following proposition from [49, Prop. 5.7]. This proposition is used to prove the theorem that follows it, and we use it to generalize these results to C^∞-rings with corners.

Proposition 2.32 *Let $\mathfrak{C}$ be a C^∞-ring, $S \subseteq \mathfrak{C}$, and let $\mathfrak{D} = \mathfrak{C}(s^{-1} : s \in S)$ be the localization of $\mathfrak{C}$ at S with projection $\pi : \mathfrak{C} \to \mathfrak{D}$, as in Definition* 2.22. *Then $(\Omega_\pi)_* : \Omega_{\mathfrak{C}} \otimes_{\mathfrak{C}} \mathfrak{D} \to \Omega_{\mathfrak{D}}$ is an isomorphism of $\mathfrak{D}$-modules.*

Finally, here is [49, Th. 5.8] which shows how pushouts of C^∞-rings give exact sequences of cotangent modules.

Theorem 2.33 *Suppose we are given a pushout diagram of C^∞-rings:*

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\beta} & \mathfrak{E} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\delta} \\ \mathfrak{D} & \xrightarrow{\gamma} & \mathfrak{F}, \end{array}$$

so that $\mathfrak{F} = \mathfrak{D} \amalg_{\mathfrak{C}} \mathfrak{E}$. Then the following sequence of $\mathfrak{F}$-modules is exact:

$$\Omega_{\mathfrak{C}} \otimes_{\mathfrak{C},\gamma\circ\alpha} \mathfrak{F} \xrightarrow{(\Omega_\alpha)_*\oplus-(\Omega_\beta)_*} \begin{array}{c}\Omega_{\mathfrak{D}} \otimes_{\mathfrak{D},\gamma} \mathfrak{F} \\ \oplus\, \Omega_{\mathfrak{E}} \otimes_{\mathfrak{E},\delta} \mathfrak{F}\end{array} \xrightarrow{(\Omega_\gamma)_*\oplus(\Omega_\delta)_*} \Omega_{\mathfrak{F}} \longrightarrow 0. \quad (2.6)$$

Here $(\Omega_\alpha)_* : \Omega_{\mathfrak{C}} \otimes_{\mathfrak{C},\gamma\circ\alpha} \mathfrak{F} \to \Omega_{\mathfrak{D}} \otimes_{\mathfrak{D},\gamma} \mathfrak{F}$ is induced by $\Omega_\alpha : \Omega_{\mathfrak{C}} \to \Omega_{\mathfrak{D}}$, and so on. Note the sign of $-(\Omega_\beta)_*$ in (2.6).

2.4 Sheaves

In this section we explain presheaves and sheaves with values in a (nice) category $\mathcal{A}$, following Godement [33] and MacLane and Moerdijk [65]. Throughout we suppose $\mathcal{A}$ is *complete*, that is, all small limits exist in $\mathcal{A}$, and *cocomplete*, that is, all small colimits exist in $\mathcal{A}$. The categories of sets, abelian groups, rings, C^∞-rings, monoids, etc., all satisfy this, as will (interior) C^∞-rings with corners. Sometimes it is helpful to suppose that objects of $\mathcal{A}$ are sets with extra structure, so there is a faithful functor $\mathcal{A} \to \mathbf{Sets}$ taking each object to its underlying set. We use presheaves and sheaves and the facts that follow to define and study local C^∞-ringed spaces and C^∞-schemes.

Definition 2.34 A *presheaf* $\mathcal{E}$ on a topological space X valued in $\mathcal{A}$ gives an object $\mathcal{E}(U) \in \mathcal{A}$ for every open set $U \subseteq X$, and a morphism $\rho_{UV} : \mathcal{E}(U) \to \mathcal{E}(V)$ in $\mathcal{A}$ called the *restriction map* for every inclusion $V \subseteq U \subseteq X$ of open sets, satisfying the conditions that

(i) $\rho_{UU} = \mathrm{id}_{\mathcal{E}(U)} : \mathcal{E}(U) \to \mathcal{E}(U)$ for all open $U \subseteq X$; and
(ii) $\rho_{UW} = \rho_{VW} \circ \rho_{UV} : \mathcal{E}(U) \to \mathcal{E}(W)$ for all open $W \subseteq V \subseteq U \subseteq X$.

A presheaf $\mathcal{E}$ is called a *sheaf* if for all open covers $\{U_i\}_{i\in I}$ of U, then

$$\mathcal{E}(U) \to \prod_{i\in I} \mathcal{E}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{E}(U_i \cap U_j)$$

forms an equalizer diagram in $\mathcal{A}$. This implies the following.

(iii) $\mathcal{E}(\emptyset) = 0$, where 0 is the final object in $\mathcal{A}$.

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If there is a faithful functor $F : \mathcal{A} \to \mathbf{Sets}$ taking an object of $\mathcal{A}$ to its underlying set that preserves limits, then a presheaf $\mathcal{E}$ valued in $\mathcal{A}$ on X is a sheaf if it equivalently satisfies the following.

(iv) (Uniqueness.) If $U \subseteq X$ is open, $\{V_i : i \in I\}$ is an open cover of U, and $s, t \in F(\mathcal{E}(U))$ with $F(\rho_{UV_i})(s) = F(\rho_{UV_i})(t)$ in $F(\mathcal{E}(V_i))$ for all $i \in I$, then $s = t$ in $F(\mathcal{E}(U))$.

(v) (Gluing.) If $U \subseteq X$ is open, $\{V_i : i \in I\}$ is an open cover of U, and we are given elements $s_i \in F(\mathcal{E}(V_i))$ for all $i \in I$ such that $F(\rho_{V_i(V_i \cap V_j)})(s_i) = F(\rho_{V_j(V_i \cap V_j)})(s_j)$ in $F(\mathcal{E}(V_i \cap V_j))$ for all $i, j \in I$, then there exists $s \in F(\mathcal{E}(U))$ with $F(\rho_{UV_i})(s) = s_i$ for all $i \in I$.

If $s \in F(\mathcal{E}(U))$ and $V \subseteq U$ is open we write $s|_V = F(\rho_{UV})(s)$.

If $\mathcal{E}, \mathcal{F}$ are presheaves or sheaves valued in $\mathcal{A}$ on X, then a *morphism* $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism $\phi(U) : \mathcal{E}(U) \to \mathcal{F}(U)$ in $\mathcal{A}$ for all open $U \subseteq X$ such that the following diagram commutes for all open $V \subseteq U \subseteq X$

$$\begin{array}{ccc} \mathcal{E}(U) & \xrightarrow{\phi(U)} & \mathcal{F}(U) \\ \downarrow{\scriptstyle \rho_{UV}} & & {\scriptstyle \rho'_{UV}}\downarrow \\ \mathcal{E}(V) & \xrightarrow{\phi(V)} & \mathcal{F}(V), \end{array}$$

where ρ_{UV} is the restriction map for $\mathcal{E}$, and ρ'_{UV} the restriction map for $\mathcal{F}$. We write $\mathrm{PreSh}(X, \mathcal{A})$ and $\mathrm{Sh}(X, \mathcal{A})$ for the categories of presheaves and sheaves on a topological space X valued in $\mathcal{A}$.

Definition 2.35 For $\mathcal{E}$ a presheaf valued in $\mathcal{A}$ on a topological space X, then we can define the *stalk* $\mathcal{E}_x \in \mathcal{A}$ at a point $x \in X$ to be the direct limit of the $\mathcal{E}(U)$ in $\mathcal{A}$ for all $U \subseteq X$ with $x \in U$, using the restriction maps ρ_{UV}.

If there is a faithful functor $F : \mathcal{A} \to \mathbf{Sets}$ taking an object of $\mathcal{A}$ to its underlying set that preserves colimits, then explicitly it can be written as a set of equivalence classes of sections $s \in F(\mathcal{E}(U))$ for any open U which contains x, where the equivalence relation is such that $s_1 \sim s_2$ for $s_1 \in F(\mathcal{E}(U))$ and $s_2 \in F(\mathcal{E}(V))$ with $x \in U, V$ if there is an open set $W \subset V \cap U$ with $x \in W$ and $s_1|_W = s_2|_W$ in $F(\mathcal{E}(W))$.

The stalk is an object of $\mathcal{A}$, and the restriction morphisms give rise to morphisms $\rho_{U,x} : \mathcal{E}(U) \to \mathcal{E}_x$. A morphism of presheaves $\phi : \mathcal{E} \to \mathcal{F}$ induces morphisms $\phi_x : \mathcal{E}_x \to \mathcal{F}_x$ for all $x \in X$. If $\mathcal{E}, \mathcal{F}$ are sheaves then ϕ is an isomorphism if and only if ϕ_x is an isomorphism for all $x \in X$.

Definition 2.36 There is a *sheafification* functor $\mathrm{PreSh}(X, \mathcal{A}) \to \mathrm{Sh}(X, \mathcal{A})$, which is left adjoint to the inclusion $\mathrm{Sh}(X, \mathcal{A}) \hookrightarrow \mathrm{PreSh}(X, \mathcal{A})$. We write $\hat{\mathcal{E}}$ for the sheafification of a presheaf $\mathcal{E}$. The adjoint property gives a morphism

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$\pi : \mathcal{E} \to \hat{\mathcal{E}}$ and a universal property: whenever we have a morphism $\phi : \mathcal{E} \to \mathcal{F}$ of presheaves on X and $\mathcal{F}$ is a sheaf, then there is a unique morphism $\hat{\phi} : \hat{\mathcal{E}} \to \mathcal{F}$ with $\phi = \hat{\phi} \circ \pi$. Thus sheafification is unique up to canonical isomorphism.

Sheafifications always exist for our categories $\mathcal{A}$, and there are isomorphisms of stalks $\mathcal{E}_x \cong \hat{\mathcal{E}}_x$ for all $x \in X$. If there is a faithful functor $F : \mathcal{A} \to \mathbf{Sets}$ taking an object of $\mathcal{A}$ to its underlying set that preserves colimits and limits, it can be constructed (as in [36, Prop. II.1.2]) by defining $\hat{\mathcal{E}}(U)$ as the subset of all functions $t : U \to \amalg_{x \in U} \mathcal{E}_x$ such that for all $x \in U$, then $t(x) = F(\rho_{V,x})(s) \in \mathcal{E}_x$ for some $s \in F(\mathcal{E}(V))$ for open $V \subset U$, $x \in V$.

If $f : X \to Y$ is a continuous map of topological spaces, we can consider *pushforwards* and *pullbacks* of sheaves by f. We will use both of these definitions when defining C^∞-schemes (with corners).

Definition 2.37 If $f : X \to Y$ is a continuous map of topological spaces, and $\mathcal{E}$ is a sheaf valued in $\mathcal{A}$ on X, then the *direct image* (or *pushforward*) sheaf $f_*(\mathcal{E})$ on Y is defined by $\big(f_*(\mathcal{E})\big)(U) = \mathcal{E}\big(f^{-1}(U)\big)$ for all open $U \subseteq V$. Here, we have restriction maps $\rho'_{UV} = \rho_{f^{-1}(U) f^{-1}(V)} : \big(f_*(\mathcal{E})\big)(U) \to \big(f_*(\mathcal{E})\big)(V)$ for all open $V \subseteq U \subseteq Y$ so that $f_*(\mathcal{E})$ is a sheaf valued in $\mathcal{A}$ on Y.

For a morphism $\phi : \mathcal{E} \to \mathcal{F}$ in $\mathrm{Sh}(X, \mathcal{A})$ we can define $f_*(\phi) : f_*(\mathcal{E}) \to f_*(\mathcal{F})$ by $\big(f_*(\phi)\big)(U) = \phi\big(f^{-1}(U)\big)$ for all open $U \subseteq Y$. This gives a morphism $f_*(\phi)$ in $\mathrm{Sh}(Y, \mathcal{A})$, and a functor $f_* : \mathrm{Sh}(X, \mathcal{A}) \to \mathrm{Sh}(Y, \mathcal{A})$. For two continuous maps of topological spaces, $f : X \to Y$, $g : Y \to Z$, then $(g \circ f)_* = g_* \circ f_*$.

Definition 2.38 For a continuous map $f : X \to Y$ and a sheaf $\mathcal{E}$ valued in $\mathcal{A}$ on Y, we define the *pullback* (*inverse image*) of $\mathcal{E}$ under f to be the sheafification of the presheaf $U \mapsto \lim_{A \supseteq f(U)} \mathcal{E}(A)$ for open $U \subseteq X$, where the direct limit is taken over all open $A \subseteq Y$ containing $f(U)$, using the restriction maps ρ_{AB} in $\mathcal{E}$. We write this sheaf as $f^{-1}(\mathcal{E})$. If $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism in $\mathrm{Sh}(Y, \mathcal{A})$, there is a *pullback morphism* $f^{-1}(\phi) : f^{-1}(\mathcal{E}) \to f^{-1}(\mathcal{F})$.

Remark 2.39 For a continuous map $f : X \to Y$ of topological spaces we have functors $f_* : \mathrm{Sh}(X, \mathcal{A}) \to \mathrm{Sh}(Y, \mathcal{A})$, and $f^{-1} : \mathrm{Sh}(Y, \mathcal{A}) \to \mathrm{Sh}(X, \mathcal{A})$. Hartshorne [36, Ex. II.1.18] gives a natural bijection

$$\mathrm{Hom}_X\big(f^{-1}(\mathcal{E}), \mathcal{F}\big) \cong \mathrm{Hom}_Y\big(\mathcal{E}, f_*(\mathcal{F})\big) \tag{2.7}$$

for all $\mathcal{E} \in \mathrm{Sh}(Y, \mathcal{A})$ and $\mathcal{F} \in \mathrm{Sh}(X, \mathcal{A})$, so that f_* is right adjoint to f^{-1}, as in §2.1.3. This will be important in several proofs later.

2.5 C^∞-schemes

We recall the definition of (local) C^∞-ringed spaces, following [49]. These allow us to define a spectrum functor and C^∞-schemes.

Definition 2.40 A *C^∞-ringed space* $\underline{X} = (X, \mathcal{O}_X)$ is a topological space X with a sheaf $\mathcal{O}_X$ of C^∞-rings on X.

A *morphism* $\underline{f} = (f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of C^∞ ringed spaces consists of a continuous map $f : X \to Y$ and a morphism $f^\sharp : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ of sheaves of C^∞-rings on X, for $f^{-1}(\mathcal{O}_Y)$ the inverse image sheaf as in Definition 2.38. From (2.7), we know f_* is right adjoint to f^{-1}, so there is a natural bijection

$$\mathrm{Hom}_X\big(f^{-1}(\mathcal{O}_Y), \mathcal{O}_X\big) \cong \mathrm{Hom}_Y\big(\mathcal{O}_Y, f_*(\mathcal{O}_X)\big). \tag{2.8}$$

We will write $f_\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ for the morphism of sheaves of C^∞-rings on Y corresponding to the morphism $f^\sharp$ under (2.8), so that

$$f^\sharp : f^{-1}(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X \quad \leftrightsquigarrow \quad f_\sharp : \mathcal{O}_Y \longrightarrow f_*(\mathcal{O}_X). \tag{2.9}$$

Given two C^∞-ringed space morphisms $\underline{f} : \underline{X} \to \underline{Y}$ and $\underline{g} : \underline{Y} \to \underline{Z}$ we can compose them to form

$$\underline{g} \circ \underline{f} = \big(g \circ f, (g \circ f)^\sharp\big) = \big(g \circ f, f^\sharp \circ f^{-1}(g^\sharp)\big).$$

If we consider $f_\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$, then the composition is

$$(g \circ f)_\sharp = g_*(f_\sharp) \circ g_\sharp : \mathcal{O}_Z \longrightarrow (g \circ f)_*(\mathcal{O}_X) = g_* \circ f_*(\mathcal{O}_X).$$

We call $\underline{X} = (X, \mathcal{O}_X)$ a *local C^∞-ringed space* if it is C^∞-ringed space for which the stalks $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ at x are local C^∞-rings for all $x \in X$. As in Definition 2.19, since morphisms of local C^∞-rings are automatically local morphisms, morphisms of local C^∞-ringed spaces $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ are just morphisms of C^∞-ringed spaces without any additional locality condition. Local C^∞-ringed spaces are called *Archimedean C^∞-spaces* in Moerdijk, van Quê, and Reyes [73, §3].

We will follow the notation of [49] and write $\mathbf{C^\infty RS}$ for the category of C^∞-ringed spaces, and $\mathbf{LC^\infty RS}$ for the full subcategory of local C^∞-ringed spaces. We write underlined upper case letters such as $\underline{X}, \underline{Y}, \underline{Z}, \ldots$ to represent C^∞-ringed spaces $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y), (Z, \mathcal{O}_Z), \ldots$, and underlined lower case letters $\underline{f}, \underline{g}, \ldots$ to represent morphisms of C^∞-ringed spaces $(f, f^\sharp), (g, g^\sharp), \ldots$. When we write '$x \in \underline{X}$' we mean that $\underline{X} = (X, \mathcal{O}_X)$ and $x \in X$. If we write '$\underline{U}$ *is open in* $\underline{X}$' we will mean that $\underline{U} = (U, \mathcal{O}_U)$ and $\underline{X} = (X, \mathcal{O}_X)$ with $U \subseteq X$ an open set and $\mathcal{O}_U = \mathcal{O}_X|_U$.

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Here is our motivating example.

Example 2.41 For a manifold X, we have a C^∞-ringed space $\underline{X} = (X, \mathcal{O}_X)$ with topological space X and its sheaf of smooth functions $\mathcal{O}_X(U) = C^\infty(U)$ for each open subset $U \subseteq X$, with $C^\infty(U)$ defined in Example 2.13. If $V \subseteq U \subseteq X$ then the restriction morphisms $\rho_{UV} : C^\infty(U) \to C^\infty(V)$ are the usual restriction of a function to an open subset $\rho_{UV} : c \mapsto c|_V$.

As the stalks $\mathcal{O}_{X,x}$ at $x \in X$ are local C^∞-rings, isomorphic to the ring of germs as in Example 2.25, then using partitions of unity we can show that $\underline{X}$ is a local C^∞-ringed space.

For a smooth map of manifolds $f : X \to Y$ with corresponding local C^∞-ringed spaces $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ as previously we define $f_\sharp(U) : \mathcal{O}_Y(U) = C^\infty(U) \to \mathcal{O}_X(f^{-1}(U)) = C^\infty(f^{-1}(U))$ for each open $U \subseteq Y$ by $f_\sharp(U) : c \mapsto c \circ f$ for all $c \in C^\infty(U)$. This gives a morphism $f_\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ of sheaves of C^∞-rings on Y. Then $\underline{f} = (f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of (local) C^∞-ringed spaces with $f^\sharp : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ corresponding to $f_\sharp$ under (2.9).

To define a spectrum functor taking a C^∞-ring to an element of $\mathbf{LC^\infty RS}$ we require the following definition.

Definition 2.42 Let $\mathfrak{C}$ be a C^∞-ring, and write $X_\mathfrak{C}$ for the set of all $\mathbb{R}$-points x of $\mathfrak{C}$, as in Definition 2.22. Write $\mathcal{T}_\mathfrak{C}$ for the topology on $X_\mathfrak{C}$ that has basis of open sets $U_c = \{x \in X_\mathfrak{C} : x(c) \neq 0\}$ for all $c \in \mathfrak{C}$. For each $c \in \mathfrak{C}$ define a map $c_* : X_\mathfrak{C} \to \mathbb{R}$ such that $c_* : x \mapsto x(c)$.

For a morphism $\phi : \mathfrak{C} \to \mathfrak{D}$ of C^∞-rings, we can define $f_\phi : X_\mathfrak{D} \to X_\mathfrak{C}$ by $f_\phi(x) = x \circ \phi$, which is continuous.

From [49, Lem. 4.15], this definition implies that $\mathcal{T}_\mathfrak{C}$ is the weakest topology on $X_\mathfrak{C}$ such that the $c_* : X_\mathfrak{C} \to \mathbb{R}$ are continuous for all $c \in \mathfrak{C}$. Also $(X_\mathfrak{C}, \mathcal{T}_\mathfrak{C})$ is a regular, Hausdorff topological space. We now define the spectrum functor.

Definition 2.43 For a C^∞-ring $\mathfrak{C}$, we will define the *spectrum* of $\mathfrak{C}$, written $\operatorname{Spec}\mathfrak{C}$. Here, $\operatorname{Spec}\mathfrak{C}$ is a local C^∞-ringed space $(X, \mathcal{O}_X)$, with X the topological space $X_\mathfrak{C}$ from Definition 2.42. If $U \subseteq X$ is open then $\mathcal{O}_X(U)$ is the set of functions $s : U \to \coprod_{x \in U} \mathfrak{C}_x$, where we write s_x for the image of x under s, such that around each point $x \in U$ there is an open subset $x \in W \subseteq U$ and element $c \in \mathfrak{C}$ with $s_y = \pi_y(c) \in \mathfrak{C}_y$ for all $y \in W$. This is a C^∞-ring with the operations Φ_f on $\mathcal{O}_X(U)$ defined using the operations Φ_f on $\mathfrak{C}_x$ for $x \in U$.

For $s \in \mathcal{O}_X(U)$, the restriction map of functions $s \mapsto s|_V$ for open $V \subseteq U \subseteq X$ is a morphism of C^∞-rings, giving the restriction map $\rho_{UV} : \mathcal{O}_X(U)$

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$\to \mathcal{O}_X(V)$. The stalk $\mathcal{O}_{X,x}$ at $x \in X$ is isomorphic to $\mathfrak{C}_x$, which is a local C^∞-ring. Hence $(X, \mathcal{O}_X)$ is a local C^∞-ringed space.

For a morphism $\phi : \mathfrak{C} \to \mathfrak{D}$ of C^∞-rings, we have an induced morphism of local C^∞-rings, $\phi_x : \mathfrak{C}_{f_\phi(x)} \to \mathfrak{D}_x$. If we let $(X, \mathcal{O}_X) = \operatorname{Spec} \mathfrak{C}$, $(Y, \mathcal{O}_Y) = \operatorname{Spec} \mathfrak{D}$, then for open $U \subseteq X$ define $(f_\phi)_\sharp(U) : \mathcal{O}_X(U) \to \mathcal{O}_Y(f_\phi^{-1}(U))$ by $(f_\phi)_\sharp(U)s : x \mapsto \phi_x(s_{f_\phi(x)})$. This gives a morphism $(f_\phi)_\sharp : \mathcal{O}_X \to (f_\phi)_*(\mathcal{O}_Y)$ of sheaves of C^∞-rings on X. Then $\underline{f}_\phi = (f_\phi, f_\phi^\sharp) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism of local C^∞-ringed spaces, where $f_\phi^\sharp$ corresponds to $(f_\phi)_\sharp$ under (2.9). Then Spec is a functor $\mathbf{C^\infty Rings}^{\rm op} \to \mathbf{LC^\infty RS}$, called the *spectrum functor*, where $\operatorname{Spec} \phi : \operatorname{Spec} \mathfrak{D} \to \operatorname{Spec} \mathfrak{C}$ is defined by $\operatorname{Spec} \phi = \underline{f}_\phi$.

This definition of spectrum functor is different to the classical Algebraic Geometry version [36, §II.2], as the topological space corresponds only to maximal ideals of the C^∞-ring, instead of also including all prime ideals. In this sense, it is coarser; however, this corresponds to the topology of a manifold as in the following example.

Example 2.44 For a manifold X then $\operatorname{Spec} C^\infty(X)$ is isomorphic to the local C^∞-ringed space $\underline{X}$ constructed in Example 2.41.

Here is [49, Lem 4.28] which shows how the spectrum functor behaves with respect to localizations and open sets. A proof is contained in [25, Lem 2.4.6], which relies on the existence of bump functions for $\mathbb{R}^n$.

Lemma 2.45 *Let $\mathfrak{C}$ be a C^∞-ring, set $\underline{X} = \operatorname{Spec} \mathfrak{C} = (X, \mathcal{O}_X)$, and let $c \in \mathfrak{C}$. If we write $U_c = \{x \in X : x(c) \neq 0\}$ as in Definition 2.42, then $U_c \subseteq X$ is open and $\underline{U}_c = (U_c, \mathcal{O}_X|_{U_c}) \cong \operatorname{Spec} \mathfrak{C}(c^{-1})$.*

We now define the global sections functor and describe its relationship to the spectrum functor.

Definition 2.46 The *global sections functor* $\Gamma : \mathbf{LC^\infty RS} \to \mathbf{C^\infty Rings}^{\rm op}$ takes $(X, \mathcal{O}_X)$ to $\mathcal{O}_X(X)$ and morphisms $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ to $\Gamma : (f, f^\sharp) \mapsto f_\sharp(Y)$, for $f_\sharp$ relating $f^\sharp$ as in (2.9).

For each C^∞-ring $\mathfrak{C}$ we can define a morphism $\Xi_\mathfrak{C} : \mathfrak{C} \to \Gamma \circ \operatorname{Spec} \mathfrak{C}$. Here, for $c \in \mathfrak{C}$ then $\Xi_\mathfrak{C}(c) : X_\mathfrak{C} \to \coprod_{x \in X_\mathfrak{C}} \mathfrak{C}_x$ is defined by $\Xi_\mathfrak{C}(c)_x = \pi_x(c) \in \mathfrak{C}_x$, so $\Xi_\mathfrak{C}(c) \in \mathcal{O}_{X_\mathfrak{C}}(X_\mathfrak{C}) = \Gamma \circ \operatorname{Spec} \mathfrak{C}$. This $\Xi_\mathfrak{C}$ is a C^∞-ring morphism as it is built from C^∞-ring morphisms $\pi_x : \mathfrak{C} \to \mathfrak{C}_x$, and the C^∞-operations on $\mathcal{O}_{X_\mathfrak{C}}(X_\mathfrak{C})$ are defined pointwise in the $\mathfrak{C}_x$. This defines a natural transformation $\Xi : \mathrm{Id}_{\mathbf{C^\infty Rings}} \Rightarrow \Gamma \circ \operatorname{Spec}$ of functors $\mathbf{C^\infty Rings} \to \mathbf{C^\infty Rings}$.

Theorem 2.47 *The functor* $\mathrm{Spec} : \mathbf{C^\infty Rings}^{\mathrm{op}} \to \mathbf{LC^\infty RS}$ *is* ***right adjoint*** *to* $\Gamma : \mathbf{LC^\infty RS} \to \mathbf{C^\infty Rings}^{\mathrm{op}}$*, and* Ξ *is the unit of the adjunction. This implies that* Spec *preserves limits as in §2.1.3. Hence if we have* C^∞*-ring morphisms* $\phi : \mathfrak{F} \to \mathfrak{D}$*,* $\psi : \mathfrak{F} \to \mathfrak{E}$ *in* $\mathbf{C^\infty Rings}$ *then their pushout* $\mathfrak{C} = \mathfrak{D} \amalg_{\mathfrak{F}} \mathfrak{E}$ *has image that is isomorphic to the fibre product* $\mathrm{Spec}\,\mathfrak{C} \cong \mathrm{Spec}\,\mathfrak{D} \times_{\mathrm{Spec}\,\mathfrak{F}} \mathrm{Spec}\,\mathfrak{E}$.

We extend this theorem to C^∞-schemes with corners in §5.3.

Remark 2.48 Our definition of spectrum functor follows [49] and Dubuc [23], and is called the *Archimedean spectrum* in Moerdijk et al. [73, §3]. They also show it is a right adjoint to the global sections functor as above.

Definition 2.49 Objects $\underline{X} \in \mathbf{LC^\infty RS}$ that are isomorphic to $\mathrm{Spec}\,\mathfrak{C}$ for some $\mathfrak{C} \in \mathbf{C^\infty Rings}$ are called *affine* C^∞*-schemes*. Elements $\underline{X} \in \mathbf{LC^\infty RS}$ that are locally isomorphic to $\mathrm{Spec}\,\mathfrak{C}$ for some $\mathfrak{C} \in \mathbf{C^\infty Rings}$ (depending upon the open sets) are called C^∞*-schemes*.

We define $\mathbf{C^\infty Sch}$ and $\mathbf{AC^\infty Sch}$ to be the full subcategories of C^∞-schemes and affine C^∞-schemes in $\mathbf{LC^\infty RS}$ respectively.

Remark 2.50 **(a)** Unlike ordinary Algebraic Geometry, affine C^∞-schemes are very general objects. All manifolds are affine, and all their fibre products are affine. But not all manifolds with corners are affine C^∞-schemes with corners.

(b) (Alternatives to C^∞-schemes.) We briefly review other generalizations of manifolds similar to C^∞-schemes. Such generalizations usually fall into a 'maps out' (based on maps $X \to \mathbb{R}$) or a 'maps in' (based on maps $\mathbb{R}^n \to X$) approach. C^∞-algebraic geometry uses 'maps out', as do the C^∞*-differentiable spaces* of Navarro González and Sancho de Salas [77], which form a subcategory of C^∞-schemes. Sikorski [84], Spallek [86], Buchner et al. [9] and González and Sancho de Salas [77] describe other 'maps out' approaches.

'Maps in' approaches include the *diffeological spaces* of Souriau [85] and Iglesias-Zemmour [39], and the various *Chen spaces* from Chen [15, 16, 17, 18]. Usually 'maps out' approaches deal well with finite limits, and 'maps in' approaches work well for infinite-dimensional spaces and quotient spaces.

2.6 Complete C^∞-rings

In ordinary Algebraic Geometry, if A is a commutative ring then $\Gamma \circ \mathrm{Spec}\, A \cong A$, and $\mathrm{Spec} : \mathbf{Rings}^{\mathrm{op}} \to \mathbf{ASch}$ is an equivalence of categories, with inverse Γ. For C^∞-rings $\mathfrak{C}$, in general $\Gamma \circ \mathrm{Spec}\,\mathfrak{C} \not\cong \mathfrak{C}$, and $\mathrm{Spec} : \mathbf{C^\infty Rings}^{\mathrm{op}} \to$

$\mathbf{AC^\infty Sch}$ is neither full nor faithful. But as in [49, Prop. 4.34], we have the following.

Proposition 2.51 *For each C^∞-ring $\mathfrak{C}$, $\operatorname{Spec}\Xi_{\mathfrak{C}} : \operatorname{Spec}\circ\Gamma\circ\operatorname{Spec}\mathfrak{C} \to \operatorname{Spec}\mathfrak{C}$ is an isomorphism in $\mathbf{LC^\infty RS}$.*

This motivates the following definition [49, Def. 4.35].

Definition 2.52 A C^∞-ring $\mathfrak{C}$ is called *complete* if $\Xi_{\mathfrak{C}} : \mathfrak{C} \to \Gamma\circ\operatorname{Spec}\mathfrak{C}$ is an isomorphism. We define $\mathbf{C^\infty Rings_{co}}$ to be the full subcategory in $\mathbf{C^\infty Rings}$ of complete C^∞-rings. By Proposition 2.51 we see that $\mathbf{C^\infty Rings_{co}}$ is equivalent to the image of the functor $\Gamma\circ\operatorname{Spec} : \mathbf{C^\infty Rings} \to \mathbf{C^\infty Rings}$, which gives a left adjoint to the inclusion of $\mathbf{C^\infty Rings_{co}}$ into $\mathbf{C^\infty Rings}$. Write this left adjoint as the functor $\Pi^{\mathrm{co}}_{\mathrm{all}} = \Gamma\circ\operatorname{Spec} : \mathbf{C^\infty Rings} \to \mathbf{C^\infty Rings_{co}}$.

An example of a non-complete C^∞-ring is the quotient $\mathfrak{C} = C^\infty(\mathbb{R}^n)/I_{\mathrm{cs}}$ of $C^\infty(\mathbb{R}^n)$ for $n > 0$ by the ideal I_{cs} of compactly supported functions, and $\Pi^{\mathrm{co}}_{\mathrm{all}}(\mathfrak{C}) = 0 \not\cong \mathfrak{C}$. The next theorem comes from [49, Prop. 4.11 and Th. 4.25].

Theorem 2.53 **(a)** $\operatorname{Spec}|_{(\mathbf{C^\infty Rings_{co}})^{\mathrm{op}}} : (\mathbf{C^\infty Rings_{co}})^{\mathrm{op}} \to \mathbf{LC^\infty RS}$ *is full and faithful, and an equivalence* $(\mathbf{C^\infty Rings_{co}})^{\mathrm{op}} \to \mathbf{AC^\infty Sch}$.

(b) *Let $\underline{X}$ be an affine C^∞-scheme. Then $\underline{X} \cong \operatorname{Spec}\mathcal{O}_X(X)$, where $\mathcal{O}_X(X)$ is a complete C^∞-ring.*

(c) *The functor $\Pi^{\mathrm{co}}_{\mathrm{all}} : \mathbf{C^\infty Rings} \to \mathbf{C^\infty Rings_{co}}$ is left adjoint to the inclusion functor $\mathrm{inc} : \mathbf{C^\infty Rings_{co}} \hookrightarrow \mathbf{C^\infty Rings}$. That is, $\Pi^{\mathrm{co}}_{\mathrm{all}}$ is a* ***reflection functor****.*

(d) *All small colimits exist in $\mathbf{C^\infty Rings_{co}}$, although they may not coincide with the corresponding small colimits in $\mathbf{C^\infty Rings}$.*

(e) $\operatorname{Spec}|_{(\mathbf{C^\infty Rings_{co}})^{\mathrm{op}}} = \operatorname{Spec}\circ\mathrm{inc} : (\mathbf{C^\infty Rings_{co}})^{\mathrm{op}} \to \mathbf{LC^\infty RS}$ *is right adjoint to* $\Pi^{\mathrm{co}}_{\mathrm{all}}\circ\Gamma : \mathbf{LC^\infty RS} \to (\mathbf{C^\infty Rings_{co}})^{\mathrm{op}}$. *Thus* $\operatorname{Spec}|_{\dots}$ *takes limits in $(\mathbf{C^\infty Rings_{co}})^{\mathrm{op}}$ (equivalently, colimits in $\mathbf{C^\infty Rings_{co}}$) to limits in $\mathbf{LC^\infty RS}$.*

Using (a), that small limits exist in the category of $\mathbf{C^\infty Rings}$, and that $\Gamma : \mathbf{LC^\infty RS} \to \mathbf{C^\infty Rings}$ is a left adjoint with image in $(\mathbf{C^\infty Rings_{co}})^{\mathrm{op}}$ when restricted to $\mathbf{AC^\infty Sch}$, then small limits in $\mathbf{C^\infty Rings_{co}}$ exist and coincide with small limits in $\mathbf{C^\infty Rings}$. As $(\mathbf{C^\infty Rings_{co}})^{\mathrm{op}} \to \mathbf{AC^\infty Sch}$ is an equivalence of categories, then $\mathbf{AC^\infty Sch}$ also has all small colimits and small limits. As Spec is a right adjoint, then limits in $\mathbf{AC^\infty Sch}$ coincide with limits in $\mathbf{C^\infty Sch}$ and $\mathbf{LC^\infty RS}$; however, it is not necessarily true that colimits in $\mathbf{AC^\infty Sch}$ coincide with colimits in $\mathbf{C^\infty Sch}$ and $\mathbf{LC^\infty RS}$.

In the following theorem we summarize results found in Dubuc [23, Th. 16], Moerdijk and Reyes [74, § II. Prop. 1.2], and the second author [49, Cor. 4.27].

Theorem 2.54 *There is a full and faithful functor* $F_{\mathbf{Man}}^{\mathbf{AC^\infty Sch}} : \mathbf{Man} \to \mathbf{AC^\infty Sch}$ *that takes a manifold* X *to the affine* C^∞*-scheme* $\underline{X} = (X, \mathcal{O}_X)$*, where* $\mathcal{O}_X(U) = C^\infty(U)$ *is the usual smooth functions on* U*. Here* $(X, \mathcal{O}_X) \cong \operatorname{Spec}(C^\infty(X))$ *and hence* $\underline{X}$ *is affine. The functor* $F_{\mathbf{Man}}^{\mathbf{AC^\infty Sch}}$ *sends transverse fibre products of manifolds to fibre products of* C^∞*-schemes.*

2.7 Sheaves of $\mathcal{O}_X$-modules on C^∞-ringed spaces

This section follows [49, §5.3], where we give the basics of sheaves of $\mathcal{O}_X$-modules for C^∞-ringed spaces. This includes the pullback of a sheaf of modules and the cotangent sheaf. Our definition of $\mathcal{O}_X$-module is the usual definition of sheaf of modules on a ringed space as in Hartshorne [36, §II.5] and Grothendieck [35, §0.4.1], using the $\mathbb{R}$-algebra structure on our C^∞-rings. The cotangent sheaf uses the cotangent modules of §2.3.

Definition 2.55 For each C^∞-ringed space $\underline{X} = (X, \mathcal{O}_X)$ we define a category $\mathcal{O}_X$-mod. The objects are *sheaves of* $\mathcal{O}_X$*-modules* (or simply $\mathcal{O}_X$*-modules*) $\mathcal{E}$ on X. Here, $\mathcal{E}$ is a functor on open sets $U \subseteq X$ such that $\mathcal{E} : U \mapsto \mathcal{E}(U)$ in $\mathcal{O}_X(U)$-mod is a sheaf as in Definition 2.34. This means we have linear restriction maps $\mathcal{E}_{UV} : \mathcal{E}(U) \to \mathcal{E}(V)$ for each inclusion of open sets $V \subseteq U \subseteq X$, such that the following commutes:

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{E}(U) & \longrightarrow & \mathcal{E}(U) \\ \downarrow{\scriptstyle \rho_{UV} \times \mathcal{E}_{UV}} & & {\scriptstyle \mathcal{E}_{UV}}\downarrow \\ \mathcal{O}_X(V) \times \mathcal{E}(V) & \longrightarrow & \mathcal{E}(V), \end{array}$$

where the horizontal arrows are module multiplication. Morphisms in $\mathcal{O}_X$-mod are sheaf morphisms $\phi : \mathcal{E} \to \mathcal{F}$ commuting with the $\mathcal{O}_X$-actions. An $\mathcal{O}_X$-module $\mathcal{E}$ is called a *vector bundle* if it is locally free, that is, around every point there is an open set $U \subseteq X$ with $\mathcal{E}|_U \cong \mathcal{O}_X|_U \otimes_{\mathbb{R}} \mathbb{R}^n$.

Definition 2.56 We define the *pullback* $\underline{f}^*(\mathcal{E})$ of a sheaf of modules $\mathcal{E}$ on $\underline{Y}$ by a morphism $\underline{f} = (f, f^\sharp) : \underline{X} \to \underline{Y}$ of C^∞-ringed spaces as $\underline{f}^*(\mathcal{E}) = f^{-1}(\mathcal{E}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$. Here $f^{-1}(\mathcal{E})$ is as in Definition 2.38, so that $\underline{f}^*(\mathcal{E})$ is a sheaf of modules on $\underline{X}$. Morphisms of $\mathcal{O}_Y$-modules $\phi : \mathcal{E} \to \mathcal{F}$ give morphisms of $\mathcal{O}_X$-modules $\underline{f}^*(\phi) = f^{-1}(\phi) \otimes \mathrm{id}_{\mathcal{O}_X} : \underline{f}^*(\mathcal{E}) \to \underline{f}^*(\mathcal{F})$.

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Definition 2.57 Let $\underline{X} = (X, \mathcal{O}_X)$ be a C^∞-ringed space. Define a presheaf $\mathcal{P}T^*\underline{X}$ of $\mathcal{O}_X$-modules on X such that $\mathcal{P}T^*\underline{X}(U)$ is the cotangent module $\Omega_{\mathcal{O}_X(U)}$ of Definition 2.29, regarded as a module over the C^∞-ring $\mathcal{O}_X(U)$. For open sets $V \subseteq U \subseteq X$ we have restriction morphisms $\Omega_{\rho_{UV}} : \Omega_{\mathcal{O}_X(U)} \to \Omega_{\mathcal{O}_X(V)}$ associated to the morphisms of C^∞-rings $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ so that the following commutes:

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \Omega_{\mathcal{O}_X(U)} & \xrightarrow{\mu_{\mathcal{O}_X(U)}} & \Omega_{\mathcal{O}_X(U)} \\ \downarrow{\scriptstyle \rho_{UV} \times \Omega_{\rho_{UV}}} & & \downarrow{\scriptstyle \Omega_{\rho_{UV}}} \\ \mathcal{O}_X(V) \times \Omega_{\mathcal{O}_X(V)} & \xrightarrow{\mu_{\mathcal{O}_X(V)}} & \Omega_{\mathcal{O}_X(V)}. \end{array}$$

Definition 2.29 implies $\Omega_{\psi \circ \phi} = \Omega_\psi \circ \Omega_\phi$, so this is a well defined presheaf of $\mathcal{O}_X$-modules. The *cotangent sheaf* $T^*\underline{X}$ *of* X is the sheafification of $\mathcal{P}T^*\underline{X}$.

The universal property of sheafification shows that for open $U \subseteq X$ we have an isomorphism of $\mathcal{O}_X|_U$-modules

$$T^*\underline{U} = T^*(U, \mathcal{O}_X|_U) \cong T^*\underline{X}|_U.$$

For $\underline{f} : \underline{X} \to \underline{Y}$ in $\mathbf{C^\infty RS}$ we have $\underline{f}^*(T^*\underline{Y}) = f^{-1}(T^*\underline{Y}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$. The universal properties of sheafification imply that $\underline{f}^*(T^*\underline{Y})$ is the sheafification of the presheaf $\mathcal{P}(\underline{f}^*(T^*\underline{Y}))$, where

$$U \longmapsto \mathcal{P}(\underline{f}^*(T^*\underline{Y}))(U) = \lim_{V \supseteq f(U)} \Omega_{\mathcal{O}_Y(V)} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U).$$

This gives a presheaf morphism $\mathcal{P}\Omega_{\underline{f}} : \mathcal{P}(\underline{f}^*(T^*\underline{Y})) \to \mathcal{P}T^*\underline{X}$ on X, where

$$(\mathcal{P}\Omega_{\underline{f}})(U) = \lim_{V \supseteq f(U)} (\Omega_{\rho_{f^{-1}(V)\,U} \circ f_\sharp(V)})_*.$$

Here, we have morphisms $f_\sharp(V) : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ from $f_\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ corresponding to $f^\sharp$ in $\underline{f}$ as in (2.9), and $\rho_{f^{-1}(V)\,U} : \mathcal{O}_X(f^{-1}(V)) \to \mathcal{O}_X(U)$ in $\mathcal{O}_X$ so that $(\Omega_{\rho_{f^{-1}(V)\,U} \circ f_\sharp(V)})_* : \Omega_{\mathcal{O}_Y(V)} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U) \to \Omega_{\mathcal{O}_X(U)} = (\mathcal{P}T^*\underline{X})(U)$ is constructed as in Definition 2.29. Then write $\Omega_{\underline{f}} : \underline{f}^*(T^*\underline{Y}) \to T^*\underline{X}$ for the induced morphism of the associated sheaves. This corresponds to the morphism $\mathrm{d}f : f^*(T^*Y) \to T^*X$ of vector bundles over a manifold X and smooth map of manifolds $f : X \to Y$ as in Example 2.30.

2.8 Sheaves of $\mathcal{O}_X$-modules on C^∞-schemes

We define the module spectrum functor MSpec as in [49, Defs. 5.16, 5.17 and 5.25], and its corresponding global sections functor, and recall their properties.

Definition 2.58 Let $\mathfrak{C}$ be a C^∞-ring and set $\underline{X} = (X, \mathcal{O}_X) = \operatorname{Spec} \mathfrak{C}$. Let $M \in \mathfrak{C}$-mod be a $\mathfrak{C}$-module. For each open subset $U \subseteq X$ there is a natural morphism $\mathfrak{C} \to \mathcal{O}_X(U)$ in $\mathbf{C^\infty Rings}$. Using this we make $M \otimes_{\mathfrak{C}} \mathcal{O}_X(U)$ into an $\mathcal{O}_X(U)$-module. This assignment $U \mapsto M \otimes_{\mathfrak{C}} \mathcal{O}_X(U)$ is naturally a presheaf $\mathcal{P}\,\mathrm{MSpec}\,M$ of $\mathcal{O}_X$-modules. Define $\mathrm{MSpec}\,M \in \mathcal{O}_X$-mod to be its sheafification.

A morphism $\mu : M \to N$ in $\mathfrak{C}$-mod induces $\mathcal{O}_X(U)$-module morphisms $M \otimes_{\mathfrak{C}} \mathcal{O}_X(U) \to N \otimes_{\mathfrak{C}} \mathcal{O}_X(U)$ for all open $U \subseteq X$, and hence a presheaf morphism, which descends to a morphism $\mathrm{MSpec}\,\mu : \mathrm{MSpec}\,M \to \mathrm{MSpec}\,N$ in $\mathcal{O}_X$-mod. This defines a functor $\mathrm{MSpec} : \mathfrak{C}\text{-mod} \to \mathcal{O}_X\text{-mod}$. It is an exact functor of abelian categories.

There is also a global sections functor $\Gamma : \mathcal{O}_X\text{-mod} \to \mathfrak{C}\text{-mod}$ mapping $\Gamma : \mathcal{E} \mapsto \mathcal{E}(X)$, where the $\mathcal{O}_X(X)$-module $\mathcal{E}(X)$ is viewed as a $\mathfrak{C}$-module via the natural morphism $\mathfrak{C} \to \mathcal{O}_X(X)$.

For any $M \in \mathfrak{C}$-mod there is a natural morphism $\Xi_M : M \to \Gamma \circ \mathrm{MSpec}\,M$ in $\mathfrak{C}$-mod, by composing $M \to M \otimes_{\mathfrak{C}} \mathcal{O}_X(X) = \mathcal{P}\,\mathrm{MSpec}\,M(X)$ with the sheafification morphism $\mathcal{P}\,\mathrm{MSpec}\,M(X) \to \mathrm{MSpec}\,M(X) = \Gamma \circ \mathrm{MSpec}\,M$. Generalizing Definition 2.52, we call M *complete* if Ξ_M is an isomorphism. Write $\mathfrak{C}\text{-mod}_{\mathrm{co}} \subseteq \mathfrak{C}$-mod for the full subcategory of complete $\mathfrak{C}$-modules.

Here is [49, Th. 5.19, Prop. 5.20, Th. 5.26, and Prop. 5.31].

Theorem 2.59 **(a)** *In Definition* 2.58*,* $\mathrm{MSpec} : \mathfrak{C}\text{-mod} \to \mathcal{O}_X\text{-mod}$ *is left adjoint to* $\Gamma : \mathcal{O}_X\text{-mod} \to \mathfrak{C}\text{-mod}$*, generalizing Theorem* 2.47*.*

(b) *There is a natural isomorphism* $\mathrm{MSpec} \circ \Gamma \Rightarrow \mathrm{Id}_{\mathcal{O}_X\text{-mod}}$*. This gives a natural isomorphism* $\mathrm{MSpec} \circ \Gamma \circ \mathrm{MSpec} \Rightarrow \mathrm{MSpec}$*, generalizing Proposition* 2.51*.*

(c) $\mathrm{MSpec}\,|_{\mathfrak{C}\text{-mod}_{\mathrm{co}}} : \mathfrak{C}\text{-mod}_{\mathrm{co}} \to \mathcal{O}_X\text{-mod}$ *is an equivalence of categories, generalizing Theorem* 2.53(a)*.*

(d) *The functor* $\Pi_{\mathrm{all}}^{\mathrm{co}} = \Gamma \circ \mathrm{MSpec} : \mathfrak{C}\text{-mod} \to \mathfrak{C}\text{-mod}_{\mathrm{co}}$ *is left adjoint to the inclusion functor* $\mathrm{inc} : \mathfrak{C}\text{-mod}_{\mathrm{co}} \hookrightarrow \mathfrak{C}\text{-mod}$*, generalizing Theorem* 2.53(c)*. That is,* $\Pi_{\mathrm{all}}^{\mathrm{co}}$ *is a* ***reflection functor****.*

(e) *There is a natural isomorphism* $T^*\underline{X} \cong \mathrm{MSpec}\,\Omega_{\mathfrak{C}}$ *in* $\mathcal{O}_X$-mod*.*

Remark 2.60 **(a)** In [49, §5.4], following conventional Algebraic Geometry as in Hartshorne [36, §II.5], the first author defined a notion of *quasi-coherent sheaf* $\mathcal{E}$ on a C^∞-scheme $\underline{X}$, which is that we may cover $\underline{X}$ by open $\underline{U} \subseteq \underline{X}$ with $\underline{U} \cong \mathrm{Spec}\,\mathfrak{C}$ and $\mathcal{E}|_{\underline{U}} \cong \mathrm{MSpec}\,M$ for $\mathfrak{C} \in \mathbf{C^\infty Rings}$ and $M \in \mathfrak{C}$-mod. But then [49, Cor. 5.22] uses Theorem 2.59(c) to show that every $\mathcal{O}_X$-module is quasi-coherent, that is, $\mathrm{qcoh}(\underline{X}) = \mathcal{O}_X$-mod, which is not true in conventional Algebraic Geometry. So here we will not bother with the language of quasi-coherent sheaves.

(b) In conventional Algebraic Geometry one also defines *coherent sheaves* [36, §II.5] to be $\mathcal{O}_X$-modules $\mathcal{E}$ locally modelled on $\operatorname{MSpec} M$ for M a finitely generated $\mathfrak{C}$-module. However, as in [49, Rem. 5.23(b)], coherent sheaves are only well-behaved on *noetherian* C^∞-schemes, and most interesting C^∞-rings, such as $C^\infty(\mathbb{R}^n)$ for $n > 0$, are not noetherian. So coherent sheaves do not seem to be a useful idea in C^∞-algebraic geometry. For example, $\operatorname{coh}(\underline{X})$ is not closed under kernels in $\mathcal{O}_X$-mod, and is not an abelian category.

Here is [49, Th. 5.32], where part (b) is deduced from Theorem 2.33.

Theorem 2.61 **(a)** *Let $\underline{f} : \underline{X} \to \underline{Y}$ and $\underline{g} : \underline{Y} \to \underline{Z}$ be morphisms of C^∞-schemes. Then in* $\mathcal{O}_X$-mod *we have*

$$\Omega_{\underline{g}\circ\underline{f}} = \Omega_{\underline{f}} \circ \underline{f}^*(\Omega_{\underline{g}}) : (\underline{g}\circ\underline{f})^*(T^*\underline{Z}) \longrightarrow T^*\underline{X}.$$

(b) *Suppose we are given a Cartesian square in* $\mathbf{C^\infty Sch}$*:*

$$\begin{CD} \underline{W} @>{\underline{f}}>> \underline{Y} \\ @V{\underline{e}}VV @VV{\underline{h}}V \\ \underline{X} @>{\underline{g}}>> \underline{Z}, \end{CD}$$

so that $\underline{W} = \underline{X} \times_{\underline{Z}} \underline{Y}$. Then the following is exact in $\mathcal{O}_W$-mod*:*

$$(\underline{g}\circ\underline{e})^*(T^*\underline{Z}) \xrightarrow{\underline{e}^*(\Omega_{\underline{g}})\oplus-\underline{f}^*(\Omega_{\underline{h}})} \underline{e}^*(T^*\underline{X})\oplus\underline{f}^*(T^*\underline{Y}) \xrightarrow{\Omega_{\underline{e}}\oplus\Omega_{\underline{f}}} T^*\underline{W} \longrightarrow 0.$$

2.9 Applications of C^∞-rings and C^∞-schemes

Since the work of Grothendieck, the theory of schemes in Algebraic Geometry has become an enormously powerful tool, and the language in which most modern Algebraic Geometry is written. As a result, Algebraic Geometers are far better at dealing with singular spaces than Differential Geometers are.

It seems desirable to have a theory of schemes in Differential Geometry – C^∞-schemes, or something similar – that could in future be used for the same purposes as schemes in Algebraic Geometry. For example, it seems very likely that many moduli spaces $\mathcal{M}$ of differential-geometric objects are naturally C^∞-schemes, as well as topological spaces.

As another example, suppose (X, g) is a Riemannian manifold, and we are interested in the moduli space $\mathcal{M}$ of some class of special embedded submanifolds $Y \subset X$, for example, minimal, or calibrated. We could imagine trying to define a compactification $\overline{\mathcal{M}}$ of $\mathcal{M}$ by regarding submanifolds $Y \subset X$

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as C^∞-subschemes of X, and taking the closure of $\mathcal{M}$ in the space of C^∞-subschemes.

For the present, to the authors' knowledge, applications of C^∞-algebraic geometry in the literature are confined to two areas: Synthetic Differential Geometry and Derived Differential Geometry, which we now discuss.

2.9.1 Synthetic Differential Geometry

Synthetic Differential Geometry is a subject in which one proves theorems about manifolds in Differential Geometry using 'infinitesimals'. It was used non-rigorously in the nineteenth century by authors such as Sophus Lie. In the 1960s William Lawvere [58] suggested a way to make it rigorous, and the subject has since been developed in detail by Anders Kock [52, 53] and others.

One supposes that the real numbers $\mathbb{R}$ can be enlarged to a 'number line' $R \supset \mathbb{R}$, a ring containing non-zero 'infinitesimal' elements $x \in R$ with $x^n = 0$ for some $n > 1$. An important rôle in the theory is played by the 'double point'

$$D = \{x \in R : x^2 = 0\}. \tag{2.10}$$

One assumes smooth functions $f : \mathbb{R} \to \mathbb{R}$, and manifolds X, can all be enlarged by infinitesimals in this way. Here are examples of how these are used.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is smooth, we can define the *derivative* $\frac{\mathrm{d}f}{\mathrm{d}x}$ by $f(x+y) = f(x) + y\frac{\mathrm{d}f}{\mathrm{d}x}$ for $y \in R$ with $y^2 = 0$.

(b) If X is a manifold, the *tangent bundle* TX is the mapping space $X^D = \mathrm{Map}_{C^\infty}(D, X)$.

(c) A *vector field* on a manifold X can be defined to be a smooth map $v : X \times D \to X$ with $v|_{X\times\{0\}} = \mathrm{id}_X$.

The theory is developed axiomatically, with axioms on the properties of infinitesimals, and theorems about manifolds (including classical results not involving infinitesimals) are proved from them. The logic is unusual: since one treats infinitesimals as ordinary points of the 'set' R, though it turns out that R is not really an honest set, then only constructive logic is allowed, and the law of the excluded middle may not be used.

For the enterprise to be at all credible, we need to know that the axioms of Synthetic Differential Geometry are consistent, as otherwise one could prove any statement from them, true or false. Consistency is proved by constructing a 'model' for Synthetic Differential Geometry, that is, a category (in fact, a topos) of spaces $\mathcal{C}$ which includes $\mathbf{Man}$ as a full subcategory, and contains other 'infinitesimal' objects such as the double point D, such that the axioms can

https://doi.org/10.1017/9781009400190.002 Published online by Cambridge University Press

be interpreted as true statements in $\mathcal{C}$, and thus proofs in Synthetic Differential Geometry can be reinterpreted as reasoning (with ordinary logic) in $\mathcal{C}$.

The connection to C^∞-algebraic geometry is that, as in Dubuc [23] and Moerdijk and Reyes [75], this category $\mathcal{C}$ may be taken to be the category $\mathbf{C^\infty Sch}$ of C^∞-schemes, with $D = \mathrm{Spec}\big(\mathbb{R}[x]/(x^2)\big)$, as a C^∞-subscheme of $\mathbb{R}$. Most early work on C^∞-schemes was directed towards proving properties of $\mathbf{C^\infty Sch}$ needed to verify consistency of various sets of axioms in Synthetic Differential Geometry.

Knowing their axioms were consistent, Synthetic Differential Geometers had little reason to study C^∞-schemes further, so the subject became inactive.

2.9.2 Derived Differential Geometry

Derived Algebraic Geometry is a generalization of classical Algebraic Geometry, in which schemes (and stacks) X are replaced by *derived schemes* (and *derived stacks*) $\boldsymbol{X}$, which have a richer geometric structure. A derived scheme $\boldsymbol{X}$ has a classical truncation $X = t_0(\boldsymbol{X})$, an ordinary scheme. The foundations were developed in the 2000s by Bertrand Toën and Gabriele Vezzosi [90, 91, 92] and Jacob Lurie [61, 62], and it has now become a major area in Algebraic Geometry. Toën [90, 91] gives accessible surveys.

Quasi-smooth derived schemes are a class of derived schemes $\boldsymbol{X}$ that behave in some ways like smooth schemes, although their classical truncations $X = t_0(\boldsymbol{X})$ may be very singular. This is known as the 'hidden smoothness' philosophy of Kontsevich [54]. A proper quasi-smooth derived $\mathbb{C}$-scheme $\boldsymbol{X}$ has a *dimension* $\dim_{\mathbb{C}} \boldsymbol{X}$, and a *virtual class* $[\boldsymbol{X}]_{\mathrm{virt}}$ in homology $H_{2\dim_{\mathbb{C}} \boldsymbol{X}}(\boldsymbol{X}, \mathbb{Z})$, which is the analogue of the fundamental class of a compact complex manifold.

Quasi-smooth derived schemes have important applications in *enumerative geometry* (as does the older notion of *scheme with obstruction theory*, which turns out to be a semi-classical truncation of a quasi-smooth derived scheme). Various moduli problems, such as moduli of stable coherent sheaves on a projective surface, have derived moduli schemes which are quasi-smooth, and the virtual class is used to define invariants 'counting' such moduli spaces.

Two characteristic features of Derived Algebraic Geometry are as follows.

(i) Derived Algebraic Geometry is always done in ∞-*categories*, not ordinary categories, as truncating to ordinary categories loses too much information. For example, the structure sheaf $\boldsymbol{\mathcal{O}}_X$ of a derived scheme is an ∞-sheaf (homotopy sheaf), but truncating to ordinary categories loses the sheaf property.
(ii) To pass from smooth algebraic geometry to Derived Algebraic Geometry,

we replace vector bundles by perfect complexes of coherent sheaves. So a smooth scheme X has tangent and cotangent bundles TX, T^*X, but a derived scheme $\boldsymbol{X}$ has a *tangent complex* $\mathbb{T}_{\boldsymbol{X}}$ and *cotangent complex* $\mathbb{L}_{\boldsymbol{X}}$, which are concentrated in degrees $[0,1]$, $[-1,0]$ if $\boldsymbol{X}$ is quasi-smooth.

We can now ask whether there is an analogous 'derived' version of Differential Geometry. In particular, can one define 'derived manifolds' and 'derived orbifolds' $\boldsymbol{X}$ which would be C^∞ analogues of quasi-smooth derived schemes? We might hope that a compact, oriented derived manifold or orbifold $\boldsymbol{X}$ would have a well-defined dimension and virtual class in homology, and could be applied to enumerative invariant problems in Differential Geometry.

In the last paragraph of [62, §4.5], Jacob Lurie outlined how to use his huge framework to define an ∞-category of derived C^∞-schemes, including derived manifolds. In 2008 Lurie's student David Spivak [87] worked out the details of this, defining an ∞-category of *derived C^∞-schemes* $\boldsymbol{X} = (X, \boldsymbol{\mathcal{O}}_X)$ which are topological spaces X with an ∞-sheaf $\boldsymbol{\mathcal{O}}_X$ of simplicial C^∞-rings, and a full ∞-subcategory of *derived manifolds*, which are derived C^∞-schemes locally modelled on fibre products $X \times_Z Y$ for X, Y, Z manifolds. Spivak also gave a list of axioms for an ∞-category of 'derived manifolds' to satisfy, and showed they hold for his ∞-category.

Some years before the invention of Derived Algebraic Geometry, Fukaya–Oh–Ohta–Ono [28, 29, 30, 31] were working on theories of Gromov–Witten invariants and Lagrangian Floer theory in Symplectic Geometry. Their theories involved giving moduli spaces $\bar{\mathcal{M}}$ of J-holomorphic curves in a symplectic manifold (X,ω) the structure of a *Kuranishi space* $\bar{\boldsymbol{\mathcal{M}}}$, and defining a virtual class/chain $[\bar{\boldsymbol{\mathcal{M}}}]_{\mathrm{virt}}$ in homology. In the 2000's there were still significant problems with the definition and theory of Kuranishi spaces, and the subject was under dispute.

When the second author read Spivak's thesis [87], he realized that *Kuranishi spaces are really derived orbifolds*. This explained the problems in the theory: vital ideas from Derived Algebraic Geometry were missing, especially the need for higher categories, as these were unknown when Kuranishi spaces were invented. The second author then developed theories of derived manifolds and derived orbifolds [42, 43, 44, 45, 48, 49, 50] with a view to applications in Symplectic Geometry, as a substitute for Fukaya–Oh–Ohta–Ono's Kuranishi spaces.

The second author found Spivak's ∞-category far too complicated to work with. So he defined a simplified version, 'd-manifolds' and 'd-orbifolds' [43, 44], which form 2-categories $\mathbf{dMan}, \mathbf{dOrb}$ rather than ∞-categories. (Although this would not work in Derived Algebraic Geometry, it turns out that

2-categories are sufficient in C^∞ geometry because of the existence of partitions of unity.) As part of the foundations of this, he developed C^∞-algebraic geometry in new directions [42, 49], in particular $\mathcal{O}_X$-modules and C^∞-stacks.

Later, the second author [45, 48, 50] found a definition of Kuranishi spaces using an atlas of charts in the style of Fukaya–Oh–Ohta–Ono, which yielded a 2-category $\mathbf{Kur}$ equivalent to $\mathbf{dOrb}$, fixing the problems with the original definition. This gives two different models for Derived Differential Geometry, one starting from derived C^∞-schemes, and one from Kuranishi spaces. To understand the relationship, observe that there are two ways to define manifolds.

(A) A manifold is a Hausdorff, second countable topological space X equipped with a sheaf $\mathcal{O}_X$ of $\mathbb{R}$-algebras (or C^∞-rings) such that $(X, \mathcal{O}_X)$ is locally isomorphic to $\mathbb{R}^n$ with its sheaf of smooth functions $\mathcal{O}_{\mathbb{R}^n}$.

(B) A manifold is a Hausdorff, second countable topological space X equipped with a maximal atlas of charts $\{(U_i, \phi_i) : i \in I\}$.

If we try to define derived manifolds by generalizing approach (A), we get some kind of derived C^∞-scheme, as in [6, 7, 8, 10, 11, 12, 13, 43, 44, 62, 87, 88, 89]; if we try to generalize (B), we get something like Kuranishi spaces in [28, 29, 30, 31, 45, 48, 50].

Derived manifolds and orbifolds are interesting for many reasons, including the following.

(a) Much of classical Differential Geometry extends nicely to the derived case.

(b) Many mathematical objects are naturally derived manifolds, for example

  (i) The solution set of $f_1(x_1, \dots, x_n) = \cdots = f_k(x_1, \dots, x_n) = 0$, where $x_1, \dots, x_n$ are real variables and $f_1, \dots, f_k$ are smooth functions.

  (ii) (Non-transverse) intersections $X \cap Y$ of submanifolds $X, Y \subset Z$.

  (iii) Moduli spaces $\mathcal{M}$ of solutions of nonlinear elliptic equations on compact manifolds. Also, if we consider moduli spaces $\mathcal{M}$ for nonlinear equations which are elliptic modulo symmetries, and restrict to objects with finite automorphism groups, then $\mathcal{M}$ is a derived orbifold.

(c) A compact, oriented derived manifold (or orbifold) $\boldsymbol{X}$ has a *virtual class* $[\boldsymbol{X}]_{\rm virt}$ in (Steenrod/Čech) homology $H_{\mathrm{vdim}\,\boldsymbol{X}}(X, \mathbb{Z})$ (or $H_{\mathrm{vdim}\,\boldsymbol{X}}(X, \mathbb{Q})$), with deformation invariance properties. Combining this with (b)(iii), we can use derived orbifolds as tools in enumerative invariant theories such as Gromov–Witten invariants in Symplectic Geometry.

Now for applications in Symplectic Geometry, especially Lagrangian Floer theory [28, 29] and Fukaya categories [3, 83], it is important to have a theory of derived orbifolds *with corners*. (Some applications involving ‘quilts’ also require derived orbifolds *with g-corners*.) To get satisfactory notions of derived manifold or orbifold with corners in the derived C^∞-scheme approach, it is necessary to go right back to the beginning, and introduce C^∞-rings and C^∞-schemes with corners. The second author pursued these ideas with his students Elana Kalashnikov [51] and the first author [25], which led to this book.

For further references on Derived Differential Geometry see Behrend–Liao–Xu [6], Borisov [7], Borisov–Noel [8], Carchedi [10], Carchedi–Roytenberg [11, 12], Carchedi–Steffens [13], and Steffens [88, 89].