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RETRACTS AND THE FIXED POINT PROBLEM FOR FINITE PARTIALLY ORDERED SETS

BY

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A partially ordered set P has the *fixed point property* if every order-preserving mapping f of P to P has a fixed point, that is, $f(a)=a$ for some $a \in P$; call P *fixed point free* if P does not have the fixed point property.

PROBLEM. *Characterize those partially ordered sets with the fixed point property.*

For those partially ordered sets that are lattices the solution is, of course, provided by the beautiful result of Tarski [**9**] and Davis [**3**]:

Let L be a lattice. Then L has the fixed point property if and only if L is a complete lattice.

In short, a lattice L that is not complete is fixed point free because it contains a chain C that is itself fixed point free and, there exists an order-preserving mapping f of L to L such that $f(L)=C$ and $f \mid C$ is the identity mapping of C. Since there is an order-preserving mapping g of C to C that is fixed point free, $g \circ f$ is a fixed point free mapping of L to L [**3**].

Apart from this result, little is known.

We call a subset Q of a partially ordered set P a *retract* of P if there is an order-preserving mapping f of P to P such that $f(P)=Q$ and $f \mid Q$ is the identity mapping of Q; the mapping f we call a *retraction mapping* of P onto Q. For instance, a lattice that is not complete contains a fixed point free chain as a retract. In fact, it is shown in [**6**] that *each maximal chain in a partially ordered set P is a retract of P.* It follows that *if P has the fixed point property then every maximal chain in P is complete.* Of course, the converse of this statement is far from true—every chain in a finite, fixed point free partially ordered set is complete.

It is easy to see that a partially ordered set P has the fixed point property if and only if every retract of P has the fixed point property. We can say somewhat more for finite partially ordered sets.

PROPOSITION 1. *A finite partially ordered set P is fixed point free if and only if there is a retract Q of P with a fixed point free automorphism.*

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Proof. If g is a retraction mapping of P onto Q and h is a fixed point free automorphism of Q then $h \circ g$ is a fixed point free mapping of P to P.

Conversely, let f be a fixed point free mapping of P to P. Since P is finite there is a positive integer n such that $f^n(P) = f^{n+1}(P)$, where $f^1 = f$ and $f^{i+1} = f \circ f^i$. Then $f' = f \mid f^n(P)$ is an automorphism of $f^n(P)$. Let $Q = f^n(P)$. There is a positive integer k such that $(f')^k$ is the identity mapping of Q. It follows that $f^{nk}(P) = Q$ and $f^{nk} \mid Q$ is the identity mapping of Q. In other words, Q is a retract of P and f' is a fixed point free automorphism of Q. □

Proposition 1 is a useful tool in studying the fixed point property for finite partially ordered sets. As an example we shall prove a result providing a sufficient condition for the fixed point property. First we require some terminology and another fact concerning retracts of finite partially ordered sets.

Let P be a partially ordered set containing no infinite chains and let $\max(P)$ ($\min(P)$) denote the set of maximal (minimal) elements of P. Call $S \subseteq P$ a *spanning* subset of P if $\max(S) \cup \min(S) \subseteq \max(P) \cup \min(P)$. Let Q be a retract of P and suppose $x \in \max(Q) - \max(P)$. Now choose $x' \in \max(P)$, $x < x'$. If $y \in Q$ and $y < x'$ then $y \le x$. Therefore, $Q' = (Q - \{x\}) \cup \{x'\} \cong Q$ and Q' is also a retract of P—if f is a retraction mapping of P onto Q then f', defined by

$$f'(z) = \begin{cases} f(z) & \text{if } f(z) \neq x \\ x' & \text{if } f(z) = x, \end{cases}$$

is a retraction mapping of P onto Q'. We summarize: *Let P be a partially ordered set containing no infinite chains and let Q be a retract of P. Then there is a spanning subset Q' of P such that $Q' \cong Q$ and Q' is a retract of P.*

For $S \subseteq P$ let $S_* = \{x \in P \mid x \le s$ for every $s \in S\}$.

THEOREM 2. *Let P be a finite partially ordered set and let S_* have the fixed point property for every nonempty subset S of* $\max(P)$. *Then P has the fixed point property.*

Proof. Let us suppose that P is fixed point free. By Proposition 1 there is a retraction mapping f of P onto a spanning subset Q of P; moreover, Q has a fixed point free automorphism g. Let $b \in \max(Q) \subseteq \max(P)$ and let $B = \{b, g(b), g^2(b), \ldots\}$. Since g is an automorphism of Q, $B \subseteq \max(Q) \subseteq \max(P)$: we claim that B_* is fixed point free.

As the empty set is (trivially) fixed point free, we may assume that $B_* \neq \phi$. Let $x \in B_*$. Then $x \le g^i(b)$ for $i = 0, 1, \ldots$ (taking $g^0(b) = b$), so $f(x) \le f(g^i(b)) = g^i(b)$ for $i = 0, 1, \ldots$, and $f(B_*) \subseteq B_* \cap Q$. Now, if $y \in B_* \cap Q$ then $g(y) \le g^i(b)$ for $i = 0, 1, \ldots$; hence, $g(B_* \cap Q) \subseteq B_* \cap Q$. We have shown that $f(B_*) = B_* \cap Q$, $f \mid B_* \cap Q$ is the identity on $B_* \cap Q$—$B_* \cap Q$ is a retract of B_*—and $g \mid B_* \cap Q$ is a fixed point free automorphism. Therefore, $g \circ f \mid B_*$ is a fixed point free mapping of B_*. □

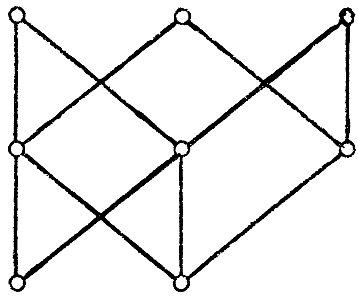

Figure 1

In [7], H. Höft and M. Höft prove that a partially ordered set P has the fixed point property if every maximal chain of P is a complete sublattice and P contains only finitely many maximal elements every nonempty subset of which has an infimum. As any finite partially ordered set with a maximum element has the fixed point property, Theorem 2 sharpens the *finite* version of this result of Höft and Höft. The converse of Theorem 2, of course, fails (see Figure 1).

Theorem 2 is inspired by a question communicated to us by J. R. Isbell who attributes it to L. Mohler. Let P be a finite partially ordered set, let $\max(P) = A \cup B$, where $A \cap B = \phi$, and for $S \subseteq P$, set $S' = \{x \in P \mid x \leq s \text{ for some } s \in S\}$. The question: *Does P have the fixed point property if each of* A', B', *and* $A' \cap B'$ *has the fixed point property*? While Theorem 2 provides a positive answer to this question in the special case that $|\max(P)| \leq 2$, K. Baclawski and A. Björner in [**1**] report an example that provides a negative answer to the general question.

A subset Q of a partially ordered set P is a *fixed point set* of P if there is an order-preserving mapping f of P to P such that $Q = \{x \in P \mid f(x) = x\}$.

Problem. *Characterize those subsets of a partially ordered set that are fixed point sets.*

Certainly a retract of P is a fixed point set of P. Nonetheless, a fixed point set need not be a retract (see Figure 2).

Again, for complete lattices the answer is at hand. Let L be a complete lattice and let K be a fixed point set of L. It is well-known that K (with the induced partial ordering) is a lattice; in fact, K is a complete lattice. It is an easy matter to show that any subset Q of a partially ordered set P is a retract of P provided that Q is a complete lattice. (This fact is implicit in G. Birkhoff [**2**, pp. 301–302].) Combining these facts yields a description of fixed point sets

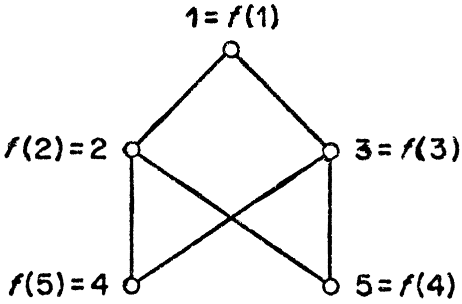

Figure 2

of complete lattices: *If L is a complete lattice and K is a subset of L then K is a fixed point set of L if and only if K is a complete lattice.*

For an integer $n \geq 3$, a *crown* is a partially ordered set $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$ in which $x_i \leq y_i$, $x_{i+1} \leq y_i$, for $i = 1, 2, \ldots, n-1$, $x_1 \leq y_n$ and $x_n \leq y_n$ are the only comparability relations (see Figure 3). A *four-crown* in a partially ordered set P is a set $\{x_1, y_1, x_2, y_2\}$ such that $x_i < y_j$, for $i, j = 1, 2$, are the only comparabilities and there is no $z \in P$ such that $x_1, x_2 \leq z \leq y_1, y_2$. D. Duffus and I. Rival [**5**] have shown that *if P is a finite, connected partially ordered set containing no crowns then a subset Q of P is a fixed point set of P if and only if Q is a retract of P.*

Let P be a finite partially ordered set. For elements $a > b$ in P, a *covers* b ($a \succ b$) if, for all $c \in P$, $a \geq c > b$ implies $a = c$; a is *irreducible* in P if a has precisely one upper cover or precisely one lower cover in P. Let $I(P)$ denote the set of irreducible elements of P. Note that P has the fixed point property if and only if $P - \{a\}$ has the fixed point property for all $a \in I(P)$ [**8**]. P is *dismantlable* (*by irreducibles*) if $P = \{a_1, a_2, \ldots, a_n\}$ and

$$a_i \in I(P - \{a_1, a_2, \ldots, a_{i-1}\})$$

for $i = 1, 2, \ldots, n-1$. A dismantlable partially ordered set has the fixed point property [**8**].

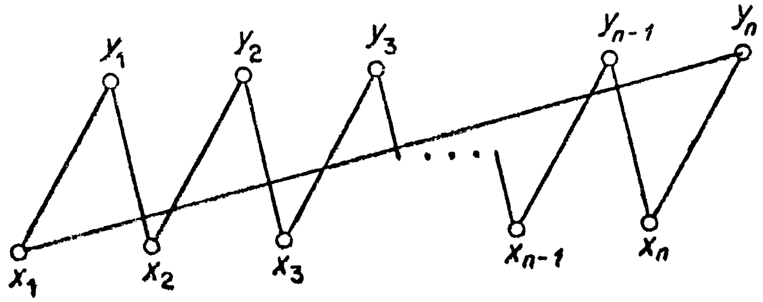

Figure 3

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A finite, connected partially ordered set containing no crowns is dismantlable [**4**]. Still, not every fixed point set of a dismantlable partially ordered set is a retract (see Figure 2).

THEOREM 3. *Let P be a dismantlable partially ordered set and let Q be a fixed point set of P. Then Q is dismantlable.*

The proof of Theorem 3 is based on two lemmas.

LEMMA 4 [**4**]. *Let P be a dismantlable partially ordered set and let $a \in I(P)$. Then $P-\{a\}$ is dismantlable.* □

LEMMA 5. *Let P be a dismantlable partially ordered set and let Q be a retract of P. Then Q is dismantlable.*

Proof. We proceed by induction on $|P|$. Let f be a retraction mapping of P onto Q. Let $a \in I(P)$. Since P is dismantlable, $P-\{a\}$ is dismantlable.

If $a \in I(P)-Q$ then Q is a retract of $P-\{a\}$, whence, by the induction hypothesis, Q is dismantlable. If $a \in I(P) \cap I(Q)$ then, with a^* as the unique upper cover of a *in* Q, define f' of $P-\{a\}$ to $Q-\{a\}$ by

$$f'(z)=\begin{cases} f(z) & \text{if } f(z)\neq a \\ a^* & \text{if } f(z)=a. \end{cases}$$

Then f' is a retraction mapping of $P-\{a\}$ onto $Q-\{a\}$. Again, the induction hypothesis implies that Q is dismantlable.

Let $a \in I(P)$, let $a \in Q-I(Q)$, and let a^* be the unique upper cover of a *in* P. Since $a \notin I(Q)$, $a^* \notin Q$. Since $a^* \geq a$, $f(a^*) \geq f(a)=a$. If $f(a^*)>a$ then a is not maximal in Q so there are distinct elements b, c in Q such that $b>a$ and $c>a$ in Q; hence $b \geq a^*$ and $c \geq a^*$ in P, so $b=f(b) \geq f(a^*)$ and $c=f(c) \geq f(a^*)$. Therefore, $f(a^*)=a$. We claim that $Q'=(Q-\{a\}) \cup \{a^*\} \cong Q$. Let $x \in Q-\{a\}$. Then $a<x$ implies $a<a^*<x$, $x<a$ implies $x<a^*$, and $a^*<x$ implies $a<x$. Let $x<a^*$. Then $x=f(x) \leq f(a^*)=a$. Hence, $Q \cong Q'$.

Since Q' is contained $P-\{a\}$, we need only show that Q' is a retract of $P-\{a\}$ in order to conclude that Q', and so Q, is dismantlable. Let a mapping f' of $P-\{a\}$ to $P-\{a\}$ be given by

$$f'(z)=\begin{cases} f(z) & \text{if } f(z)\neq a \\ a^* & \text{if } f(z)=a. \end{cases}$$

Then $f(P-\{a\})=Q'$ and, since $f(a^*)=a$, $f' \mid Q'$ is the identity mapping on Q'. It is straightforward to show that f' is order-preserving. □

Proof of Theorem 3. We proceed by induction on $|P|$. Let f be an order-preserving mapping of P to P such that $Q=\{x \in P \mid f(x)=x\}$. As in the proof of Proposition 1, there is a positive integer n such that $f^n(P)$ is a retract of P.

Observe that $Q \subseteq f^n(P)$ and, by Lemma 5, $f^n(P)$ is dismantlable. If f is not an automorphism of P then $|f^n(P)| < |P|$ and, by induction, Q is dismantlable. Therefore, f is an automorphism of P.

Suppose that $a \in I(P)$ and $a \notin Q$. Since f is an automorphism and P is finite, $f^i(a) \in I(P)$; in fact, since $f^i(a)$ is non-comparable with $f^j(a)$ for $f^i(a) \neq f^j(a)$, $f^i(a) \in I(P - \{a, f(a), \ldots, f^{i-1}(a)\})$, for $i = 0, 1, \ldots$. By Lemma 4, $P' = P - \{a, f(a), f^2(a), \ldots\}$ is dismantlable. Again, the induction hypothesis applied to P' yields that Q is dismantlable. Therefore, we may assume that $I(P) \subseteq Q$. Let $a \in I(P)$ and let a_* be the unique lower cover of a in P. Since f is an automorphism, $f(a_*) \prec f(a) = a$; it follows that $a_* \in Q$. Let $P'' = P - \{a\}$ and let $f'' = f \mid P''$. Then f'' is an automorphism of P'' and $\{x \in P'' \mid f''(x) = x\} = Q - \{a\}$. We have that P'' is dismantlable, whence by the induction hypothesis, $Q - \{a\}$ is dismantlable. Since a_* is the unique lower cover of a in P and $a_* \in Q$, $a \in I(Q)$. Therefore, Q is dismantlable. □

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