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THE DUALITY BETWEEN FLOW CHARTS AND CIRCUITS

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This paper contains a precise description of the duality between the formal evolutions of flow charts and of circuits. In addition, it contains a new description of the free category-with-products on a multigraph as a familially representable construction.

0. INTRODUCTION

If A is a set of elementary actions then the elements of the free monoid A^* may be thought of as *formal evolutions* of the set of actions. This can be generalised. Given a *graph* $\mathbf{G}$ of elementary actions the arrows in the *free category* – with appropriate structure – on $\mathbf{G}$ may again be thought of as the formal evolutions of the elementary actions, taking into account the specificity of the graph.

This paper is concerned in particular with the formal evolution of both flow charts and of digital circuits. The appropriate notion of graph in *both cases* is multigraph. The formal evolutions of a flow chart are arrows in the *free category-with-sums* on a multigraph; the formal evolutions of a circuit are arrows in the *free category-with-products* on a multigraph.

As a result there is a precise categorical duality between the evolution of flow charts and of circuits. Roughly speaking, this is the duality between terms and trees.

In addition to this duality theorem, we give a new description of the free category-with-products on a multigraph. Diers [2] introduced the notion of a locally representable functor, that is, a functor represented by a family of objects rather than one object. Johnson and Walters, calling the notion instead *familially representable functor*, found further examples (see [4], [5]). A simple example is the free-monoid functor. The family of objects in this case is the natural numbers, and the free monoid on A may be thought of as $\{n \to A; n \text{ a natural number}\}$. In section 5 we indicate that the free category-with-products on a multigraph is similarly familially representable – an idea suggested by the well-known description in terms of families of the free category-with-products on a *category*.

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The importance in computer science of categories with sums, and with products, has long been recognised – for example, Elgot in [**3**] treats flow charts in terms of sums. However the analysis in terms of multigraphs and the resulting precise duality mentioned above seems to be new. Note that understanding the *formal* evolution of flow charts and circuits is the first stage towards understanding, and provides a language for, *actual* evolution. Actual evolution involves models, categories which are not free, and notions of time (see [**8**]).

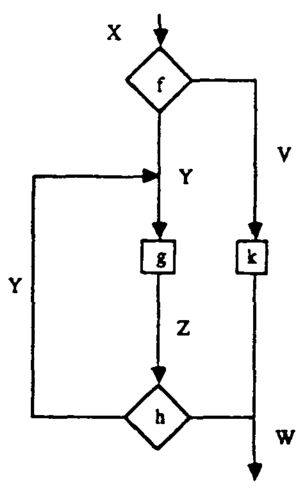
1. Flow charts and circuits as multigraphs

Definition: A multigraph $\mathbf{G}$ is a finite family of finite sets $G_0, G_1, \cdots, G_n$ and G_* together with, for each $k = 0, 1, \cdots, n$, a family of functions $d_0, d_1, \cdots, d_k : G_k \to G_*$. The elements of G_k are called *arrows*, and the elements of G_* are called *objects*.

Example 1. *Flow Charts.*

In dealing with flow charts it is useful to represent the arrows of a multigraph in a particular way – called the *additive notation*. If $f \in G_k$ and $d_0 f = X, d_1 f = Y_1, d_2 f = Y_2, \cdots, d_k f = Y_k$ we denote f by $f : X \to Y_1 + Y_2 + \cdots Y_k$.

Now, underlying a flow chart there is a multigraph. We illustrate with a simple example.



Notice that we have given names to the edges of the flow-chart – these are objects of the multigraph underlying the flowchart. There are four arrows in the multigraph: f, g, h, k. In fact, $G_1 = \{g, k\}$, $G_2 = \{f, h\}$, and $f : X \to Y + V$, $h : Z \to Y + W$, $g : Y \to Z$, $k : V \to W$.

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The additive notation is apt since, for example, the output state space of f is a disjoint sum of the state spaces corresponding to the edges Y and V (see for example [1]).

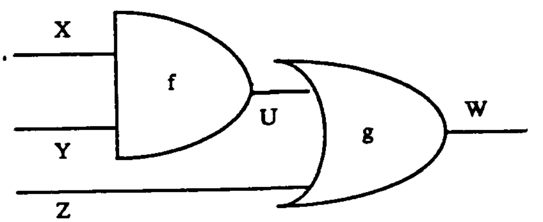
In dealing with circuits it is useful to represent the arrows of a multigraph in another way – called the *multiplicative notation.* If $f \in G_k$ and $d_0 f = Y, d_1 f = X_1, d_2 f = X_2, \cdots, d_k f = X_k$ we denote f by

$$f : X_1 \times X_2 \times \cdots X_k \to Y,$$

or more briefly $f : X_1 X_2 \cdots X_k \to Y$.

Now, underlying a circuit there is a multigraph. We illustrate with two simple examples.

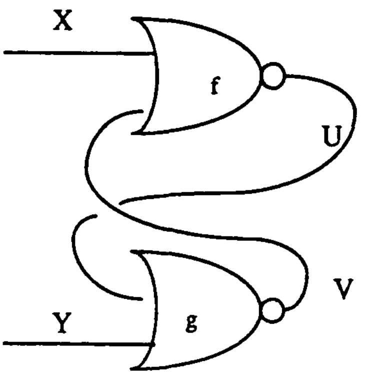
EXAMPLE 2. *A combinatorial circuit.*



Notice that we have given names to the edges of the circuit – these are objects of the multigraph underlying the circuit. There are two arrows in the multigraph: f, g. In fact, $G_2 = \{f, g\}$, and $f : X \times Y \to U$, $g : U \times Z \to V$.

The multiplicative notation is apt since, for example, the input state space of f is a product of the state spaces corresponding to the edges X and Y (see for example [8]).

EXAMPLE 3. *A circuit with feedback.*



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The multigraph corresponding to this circuit has four objects X, Y, U, V and two arrows $f : XV \to U$, $g : UY \to V$.

2. The duality theorem

The precise definition of *free category-with-products* $\mathcal{F}_\times \mathbf{G}$ on multigraph $\mathbf{G}$ – as well as a concrete description of it using *terms* – is given in [7]. After recalling this definition briefly we will give the definition of *free category-with-sums on a multigraph* and then prove the categorical duality at the basis of this paper.

Let $\mathcal{CAT}_\times$ be the 2-category of categories-with-products and the usual product-preserving functors, and let $\mathcal{MGPH}$ be the category of multigraphs. In [7] a 2-functor $\mathcal{U}_\times : \mathcal{CAT}_\times \to \mathcal{CAT}(\mathcal{MGPH})$ is described. Then $\mathcal{F}_\times \mathbf{G}$ satisfies the universal property

$$\mathcal{CAT}_\times(\mathcal{F}_\times \mathbf{G}, \mathbf{C}) \simeq \mathcal{CAT}(\mathcal{MGPH})(\mathbf{G}, \mathcal{U}_\times \mathbf{C}).$$

($\mathbf{C}$ is a category with products; on the right-hand-side $\mathbf{G}$ is the multigraph regarded as a discrete category object in $\mathcal{MGPH}$.) That is, $\mathcal{F}_\times$ is a partial left adjoint, in an appropriate 2-dimensional sense, of $\mathcal{U}_\times$.

The definition of free category-with-sums is closely related. Let $\mathcal{CAT}_+$ be the 2-category of categories-with-sums and the usual sum-preserving functors. Then the assignment to a category of its dual category is an isomorphism $\mathcal{I} : \mathcal{CAT}_+ \to \mathcal{CAT}_\times^{op}$ (the dual obtained by reversing 2-cells), and it is also an isomorphism $\mathcal{J} : \mathcal{CAT}(\mathcal{MGPH}) \to [\mathcal{CAT}(\mathcal{MGPH})]^{op}$. Then define the forgetful 2-functor $\mathcal{U}_+ : \mathcal{CAT}_+ \to \mathcal{CAT}(\mathcal{MGPH})$ by $\mathcal{U}_+ = \mathcal{J}^{-1} \circ \mathcal{U}_\times^{op} \circ \mathcal{I}$; that is, $\mathcal{U}_+(\mathbf{C}) = [\mathcal{U}_\times(\mathbf{C}^{op})]^{op}$. Then the free category-with-sums $\mathcal{F}_+ \mathbf{G}$ on multigraph $\mathbf{G}$ is defined by the universal property

$$\mathcal{CAT}_+(\mathcal{F}_+ \mathbf{G}, \mathbf{C}) \simeq \mathcal{CAT}(\mathcal{MGPH})(\mathbf{G}, \mathcal{U}_+ \mathbf{C}).$$

($\mathbf{C}$ is a category with sums; on the right-hand-side $\mathbf{G}$ is the multigraph regarded as a discrete category object in $\mathcal{MGPH}$.)

Duality Theorem. $\mathcal{F}_+ \mathbf{G} \simeq (\mathcal{F}_\times \mathbf{G})^{op}$.

Proof: The result is immediate from the definition of $\mathcal{F}_+$ as a composite of $\mathcal{F}_\times$ with isomorphisms; to be specific we spell out the details.

$$\begin{aligned}
\mathcal{CAT}_+(\mathcal{F}_+ \mathbf{G}, C) &\simeq \mathcal{CAT}(\mathcal{MGPH})(\mathbf{G}, \mathcal{U}_+ \mathbf{C}) \\
&\cong \mathcal{CAT}(\mathcal{MGPH})(\mathbf{G}^{op}, (\mathcal{U}_\times \mathbf{C}^{op})^{op}) \quad (\text{since } \mathbf{G}^{op} = \mathbf{G}) \\
&\cong [\mathcal{CAT}(\mathcal{MGPH})]^{op}(\mathbf{G}, \mathcal{U}_\times \mathbf{C}^{op}) \\
&\cong [\mathcal{CAT}(\mathcal{MGPH})(\mathbf{G}, \mathcal{U}_\times \mathbf{C}^{op})]^{op} \\
&\simeq [\mathcal{CAT}_\times(\mathcal{F}_\times \mathbf{G}, \mathbf{C}^{op})]^{op} \\
&\cong \mathcal{CAT}_\times^{op}(\mathcal{F}_\times \mathbf{G}, \mathbf{C}^{op}) \\
&\cong \mathcal{CAT}_+((\mathcal{F}_\times \mathbf{G})^{op}, \mathbf{C}).
\end{aligned}$$

$\square$

It is instructive to examine this duality at the level of arrows. Consider the classical description of the free category-with-products $\mathcal{F}_\times \mathbf{G}$ on a multigraph G – see [7] for a more details. Briefly, the objects of $\mathcal{F}_\times \mathbf{G}$ are words $X_1 X_2 \cdots X_m$ in the objects of $\mathbf{G}$. Now regard the arrows of the multigraph as function symbols. The arrows

$$X_1 X_2 \cdots X_m \to Y_1 Y_2 \cdots Y_n$$

are n-tuples of terms built out of the function symbols, and with variables amongst the variables $x_1, x_2, \cdots, x_n$.

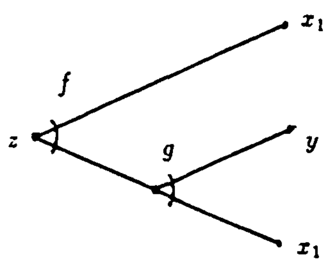
For example, if $\mathbf{G}$ is the multigraph with objects X, Y, Z and two arrows $f : X^2 \to Z$, $g : YX \to X$ then

$$f(x_1, g(y, x_1)) : X^2 Y \to Z$$

is an arrow of $\mathcal{F}_\times \mathbf{G}$. This arrow might be more clearly denoted $z = f(x_1, g(y, x_1))$. By the duality theorem, $\mathcal{F}_+ \mathbf{G}$ has objects which are words in the objects of $\mathbf{G}$, but now we denote them additively. The arrows of $\mathbf{G}$ written additively are $f : Z \to X + X$, $g : X \to Y + X$. The term we have just described is an arrow in $\mathcal{F}_+ \mathbf{G}$ but in the reverse direction

$$Z \to X + X + Y.$$

It is useful to use a different notation – reflecting the interpretation in **Sets** – for the arrows as well as the objects, and to regard the arrow as a 'tree of choices' rather than as a term.



We write the arrow as:

$$\begin{array}{lll} \text{either} & f(z) = x_1 & \\ \text{or} & (\text{ either} & g(f(z)) = y \\ & \quad\text{or} & g(f(z)) = x_1). \end{array}$$

You are meant to read this as follows. The arrow has three formulas depending on where f and g land. Notice that the variables allow you to specify which copy of X the arrow lands in.

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3. The formal evolution of circuits

We hope to persuade you by some examples that it is appropriate to view the arrows in the free category-with-products on the multigraph of a circuit as the formal evolutions of the circuit.

Example 1. Consider the circuit in section 1, example 2. We now think of the name attached to an edge as the state space of the edge. Then the state space of the whole circuit is $X \times Y \times Z \times U \times W$, and hence an evolution of the whole circuit is an arrow

$$XYZUW \to XYZUW.$$

Two examples in $\mathcal{F}_\times \mathbf{G}$ are

$$(x, y, z, f(x,y), g(u,z)) \text{ and}$$
$$(x, y, z, f(x,y), g(f(x,y),z)),$$

which clearly describe two possible evolutions of the circuit.

Of course the endomorphisms of $XYZUW$ are not the only arrows in $\mathcal{F}_\times \mathbf{G}$. For example, there is exactly one arrow in $\mathcal{F}_\times \mathbf{G}$ from XYZ to W, namely

$$g(f(x,y),z) : XYZ \to W.$$

Such an arrow is clearly important in the evolution of the circuit, and deserves to be considered as an evolution of 'parts' of the circuit.

It is less clear that arrows which have a repetition of X, Y, Z, U or W in the domain or codomain should be called evolutions of the circuit since they are not physically realised; however, the necessity of the free category-with-products construction suggests that they should also be included as evolutions.

Example 2. Consider the circuit in section 1, example 3. The feedback in this circuit is, to a first approximation, described by the arrows

$$\alpha = (x, y, f(x,v), g(u,y)) : XYUV \to XYUV,$$

and $\alpha \circ \alpha$, $\alpha \circ \alpha \circ \alpha, \cdots$.

It can be better described by adding constants $0,1 : 1 \to X$, $0,1 : 1 \to Y$, to the circuit. Then it is possible to describe phenomena which involve (externally) changing the states of X and Y, and the order in which the atomic actions are applied. It is instructive to consider the meaning of the following four different arrows from UV to UV:

$$(f(1,v), g(u,0))$$
$$(f(1,v), g(f(1,v),0))$$
$$(f(1,g(u,0)), g(f(1,v),0))$$
$$(f(0,g(f(1,v),0)), g(f(1,g(u,0)),0)).$$

4. The formal evolution of flow charts

We hope to persuade you that it is appropriate to regard the arrows in the free category-with-sums on the multigraph of a flow chart as the formal evolutions of the flow chart.

The free category-with-sums $\mathcal{F}_+\mathbf{G}$ on a multigraph $\mathbf{G}$ is the dual of $\mathcal{F}_\times\mathbf{G}$. As we have mentioned we can interpret the arrows in the free category-with-sums are 'trees of choices'. Consider the flow chart in Section 1, Example 1. The state space of the whole flow chart is $X+Y+Z+V+W$, so that evolutions of the whole flow chart are endomorphisms of $X+Y+Z+V+W$. We give just one example of an a partial evolution $X \to Y+W$:

$$\begin{array}{llll} \text{either} & \{\text{either} & [\text{either} & h(g(h(g(f(x))))) = y \\ & & \quad\ \text{or} & h(g(h(g(f(x))))) = w] \\ & \quad\ \text{or} & h(g(f(x))) = w\} & \\ \text{or} & k(f(x)) = w. & & \end{array}$$

5. A new description of the free category-with-products on a multigraph

We give briefly a new description of the free categories-with-products functor

$$\mathcal{F}_\times : \mathcal{MGPH} \to \mathcal{CAT}_\times \subseteq \mathcal{CAT},$$

in terms of the notion of 'familially representable' functors; further details are given in [**5**]. In order to say when a functor $\mathcal{F} : \mathbf{C} \to \mathcal{CAT}$ is familially representable we need the notion of a 'cocategory family' in $\mathbf{C}$. This is a category $\mathbf{M}$ (whose objects and arrows we will denote $m,\ n,\ p, \cdots,\ f : m \to n,\ g : n \to p, \cdots$) and two families of objects of $\mathbf{C}$, namely I_m (m an object of $\mathbf{C}$) and I_f (f an arrow of $\mathbf{C}$). In addition there are various associated arrows of $\mathbf{C}$, namely $s_m : I_m \to I_f$, $t_n : I_n \to I_f$, $i_m : I_{1_m} \to I_m$, and $c_{f,g} : I_{g\circ f} \to I_f \underset{I_n}{+} I_g$. The idea is that s, t are cosource and cotarget maps, i is coidentity, and c is cocomposition. These data are required to satisfy obvious coidentity and coassociativity laws.

Now given such a cocategory family in $\mathbf{C}$ and an object X of $\mathbf{C}$ there is an induced category with morphism set

$$\sum_f Hom_{\mathbf{C}}(I_f, X)$$

and object set

$$\sum_m Hom_{\mathbf{C}}(I_m, X).$$

As we vary X we get a functor $\mathbf{C} \to \mathcal{CAT}$; any functor isomorphic to a functor induced in such a way is called 'familially representable'.

We claim that $\mathcal{F}_\times$ is familially representable. Let us describe the appropriate category $\mathbf{M}$, and families I. Consider the multigraph $\mathbf{T}$ with one object X and, to each natural number k one arrow $\mu_k : X^k \to X$ – that is, the terminal multigraph. Let $\mathbf{M}$ be the free category-with-products on $\mathbf{T}$. Its objects are powers of X; we may identify X^m with m. An arrow from m $(=X^m)$ to n $(=X^n)$ is an n-tuple of terms constructed using μ's and with variables out of $x_1, x_2, \cdots, x_m$. Let I_m be the multigraph with m objects $1, 2, \cdots, m$ and no arrows. If $f : m \to n$ is an arrow in $\mathbf{M}$ let I_f be the multigraph whose objects are the subterms of f, including all the variables $x_1, x_2, \cdots, x_m$, and whose arrows are μ's which build a subterm out of subterms one level below. Note that each variable is counted only once as an object, whereas a subterm is counted as many times as it occurs in f. For example, the multigraph corresponding to the term

$$(\mu_2(x_2, x_1)), \mu_2(x_1, \mu_2(x_2, x_1)) : 3 \to 2$$

has objects x_1, x_2, x_3, two copies of $\mu_2(x_2, x_1)$, $\mu_2(x_1, \mu_2(x_2, x_1))$; and arrows

$$\begin{aligned} &\mu_2 : x_2 x_1 \to \mu_2(x_2, x_1) \\ &\mu_2 : x_2 x_1 \to \mu_2(x_2, x_1) \\ &\mu_2 : x_1 \mu_2(x_2, x_1) \to \mu_2(x_1, \mu_2(x_2, x_1)). \end{aligned}$$

Now s_m takes k to x_k, and t_m takes k to the k-th highest-level subterm. Composition is the identity.

Roughly, the familial representability of $\mathcal{F}_\times$ amounts to the fact that a general term is a labelling of a term in $\mathbf{M}$.

6. Conclusions

In view of the identification in Section 3, Section 4 of formal evolutions of flows and circuits as arrows in certain free categories we are now able to interpret the duality theorem of Section 2 as *the category of formal evolutions of a circuit is the dual of the category of formal evolutions of a flow chart with the same multigraph.*

Remark. Although circuits have been discussed entirely in terms of products (and products are sufficient to understand their dynamics, see [**8**]), in fact the state space of the simplest control line is $1 + 1$ so that the control aspect of circuits is achieved by sums which are not explicit in the multigraph. Similarly, although flow-charts have been discussed entirely in terms of sums, in practice the state space of an edge of a flow chart is a product not explicit in the multigraph. A richer theory, hinted at in [**6**] and to be discussed in future papers, will make explicit both sums and products and the relation between them.

References

[1] M.A. Arbib and E.G. Manes E.G., *Algebraic approaches to program semantics* (Springer-Verlag, Berlin, Heidelberg, New York, 1986).

[2] Y. Diers, *Doctoral Thesis* (Université de Paris VI, 1977).

[3] C.C. Elgot, *Algebraic theories and program schemes, Symposium on the semantics of algorithmic languages* (Springer-Verlag, Berlin, Heidelberg, New York, 1971).

[4] M.S.J. Johnson, *Pasting Diagrams*, Ph.D. thesis (University of Sydney, 1987).

[5] M.S.J. Johnson and R.F.C. Walters, 'Algebra objects and algebra families for finite limit theories', in *Pure Mathematics Department Research Reports 89-12* (University of Sydney, 1989).

[6] R.F.C. Walters, 'Data types in distributive categories', *Bull. Austral. Math. Soc.* 40 (1989), 79–82.

[7] R.F.C. Walters, 'The free category with products on a multigraph', *J. Pure Appl. Algebra* (to appear).

[8] R.F.C. Walters, *Categories and computer science, 21 lectures* (Pure Mathematics Department, University of Sydney., 1988).

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