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AKCOGLU'S ERGODIC THEOREM FOR UNIFORM SEQUENCES

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1. Introduction. Let $(X, F, \mathfrak{p})$ be a sigma-finite measure space. In what follows we assume p fixed, $1 < p < \infty$. Let T be a contraction of $L_p(X, F, \mu)$ ($||T||_p \leqq 1$). If $f \geqq 0$ implies $Tf \geqq 0$ we will say that T is positive. In this paper we prove that if $\{k_i\}_{i=1}^{\infty}$ is a uniform sequence (see Section 2 for definition) and T is a positive contraction of L_p, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} T^{k_i} f(x)$$

exists and is finite almost everywhere for every $f \in L_p(X, F, \mu)$.

2. Preliminaries. We begin with describing the construction of a uniform sequence as given in [**2**]. Let Ω be a compact metric space, B the collection of Borel subsets of Ω, and ϕ a homomorphism of Ω such that $\{\phi^n\}$, n a positive integer, is an equicontinuous set of mappings. The system (Ω, ϕ) is then called *uniformly L stable*. We assume that Ω possesses a dense orbit, and it then follows (see [**2**]) that there exists a ϕ invariant probability measure on (Ω, B) which we denote by ν, such that for any $w \in \Omega$, and any continuous function f on Ω,

$$\int f d\nu = \lim_{n} \frac{1}{n} \sum_{t=0}^{n-1} f(\phi^t w).$$

Such a system will be called *strictly L stable*.

If $Y \in B$ and $y \in \Omega$, then we define the i*th entry time* $k_i(y, Y)$ of y into Y recursively as:

$$k_1(y, Y) = \min\{i \geqq 1: \phi^i y \in Y\}$$
$$k_i(y, Y) = \min\{j > k_{i-1}(y, Y): \phi^j y \in T\} \qquad i > 1$$

allowing infinity as a value.

Definition. A sequence $\{k_i\}_{i=1}^{\infty}$ of natural numbers will be called *uniform* if there exist:

(1) a strictly L stable system (Ω, B, ν, ϕ)
(2) a $Y \in B$ such that $\nu(Y) > 0 = \nu(\partial Y)$ and
(3) a point $y \in \Omega$ such that $k_i = k_i(y, Y)$ for each $i \geqq 1$.

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The (Ω, B, ν, ϕ), Y and y in the above definition will be called the *apparatus* connected with the uniform sequence $\{k_i\}_{i=1}^{\infty}$. We will also need Akcoglu's ergodic theorem ([**1**]) which we will state next.

THEOREM. *Let (X, F, μ) be a σ-finite measure space, T a positive contraction of $L_p(X, F, \mu)$, some p, $1 < p < \infty$. Then*

$$\lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x)$$

exists and is finite almost everywhere for all $f \in L_p(X, F, \mu)$.

3. Result.

THEOREM. *Let (X, F, μ) be a σ-finite measure space, T a positive contraction of $L_p(X, F, \mu)$, p fixed $1 < p < \infty$, $\{k_i\}_{i=1}^{\infty}$, a uniform sequence. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} T^{k_i} f$$

exists and is finite a.e. for all f in $L_p(X, F, \mu)$.

Proof. We adapt the proof of Theorem 1 of [**4**]. Let (Ω, B, ν, ϕ) and y, Y be the apparatus connected with the uniform sequence, $X_1 \in F$ $\mu(X_1) < \infty$. Let

$$(\Omega', B', \nu') = (\Omega, B, \nu) \times (X, F, \mu),$$

Φ the operator on $L_p(\Omega, B, \nu)$ defined by

$$\Phi f = f \circ \phi,$$

T' the operator induced on $L_p(\Omega', B', \nu')$ by defining

$$T'(f \cdot g) = \Phi f \cdot Tg$$

where

$$f \in L_p(\Omega, B, \nu) \text{ and } g \in L_p(X, F, \mu).$$

Then T' is a positive contraction of $L_p(\Omega', B', \nu')$. Let $f \in L_p(X, F, \mu)$, $f \geqq 0$ and $\epsilon > 0$. As in the proof of Theorem 1 of [**2**] there exists open subsets Y_1, Y_2 and W of Ω such that

(1) $Y_1 \subseteq Y \subseteq Y_2$
(2) $\nu(Y_2 - Y_1) < \epsilon$
(3) $y \in W$
(4) for any $w \in W$ and any $n \geqq 0$,

$$1_{Y_1}(\phi^n w) \leqq 1_Y(\phi^n y) \leqq 1_{Y_2}(\phi^n w).$$

Put

$$g_1(x, w) = f(x) 1_{Y_1}(w)$$
$$g_2(x, w) = f(x) 1_{Y_2}(w).$$

Akcoglu's ergodic theorem ([**1**]) implies

$$\bar{g}_i(x, w) = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} T'^k g_i(x, w)$$

exists and is finite a.e. for $i = 1, 2$.
The mean ergodic theorem ([**3**], p. 54) implies

$$\lim_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} T'^k g_i - \bar{g}_i \right\|_p = 0 \qquad \text{for } i = 1, 2.$$

We will need

$$\lim_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} T'^k (g_2 - g_1) - (\bar{g}_2 - \bar{g}_1) \right\|_1 = 0.$$

To show this, recall $\mu(X_1) < \infty$. Then

$$1 \in L_q(X_1 \times W, B', \nu')$$

and we have

$$\left| \int_{X_1 \times W} (\bar{g}_2 - \bar{g}_1) d\nu' - \int_{X_1 \times W} \left(\frac{1}{n} \sum_{k=0}^{n-1} T^k f(x) 1_{Y_2 - Y_1}(\phi^k w) \right) d\nu' \right|$$

$$= \left| \int_{X_1 \times W} \left(\bar{g}_2(x, w) - \frac{1}{n} \sum_{k=0}^{n-1} T'^k g_2(x, w) \right) d\nu' \right.$$

$$\left. + \int_{X_1 \times W} \left(\frac{1}{n} \sum_{k=0}^{n-1} T'^k g_1(x, w) - \bar{g}_1(x, w) \right) d\nu' \right|$$

$$\leqq \left(\left\| \bar{g}_2 - \frac{1}{n} \sum_{k=0}^{n-1} T'^k g_2(x, w) \right\|_p + \left\| \bar{g}_1 - \frac{1}{n} \sum_{k=0}^{n-1} T'^k g_1(x, w) \right\|_p \right)$$

$$\times [\nu'(X_1 \times W)]^{1/q}.$$

Therefore

$$\lim_n \int_{X_1 \times W} \left(\frac{1}{n} \sum_{k=0}^{n-1} T^k f(x) 1_{Y_2 - Y_1} \phi^k(w) d\nu' \right) = \int_{X_1 \times W} (\bar{g}_2 - \bar{g}_1) d\nu'.$$

Then

$$\int_{X_1 \times W} (\bar{g}_2 - \bar{g}_1) d\nu' = \int_{X_1 \times W} \left(\lim_n \frac{1}{n} \sum_{k=0}^{n-1} T'^k f(x) 1_{Y_2}(\phi^k(w)) \right.$$

$$\left. - \frac{1}{n} \sum_{k=0}^{n-1} T'^k f(x) 1_{Y_1} \phi^k(w) \right) d\nu'.$$

Put

$$S(x, w) = S(x) = \limsup_n \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x) 1_Y(\phi^k y)$$

$$s(x, w) = s(x) = \liminf_n \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x) 1_Y(\phi^k y).$$

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Then $\bar{g}_1(x, w) \leqq s(x) \leqq S(x) \leqq \bar{g}_2(x, w)$ almost everywhere on $X_1 \times W$. We want to show $S(x) = s(x)$ a.e., or

$$\int_{X_1} S(x) - s(x) = 0.$$

But

$$\int_{X_1} (S(x, w) - s(x, w))d\mu = \frac{1}{\nu(w)} \int_{X_1 \times W} (S(x, w) - s(x, w)d\nu'$$
$$\leqq \frac{1}{\nu(w)} \int_{X_1 \times W} (\bar{g}_2 - \bar{g}_1)d\nu' = \frac{1}{\nu(w)} \lim_n \int_{X_1 \times W} \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x) 1_{y_2 - y_1}$$
$$\times (\phi^k(w))d\nu' = \frac{1}{\nu(w)} \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_{X_1} T^k f(x) d\mu \int_W 1_{y_2 - y_1} \phi^k(w) d\nu'$$
$$\leqq \frac{1}{\nu(w)} \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} ||f||_p \mu(X_1)^{1/q} \int_W 1_{y_2 - y_1} \phi^k(w) d\nu$$
$$\leqq \frac{1}{\nu(w)} ||f||_p \mu(X_1)^{1/q} \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_W 1_{y_2 - y_1} \phi^k(w) d\nu.$$

Now using the mean ergodic theorem, the Birkoff ergodic theorem and the ergodicity of ϕ, we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_W 1_{y_2 - y_1} \phi^k(w) d\nu = \int_W \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{y_2 - y_1} \phi^k(w) d\nu$$
$$= \int_W \nu(y_2 - y_1) d\nu = \nu(w)\nu(y_2 - y_1).$$

So

$$\int_{X_1} (S(x, w) - s(x, w))d\mu \leqq \frac{1}{\nu(w)} ||f||_p \mu(X_1)^{1/q} \nu(y_2 - y_1)\nu(w)$$
$$< ||f||_p \mu(X_1)^{1/q} \epsilon.$$

Since $\epsilon > 0$ was arbitrary, and $\mu(X_1) < \infty$, we have $S(X) = s(x)$ almost everywhere on X_1, and since X is σ-finite, we have

$$S(X) = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} T f(x) 1_Y(\phi^k y)$$

exists and is finite a.e. However,

$$\lim \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) 1_Y(\phi^k y) = \lim \frac{1}{n} \sum_{i=1}^{n-1} \chi_{\{i : k_i \leqq n\}} T^{k_i} f(x),$$

and in [**2**] it is shown that

$$\lim_n \frac{n}{|\{i : k_i \leqq n\}|} \quad \text{exists.}$$

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Hence

$$\lim_n \frac{1}{n}\sum_{k=0}^{n-1} T^{k_i}f(x) = \lim_n \frac{n}{|\{i:k_i \leqq n\}|}\cdot\frac{1}{n}\sum_{k=0}^{n-1}\chi_{\{i:k_i\leqq n\}}T^{k_i}f(x).$$

exists and is finite almost everywhere. This concludes the proof of the theorem.

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