THE DENSEST PACKING OF 9 CIRCLES IN A SQUARE

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Packing problems of this kind are obviously equivalent to the problems of placing k (here 9) points in a unit square such that the minimum distance between any two of them be as large as possible. The solutions of these problems are known for $2 \leq k \leq 9$. The largest possible minimum distances m_k are given in table 1, and the corresponding "best" configurations shown in figure 1.

k	m_k		
2	$\sqrt{2}$	$\approx$	1.414
3	$\sqrt{6} - \sqrt{2}$	$\approx$	1.035
4	1	=	1.000
5	$\sqrt{2} / 2$	$\approx$	0.707
6	$\sqrt{13} / 6$	$\approx$	0.601
7	$2(2 - \sqrt{3})$	$\approx$	0.536
8	$(\sqrt{6} - \sqrt{2}) / 2$	$\approx$	0.518
9	1/2	=	0.500

Table 1

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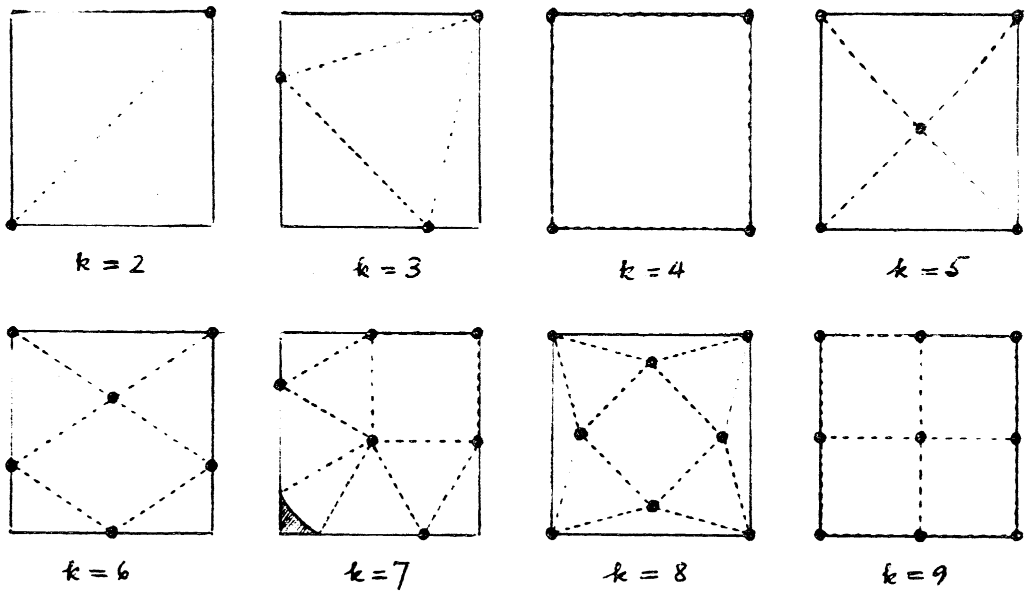

Figure 1

The cases $k = 2, 3, 4,$ and 5 are solved easily. For $k = 6$ R.L. Graham obtained the solution recently. The case $k = 8$ is treated in a separate paper [1]. A proof for $k = 7$ has also been found by the author; although using essentially the same methods it is much more complicated. The case is interesting because the best configuration is not unique, one point being free to be placed anywhere in the shaded area.

For $k = 9$, which we shall solve here, the best configuration is easily guessed, and the conjecture indeed not difficult to prove. We have to show that, for any nine points P_i $(1 \leq i \leq 9)$ of a closed unit square,

$$\min_{1 \leq i < j \leq 9} d(P_i, P_j) \leq \frac{1}{2} \equiv m_9$$

and that equality holds only for the conjectured configuration. ($d(P_i, P_j)$ denotes the distance between P_i and P_j.)

Let S be any set of nine points P_i $(1 \leq i \leq 9)$ of a closed unit square with

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(1) $$\min_{1\le i<j\le 9} d(P_i, P_j) \ge \frac{1}{2}.$$

We shall show that there is just one such set; namely the conjectured one, for which in (1) obviously equality holds.

(1) The unit square may be covered by 9 closed squares Q_{0i} $(1 \le i \le 9)$ of side $s_0 = \frac{1}{3}$. Their diameter is $\frac{\sqrt{2}}{3} < \frac{1}{2}$, so by (1) in each of them there can be at most one point of S. Since there are as many points P_i as squares Q_{0i}, in each of the squares there must lie exactly one point of S: $P_i \in Q_i$ $(1 \le i \le 9)$.

(2) We shall now indicate a procedure by which the location of the points P_i may be restricted to squares $Q_{1i} \subset Q_{0i}$ with side $s_1 < s_0$. Iterating the process, every P_i is successively confined to squares Q_{ni} of sides s_{ni} $(n = 0, 1, 2, \dots)$ $(s_0 > s_1 > s_2 > \dots)$. In every stage the square Q_{ni} is in perspective to the unit square with respect to the conjectured position of P_i.

In every step the same method is applied. Every square Q_{ni} can be reduced by its closest neighbour squares in the following way (see figure 2). Consider a rectangle, of side s_n

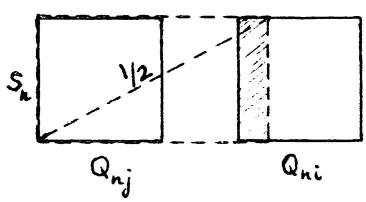

Figure 2

and diagonal $\frac{1}{2}$, which contains the neighbour square Q_{nj} and as much as possible of the square Q_{ni} to be reduced. Excluding the side which lies in Q_{ni}, by (1) the rectangle can contain at most one point of S. Since it contains $P_j \in Q_{nj}$, the region in the rectangle (shaded) can be excluded from Q_{ni} as possible location of P_i.

(3) By this method the four "corner" squares (i. e. the squares Q_{ni} which contain the vertices of the unit square) are reduced least, because they have only two closest neighbours. For sake of simplicity we shall also reduce the five other squares only as much as the corner squares, in order to obtain again nine squares $Q_{n+1,i}$ of equal size. Thus it is sufficient to investigate the effect of the reducing process on a corner square. s_{n+1} is found by (see figure 3)

$$s_n^2 + \left(\frac{1}{2} + \frac{1}{2}s_n - s_{n+1}\right)^2 = \left(\frac{1}{2}\right)^2$$

and therefore

$$s_{n+1} = \frac{1}{2}\left(1 + s_n - \sqrt{1 - 4s_n^2}\right).$$

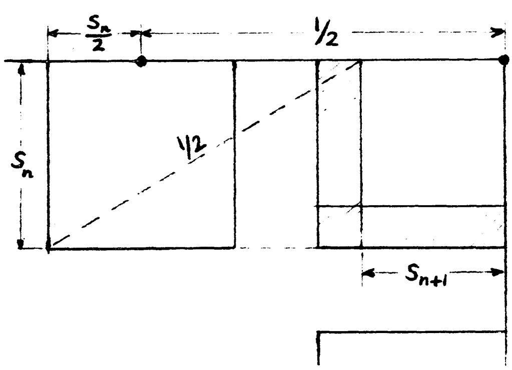

Figure 3

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(4) We shall now show that $s_n \to 0$ as $n \to \infty$. As a consequence the points P_i must lie at the conjectured positions. In fact

$$s_{n+1} = \frac{1}{2} \frac{(1 + s_n)^2 - (1 - 4s_n^2)}{1 + s_n + \sqrt{1 - 4s_n^2}}$$

and

$$\frac{s_{n+1}}{s_n} = \frac{2 + 5 s_n}{2 + 2s_n + 2\sqrt{1 - 4s_n^2}} .$$

Since $s_n \leq s_0 = \frac{1}{3}$, we have

$$\frac{s_{n+1}}{s_n} \leq \frac{2 + 2s_n + 1}{2 + 2s_n + 2\sqrt{1 - 4/9}} = \frac{9 + 6s_n}{6 + 6s_n + 2\sqrt{5}}$$

$$< \frac{9 + 6s_n}{10 + 6 s_n} = 1 - \frac{1}{10 + 6s_n} \leq 1 - \frac{1}{12} ,$$

and the proof is complete.

REFERENCE

1. J. Schaer and A. Meir, On a geometric extremum problem, Can. Math. Bull. 8 (1965), 21-27.

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