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SOME BIFURCATION PROBLEMS IN CHOLESTERIC LIQUID CRYSTAL THEORY

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1. Introduction

A liquid crystal is a transversely isotropic liquid consisting of large, relatively rigid, elongated molecules which align more or less parallel to their neighbours. Three distinct types of liquid crystal occur, namely nematic, cholesteric and smectic. In the absence of any external influences, nematics tend to orientate with their anisotropic axis uniformly aligned, whereas cholesterics prefer a characteristic helical configuration and smectics are more highly organised in layered structures. However, it is possible to influence the orientation of the anisotropic axis by a variety of external means. In particular, solid surfaces affect the alignment through the action of surface torques, while electromagnetic fields exert body torques which tend to align the anisotropic axis either parallel or perpendicular to the applied field. Detailed descriptions of the physical properties of liquid crystals may be found in the books by de Gennes [**1**] and Chandrasekhar [**2**] and the review by Stephen and Straley [**3**].

There have been a variety of experiments to investigate the competition between the orientational effects of electromagnetic fields and solid boundaries. One of the first, performed by Freedericksz and Zvolina [**4**], involves a sample of nematic liquid crystal at rest in a small gap between parallel plates, where suitable prior treatment of the solid surfaces leads to an initial alignment of the anisotropic axis uniformly parallel to the solid boundaries. Upon applying a magnetic field perpendicular to the plane of the plates, there is no distortion of the initial configuration until the field strength exceeds a critical value when one observes a smooth transition to a perturbed configuration in which the anisotropic axis tilts in the direction of the field. A somewhat similar effect of practical interest occurs by first rotating one plate in its own plane relative to the other. This produces an initial configuration in which the anisotropic axis is everywhere parallel to the solid boundaries, being constant in any plane parallel to the plates but varying uniformly with distance across the gap. This uniformly twisted nematic structure is employed in the display device described by Schadt and Helfrich [**5**]. When a sample of cholesteric liquid crystal is placed between parallel plates, the initial orientation of the anisotropic axis commonly exhibits the uniformly twisted configuration described above. For technical reasons cholesteric materials are of importance in display devices.

Here we examine possible equilibrium configurations that are relevant to the Freedericksz experiment in which a cholesteric liquid crystal is subjected to either a magnetic or electric field applied perpendicular to the plane of the plates. As is common in calculations for nematics, one first employs the strong anchoring condition in which the boundary alignment is prescribed. The analysis is then repeated for the case when a particular form of weak anchoring or couple stress boundary condition is adopted. We predict a variety of critical field strengths at which distortion may commence, and determine possible orientation patterns once the appropriate critical value is exceeded. The mathematical formulation of such problems result in some rather interesting examples of bifurcation phenomena in the theory of non-linear ordinary differential equations. At the same time it confirms in greater detail predictions made by Raynes [6].

2. Basic equations

Continuum theory introduces a unit vector **n**, called the director, to represent the orientation of the anisotropic axis, and detailed accounts of this theory are presented in the reviews by Ericksen [7] and Leslie [8]. This paper seeks possible solutions relevant to the Freedericksz experiment in which a thin layer of cholesteric liquid crystal confined between parallel plates, a distance $2l$ apart, is subjected to an electromagnetic field applied perpendicular to the plane of the plates.

Considering orientation patterns of the form

$$n_x=\cos\theta(z)\cos\phi(z),\quad n_y=\cos\theta(z)\sin\phi(z),\quad n_z=\sin\theta(z), \tag{2.1}$$

pertinent continuum equations describing the static isothermal behaviour of incompressible cholesteric liquid crystals are

$$(\partial W/\partial\theta')'-\partial W/\partial\theta=0,\quad (\partial W/\partial\phi')'=0, \tag{2.2}$$

where the prime denotes differentiation with respect to z. Here

$$W=W_d(\theta,\theta',\phi')+W_f(\theta), \tag{2.3}$$

where W_d represents the free energy per unit volume and is given by

$$2W_d=f(\theta)(\theta')^2+g(\theta)(\phi')^2-2k_2\tau\cos^2\theta\phi'+k_2\tau^2 \tag{2.4}$$

with

$$f(\theta)=k_1\cos^2\theta+k_3\sin^2\theta,\quad g(\theta)=\cos^2\theta(k_2\cos^2\theta+k_3\sin^2\theta). \tag{2.5}$$

In equations (2.4) and (2.5) k_1, k_2, k_3 and τ are constant material parameters. W_f is the energy per unit volume associated with the electromagnetic field and for a magnetic field

$$2W_f=-\chi_a H^2\sin^2\theta-\chi_\perp H^2, \tag{2.6}$$

where H is the constant magnetic field strength and χ_a and $\chi_\perp$ are constant material parameters. For an electric field

$$2W_f = -D^2/(\varepsilon_{\|}\sin^2\theta - \varepsilon_{\perp}\cos^2\theta), \tag{2.7}$$

D being related to the constant voltage V applied across the plates through the relation

$$V = D\int_{-l}^{l}(\varepsilon_{\|}\sin^2\theta + \varepsilon_{\perp}\cos^2\theta)^{-1}\,dz, \tag{2.8}$$

where $\varepsilon_{\|}$ and $\varepsilon_{\perp}$ are constant material parameters. It is customary to assume that

$$k_1 > 0, \quad k_2 > 0, \quad k_3 > 0 \quad \text{and} \quad \chi_a > 0. \tag{2.9}$$

We also note that, since χ_a is generally small compared with $\chi_{\perp}$, one assumes that the interaction between a magnetic field and the liquid crystal is negligible and hence the field strength is constant throughout the sample. However, as Deuling [9] discusses, ε_a defined by

$$\varepsilon_a = \varepsilon_{\|} - \varepsilon_{\perp} \tag{2.10}$$

is not small compared with $\varepsilon_{\perp}$. Hence the interaction between an electric field and the liquid crystal cannot be ignored and the field strength does not remain constant throughout the layer. For this reason the problems concerning magnetic and electric fields must be considered separately.

Employing a strong anchoring condition, where prior treatment of the bounding surfaces dictates a prescribed orientation at the plates, one seeks solutions of equations (2.2) subject to the boundary conditions

$$\left.\begin{aligned} &\theta(-l) = 0, \quad \theta(l) = 0 \quad \text{or} \quad n\pi, \\ &\phi(-l) = -\phi_0, \quad \phi(l) = \phi_0 \quad \text{or} \quad \phi_0 + m\pi, \end{aligned}\right\} \tag{2.11}$$

where m and n are integers. Given that a variety of solutions are possible, we follow Dafermos [10] and assume that the solution most likely to occur in practice is that which minimises the energy function

$$E = \int_{-l}^{l} W\,dz. \tag{2.12}$$

His experience suggests that possible solutions in which the range of θ exceeds $\pi/2$ are associated with energies larger than those whose ranges do not. In addition, it can be shown that if $\theta(z)$ is a solution then so is $-\theta(z)$. For these reasons, we simplify the analysis throughout by assuming that $\theta \in [0, \pi/2]$.

As an alternative to the strong anchoring condition, one might employ a weak anchoring or couple stress boundary condition at the plates. For definiteness, we adopt at both surfaces the relatively simple interfacial energy

$$w=A(\mathbf{v}\cdot\mathbf{n})^2+B, \qquad A>0, \tag{2.13}$$

where A and B are constants and $\mathbf{v}$ is the outward unit normal, although some generalisations are possible. With this choice, appropriate couple stress conditions are

$$g(\theta)\phi'=k_2\tau\cos^2\theta, \quad f(\theta)\theta'\pm A\sin 2\theta=0 \quad \text{on} \quad z=\pm l. \tag{2.14}$$

We now anticipate that the solution most likely to appear is that which minimises the energy function

$$\tilde{E}(\bar{\theta})=\int_{-l}^{l} W\,dz+2A\sin^2\bar{\theta}, \tag{2.15}$$

where $\bar{\theta}$ is the boundary value of θ.

3. Problem 1. Magnetic field with strong anchoring

One obvious solution of equations (2.2) which satisfies the boundary conditions (2.11) is the uniformly twisted planar configuration

$$\theta=0, \quad \phi=\phi_0 z/l, \tag{3.1}$$

and other similar possible configurations are

$$\theta=0, \quad \phi=\{(2\phi_0+m\pi)z+ml\pi\}/2l. \tag{3.2}$$

Of these solutions, (3.1) is associated with the least energy provided

$$\phi_0-\pi/4<l\tau<\phi_0+\pi/4. \tag{3.3}$$

For definiteness, we select a fixed value for ϕ_0 and given that applications employ a $\pi/2$ twist we choose ϕ_0 to be $\pi/4$. As a consequence of (3.3), τ is taken to be positive in this paper so that configurations of the form (3.2) may be ignored in the following analysis.

Apart from the basic initial alignment (3.1), non-parallel twisted distortions in which both θ and ϕ vary with z are also possible. As Leslie [**11**] discusses, equation $(2.2)_2$ integrates immediately to give

$$g(\theta)\phi'=k_2\tau\cos^2\theta+a, \tag{3.4}$$

while a suitable combination of $(2.2)_1$ and $(2.2)_2$ readily integrates to yield

$$f(\theta)(\theta')^2+g(\theta)(\phi')^2+\chi_a H^2\sin^2\theta=b, \tag{3.5}$$

where a and b are constants of integration. Symmetry considerations suggest that one examines distortions in which

$$\theta(z)=\theta(-z) \tag{3.6}$$

with

$$\theta'(o)=0, \quad \theta(o)=\theta_m, \tag{3.7}$$

where θ_m is a positive parameter to be determined. Using (3.4) and (3.7) in (3.5) and then integrating, one obtains z in the form

$$z+l=\int_0^\theta \{f(\zeta)/F(\zeta,\theta_m)\}^{\frac{1}{2}}\,d\zeta, \qquad -l\leqq z\leqq 0 \tag{3.8}$$

with

$$F(\zeta,\theta_m)\equiv\chi_a H^2(\sin^2\theta_m-\sin^2\zeta)+(a+k_2\tau\cos^2\theta_m)^2/g(\theta_m)$$
$$-(a+k_2\tau\cos^2\zeta)^2/g(\zeta). \tag{3.9}$$

From (2.11), (3.4) and (3.6), it follows that

$$\phi(z)=-\phi(-z), \quad \phi(o)=0, \tag{3.10}$$

and an integration of (3.4) now gives

$$\phi+\phi_0=\int_0^\theta \{f(\zeta)/F(\zeta,\theta_m)\}^{\frac{1}{2}}(a+k_2\tau\cos^2\zeta)/g(\zeta)\,d\zeta, \qquad -\phi_0\leqq\phi\leqq 0. \tag{3.11}$$

Equations (3.6), (3.8), $(3.10)_1$ and (3.11) give the complete solution, provided θ_m and a satisfy the relations

$$\begin{aligned} l&=\int_0^{\theta_m} \{f(\theta)/F(\theta,\theta_m)\}^{\frac{1}{2}}\,d\theta,\\ \phi&=\int_0^{\theta_m} \{f(\theta)/F(\theta,\theta_m)\}^{\frac{1}{2}}(a+k_2\tau\cos^2\theta)/g(\theta)\,d\theta. \end{aligned} \tag{3.12}$$

With the change of variable (cf. Dafermos [**10**])

$$\sin\lambda=\sin\theta/\sin\theta_m, \tag{3.13}$$

in (3.12), l and ϕ_0 are determined by the equations

$$\begin{aligned} l&=\int_0^{\pi/2} \{f(\theta)/G(\theta,\theta_m)\}^{\frac{1}{2}}/\cos\theta\,d\lambda,\\ \phi_0&=\int_0^{\pi/2} \{f(\theta)/G(\theta,\theta_m)\}^{\frac{1}{2}}(a+k_2\tau\cos^2\theta)/(g(\theta)\cos\theta)\,d\lambda, \end{aligned} \tag{3.14}$$

where

$$G(\theta,\theta_m)\equiv F(\theta,\theta_m)/(\sin^2\theta_m-\sin^2\theta). \tag{3.15}$$

Taking limits as θ_m tends to zero in (3.14) now results in a critical magnetic field strength H_c, given by

$$\chi_a H_c^2 l^2 = k_1\pi^2/4+(k_3-2k_2)\phi_0^2+2k_2\tau l\phi_0. \tag{3.16}$$

At this value a smooth transition from the configuration (3.1) to that described by (3.8) and (3.11) is possible and it follows that a real H_c exists if and only if either

$$k_3 \geqq 2k_2 \quad \text{or} \quad \phi_0 \leqq k_2\tau l[1+\{1+k_1(2k_2-k_3)/k_2^2l^2\tau^2\}^{\frac{1}{2}}]/(2k_2-k_3). \tag{3.17}$$

A necessary condition for the non-parallel distortion to appear as H exceeds H_c is

$$d(\chi_a H^2)/d\beta|_{\beta=0}>0, \qquad \beta\equiv\sin^2\theta_m, \tag{3.18}$$

and differentiation of the relations (3.14) with respect to β yields the result

$$2\chi_a l^2(dH^2/d\beta)_{\beta=0}=k_3\pi^2/4-\{p\phi_0^2+2k_2(k_3-2k_2)l\tau\phi_0+k_2^2l^2\tau^2\}/k_2, \tag{3.19}$$

where

$$p\equiv k_3^2-k_2k_3+k_2^2. \tag{3.20}$$

Hence it follows that (3.18) obtains if and only if

$$\left.\begin{aligned} &\phi_0<k_2(k_3-2k_2)l\tau(-1+\sqrt{1+q})/p, && k_3>2k_2,\\ &\phi_0<\{(\pi^2-2l^2\tau^2)/6\}^{\frac{1}{2}}, && k_3=2k_2,\\ &\phi_0<k_2(2k_2-k_3)l\tau(1+\sqrt{1+q})/p, && k_3<2k_2, \end{aligned}\right\} \tag{3.21}$$

where

$$q\equiv p\{k_3\pi^2/4k_2l^2\tau^2-1\}/(k_3-2k_2)^2, \tag{3.22}$$

provided q is positive. In the special case when

$$\phi_0=l\tau=\pi/4, \tag{3.23}$$

one deduces from (2.9), (3.16) and (3.19) that a real H_c exists and (3.18) holds provided

$$(\sqrt{17}-3)k_3<4k_2, \tag{3.24}$$

this having some relevance to applications.

The difference between the total energies E_1 and E_0 per unit area associated with the configurations described by (3.8) and (3.11) and (3.1), respectively, is given by

$$\Delta E = E_1 - E_0 = \int_0^{\pi/2} \left\{ \chi_a H^2 \beta \cos 2\lambda + \frac{a^2}{g(\theta_m)} + \frac{2k_2\tau\phi_0}{l} - \frac{k_2\phi_0^2}{l^2} + 2ak_2\tau\left(\frac{\cos^2\theta_m}{g(\theta_m)} - \frac{\cos^2\theta}{g(\theta)}\right) + k_2^2\tau^2\left(\frac{\cos^4\theta_m}{g(\theta_m)} - \frac{2\cos^4\theta}{g(\theta)}\right)\right\} \times \left\{\frac{\beta f(\theta)}{F(\theta,\theta_m)}\right\}^{\frac{1}{2}} \frac{\cos\lambda}{\cos\theta}\, d\lambda. \tag{3.25}$$

Differentiation of the relations (3.25) and (3.14) readily yields

$$\left.\frac{d(\Delta E)}{d\beta}\right|_{\beta=0} = 0, \tag{3.26}$$

and a further differentiation leads to the result

$$\left.\frac{4ld^2(\Delta E)}{d\beta^2}\right|_{\beta=0} = -k_3\pi^2/4 + \{p\phi_0^2 + 2k_2(k_3 - 2k_2)l\tau\phi_0 + k_2^2 l^2\tau^2\}/k_2. \tag{3.27}$$

One immediately observes that the conditions required to ensure (3.18) holds are identical to those required to render ΔE negative for values of θ_m in the neighbourhood of θ_m equal zero. For fixed values of l and ϕ_0, we therefore anticipate a smooth transition between the two configurations as H exceeds H_c, given by (3.16), provided

$$k_2k_3\pi^2 - 4\{p\phi_0^2 + 2k_2(k_3 - 2k_2l\tau\phi_0) + k_2^2l^2\tau^2\} > 0. \tag{3.28}$$

If $H(\theta_m)$, as defined by (3.14), increases monotonically with θ_m, one expects the non-parallel distortion to persist so long as ΔE remains negative. Finally we note that H tends to infinity as θ_m tends to $\pi/2$.

4. Problem 2. Electric field with strong anchoring

Again we are concerned with finding solutions of (2.2) subject to the conditions (2.11) and one such obvious solution is the configuration (3.1). In seeking non-parallel distortions, the relevant equations are (3.4) and

$$f(\theta)(\theta')^2 + g(\theta)(\phi')^2 - D^2/(\varepsilon_\parallel \sin^2\theta + \varepsilon_\perp \cos^2\theta) = c, \tag{4.1}$$

where c is a constant of integration. By an analysis parallel to that described in Section 3, one obtains a critical value D_c and its derivative given by

$$\varepsilon_a l^2 D_c^2/\varepsilon_\perp^2 = k_1\pi^2/4 + (k_3 - 2k_2)\phi_0^2 + 2k_2 l\tau\phi_0 \tag{4.2}$$

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and

$$(2\varepsilon_a l^2/\varepsilon_\perp^2)\frac{dD^2}{d\beta}\bigg|_{\beta=0}=3\varepsilon_a^2 l^2 D_c^2/\varepsilon_\perp^3+k_3\pi^2/4-\{p\phi_0^2+2k_2(k_3-2k_2)l\tau\phi_0+k_2^2 l^2\tau^2\}/k_2, \quad (4.3)$$

where ε_a is assumed to be positive. Utilising (2.8) in (4.2) and (4.3) yields a critical voltage V_c defined by

$$\varepsilon_a V_c^2=k_1\pi^2+4\phi_0^2(k_3-2k_2)+8k_2 l\tau\phi_0 \quad (4.4)$$

with

$$\frac{2d}{d\beta}(V^2\varepsilon_a)\bigg|_{\beta=0}=\varepsilon_a^2 V_c^2/\varepsilon_\perp+k_3\pi^2-4\{p\phi_0^2+2k_2(k_3-2k_2)l\tau\phi_0+k_2^2 l^2\tau^2\}/k_2. \quad (4.5)$$

Here

$$\Delta E=\tfrac{1}{2}\int_{-l}^{l}\{-D^2/(\varepsilon_\perp+\varepsilon_a\beta)+V^2\varepsilon_\perp l^{-2}/4+Q\}\,dz, \quad (4.6)$$

where

$$Q=\frac{a^2}{g(\theta_m)}+2ak_2\tau\left(\frac{\cos^2\theta_m}{g(\theta_m)}-\frac{\cos^2\theta}{g(\theta)}\right)+k_2^2\tau^2\left(\frac{\cos^4\theta_m}{g(\theta_m)}-\frac{2\cos^4\theta}{g(\theta)}\right)$$
$$+\frac{2k_2\tau\phi_0}{l}-\frac{k_2\phi_0^2}{l^2}, \quad (4.7)$$

and after a lengthy but straightforward calculation, one eventually obtains the results

$$\frac{d}{d\beta}(\Delta E)\bigg|_{\beta=0}=0, \quad \frac{d^2}{d\beta^2}(\Delta E)\bigg|_{\beta=0}=\frac{1}{8l}\frac{d}{d\beta}(V^2\varepsilon_a)\bigg|_{\beta=0}. \quad (4.8)$$

One notes that the condition for V to be a monotonic increasing function of θ_m in the neighbourhood of θ_m equal zero is identical to the condition for the energy associated with the non-parallel distortion to be less than that associated with the initial configuration (3.1). Hence for given values of l and ϕ_0, we expect a smooth transition between the two configurations as V exceeds V_c, given by (4.4), provided

$$k_2k_3\pi^2-4\{p\phi_0^2+2k_2(k_3-2k_2)l\tau\phi_0+k_2^2\tau^2 l^2\}+k_2\varepsilon_a^2 V_c^2/\varepsilon_\perp>0. \quad (4.9)$$

This condition is obviously less or more restrictive than (3.28) depending on whether $\varepsilon_\perp$ is positive or negative.

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5. Problem 3. Magnetic field with weak anchoring

The uniformly twisted planar configuration

$$\theta = 0, \quad \phi' = \tau \tag{5.1}$$

is one obvious solution of (2.2) subject to the conditions (2.14). In addition, non-parallel distortions of the form described by (3.6), (3.7) and (3.10) are also possible. Assuming

$$\theta(\pm l) = \bar{\theta} \quad \text{and} \quad \phi(\pm l) = \pm \bar{\phi}, \tag{5.2}$$

it can be shown that θ_m and $\bar{\theta}$ are related through the equation

$$k_2^2\tau^2\{\cos^4\theta_m/g(\theta_m) - \cos^4\bar{\theta}/g(\bar{\theta})\} + \chi_a H^2(\sin^2\theta_m - \sin^2\bar{\theta}) = A^2\sin^2 2\bar{\theta}/f(\bar{\theta}) \tag{5.3}$$

and for small values of θ_m

$$\bar{\theta}^2 = (\chi_a H^2 - k_3\tau^2)\theta_m^2/(\chi_a H^2 - k_3\tau^2 + 4A^2/k_1) + O(\theta_m^4). \tag{5.4}$$

Two distinct types of solution must be investigated. In one

$$\chi_a H^2 - k_3\tau^2 > 0 \quad \text{and} \quad \pi/2 \geqq \theta_m \geqq \bar{\theta} \geqq 0, \tag{5.5}$$

while in the other

$$k_3\tau^2 - \chi_a H^2 - 4A^2/k_1 > 0 \quad \text{and} \quad \pi/2 \geqq \bar{\theta} \geqq \theta_m \geqq 0. \tag{5.6}$$

If (5.5) holds, a possible solution is again determined by (3.6), (3.8), (3.10) and (3.11) provided one sets a identically equal to zero and replaces the lower limit in the integrals by $\bar{\theta}$. With the change of variable (3.13), one finds

$$l = \int_{\bar{\lambda}}^{\pi/2} \left\{ \frac{k_1(1+m_1\beta\sin^2\lambda)(1+m_2\beta\sin^2\lambda)(1+m_2\beta)}{\chi_a H^2(1+m_2\beta\sin^2\lambda)(1+m_2\beta) - k_3\tau^2} \right\}^{\frac{1}{2}} \frac{d\lambda}{(1-\beta\sin^2\lambda)^{\frac{1}{2}}}, \tag{5.7}$$

where

$$\sin\bar{\lambda} = \sin\bar{\theta}/\sin\theta_m; \; m_i = k_3/k_i - 1, \qquad (i = 1, 2). \tag{5.8}$$

Taking the limit as β tends to zero in (5.7) yields a critical magnetic field strength H_{1c} defined by

$$\chi_a H_{1c}^2 = k_3\tau^2 + k_1(\pi/2 - \bar{\lambda}_c)^2/l^2, \; \bar{\lambda}_c = \sin^{-1}\{(\chi_a H_{1c}^2 - k_3\tau^2)/(4A^2/k_1 + \chi_a H_{1c}^2 - k_3\tau^2)\}^{\frac{1}{2}}. \tag{5.9}$$

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Differentiation of (5.7) with respect to β leads to the result that (3.18) is valid if and only if

$$\frac{\sin 2\bar{\lambda}_c}{2}\left(\frac{k_3}{k_1}-\frac{3k_3\tau^2 m_2}{\chi_a H_{1c}^2-k_3\tau^2}\right)\left(\frac{\pi}{4}-\frac{\bar{\lambda}_c}{2}\right)+\frac{\sin^2 2\bar{\lambda}_c}{8}\left(\frac{k_3}{k_1}-\frac{k_3\tau^2 m_2}{\chi_a H_{1c}^2-k_3\tau^2}\right)$$
$$+\frac{(k_3 m_2\tau^2-4k_3A^2/k_1^2)(\chi_a H_{1c}^2-k_3\tau^2)^2}{(4A^2/k_1+\chi_a H_{1c}^2-k_3\tau^2)^3}-\frac{k_3 m_2\tau^2}{(4A^2/k_1+\chi_a H_{1c}^2-k_3\tau^2)}>0. \tag{5.10}$$

The energy difference per unit area $\Delta\tilde{E}$ between the two solutions is now

$$\Delta\tilde{E}=\tilde{E}(\bar{\theta})-\tilde{E}(0)=\int_{\bar{\lambda}}^{\pi/2}\left\{\chi_a H^2\beta\cos 2\lambda+k_2^2\tau^2\left(\frac{\cos^4\theta_m}{g(\theta_m)}-\frac{2\cos^4\theta}{g(\theta)}+k_2^{-1}\right)\right\}$$
$$\times\left\{\frac{\beta f(\theta)}{F(\theta,\theta_m)}\right\}^{\frac{1}{2}}\frac{\cos\lambda\, d\lambda}{\cos\theta}+2A\sin^2\bar{\theta}, \tag{5.11}$$

and differentiating this equation twice with respect to β yields the results

$$\left.\frac{d(\Delta\tilde{E})}{d\beta}\right|_{\beta=0}=0 \quad\text{and}\quad \left.\frac{d^2(\Delta\tilde{E})}{d\beta^2}\right|_{\beta=0}=\int_{\bar{\lambda}_c}^{\pi/2}\frac{d^2Q_1}{d\beta^2}d\lambda-2\left.\left(\frac{dQ_1}{d\beta}\frac{d\bar{\lambda}}{d\beta}\right)\right|_{\beta=0}+2A\left.\frac{d^2(\sin^2\bar{\theta})}{d\beta^2}\right|_{\beta=0}, \tag{5.12}$$

where

$$Q_1=k_1^{\frac{1}{2}}\beta\frac{\{[\chi_a H^2(1+m_2\beta)(1+m_2\beta\sin^2\lambda)-k_3\tau^2]\cos 2\lambda+k_3 m_2\tau^2\beta\sin^2\lambda\}(1+m_1\beta\sin^2\lambda/2)(1+\beta\sin^2\lambda/2)}{\{\chi_a H^2(1+m_2\beta)(1+m_2\beta\sin^2\lambda)-k_3\tau^2\}^{\frac{1}{2}}(1+m_2\beta)^{\frac{1}{2}}(1+m_2\beta\sin^2\lambda)^{\frac{1}{2}}}. \tag{5.13}$$

For fixed l, one thus expects a smooth transition between configurations as H exceeds H_{1c} provided (5.5) and (5.10) are satisfied and the right-hand side of $(5.12)_2$ is negative, these results agreeing with those of Leslie [11] in the limit as A tends to infinity. If H increases monotonically with θ_m, the non-parallel distortion persists so long as $\Delta\tilde{E}$ remains negative and (5.3) yields the result

$$\lim_{\theta_m\to\pi/2}(\cos\bar{\theta}/\cos\theta_m)=\{(\chi_a H_2^2-k_2^2\tau^2/k_3)/(\chi_a H_2^2-k_2^2\tau^2-4A^2/k_3)\}^{\frac{1}{2}},$$
$$H_2=\lim_{\theta_m\to\pi/2}H(\theta_m), \tag{5.14}$$

where $\bar{\theta}$ tends to $\pi/2$ with θ_m. Obviously a necessary condition for this to happen is

$$k_3\chi_a H_2^2>k_2^2\tau^2+4A^2. \tag{5.15}$$

Employing the change of variable

$$\cosh\lambda = \cos\theta/\cos\theta_m \tag{5.16}$$

in the appropriate form of $(3.12)_1$, one can show that H_2 is defined by

$$l = \frac{k_3^{\frac{1}{2}}}{(\chi_a H_2^2 k_3 - k_2^2\tau^2)^{\frac{1}{2}}} \cosh^{-1}\{(\chi_a H_2^2 k_3 - k_2^2\tau^2)/(\chi_a H_2^2 k_3 - k_2^2\tau^2 - 4A^2)\}^{\frac{1}{2}}. \tag{5.17}$$

Thus the director becomes everywhere perpendicular to the plates as H exceeds the finite value H_2, provided the appropriate conditions are satisfied. This phenomenon is of interest, since it is clearly impossible when there is strong anchoring at the plates, as first noted in a simpler context by Nehring, Kmetz and Scheffer [12].

If (5.6) obtains, a possible non-parallel solution is

$$z = \int_{\theta_m}^{\theta} \{f(\zeta)/F(\zeta,\theta_m)\}^{\frac{1}{2}}\,d\zeta, \qquad 0 \leqq z \leqq l, \tag{5.18}$$

with

$$l = \int_{\theta_m}^{\bar\theta} \{f(\zeta)/F(\zeta,\theta_m)\}^{\frac{1}{2}}\,d\zeta, \tag{5.19}$$

where ϕ and $\bar\phi$ are found by using (5.18) in (3.4) together with the conditions $(2.14)_1$. With the change of variable

$$\cosh\lambda = \sin\theta/\sin\theta_m, \tag{5.20}$$

(5.19) becomes

$$l = \int_0^{\bar\lambda_1} \frac{f^{\frac{1}{2}}(\theta)}{\cos\theta}\{-\chi_a H^2 + k_2^2 k_3 \tau^2 \cos^4\theta\cos^4\theta_m/g(\theta)g(\theta_m)\}^{-\frac{1}{2}}\,d\lambda,$$

$$\bar\lambda_1 = \cosh^{-1}(\sin\bar\theta/\sin\theta_m), \tag{5.21}$$

and hence the critical relationship between l and H is found to be

$$l = \left\{\frac{k_1}{k_3\tau^2 - \chi_a H^2}\right\}^{\frac{1}{2}} \cosh^{-1}\left\{\frac{\chi_a H^2 - k_3\tau^2}{\chi_a H^2 - k_3\tau^2 + 4A^2/k_1}\right\}^{\frac{1}{2}}. \tag{5.22}$$

However, since one can show here that

$$\frac{d}{d\beta}(\Delta\tilde{E})\bigg|_{\beta=0} > 0, \tag{5.23}$$

the non-parallel configuration is not expected to appear in preference to the initial alignment (5.1) and so such solutions may be ignored.

6. Problem 4. Electric field with weak anchoring

The initial alignment (5.1) is obviously one solution of (2.2) subject to (2.14). In seeking non-parallel solutions of the form described above, it follows from the field equations (3.4) and (5.1) and the boundary conditions that $\bar{\theta}$ and θ_m must satisfy the relationship

$$k_2^2\tau^2\left\{\frac{\cos^4\theta_m}{g(\theta_m)}-\frac{\cos^4\bar{\theta}}{g(\bar{\theta})}\right\}+\frac{D^2\varepsilon_a(\sin^2\theta_m-\sin^2\bar{\theta})}{(\varepsilon_\perp+\varepsilon_a\sin^2\bar{\theta})(\varepsilon_\perp+\varepsilon_a\sin^2\theta_m)}=\frac{A^2\sin^2 2\bar{\theta}}{f(\bar{\theta})}. \tag{6.1}$$

Again two types of configuration must be considered. If

$$\varepsilon_a D^2/\varepsilon_\perp^2-k_3\tau^2>0 \quad \text{and} \quad \pi/2\geqq\theta_m\geqq\bar{\theta}\geqq 0, \tag{6.2}$$

a parallel analysis to that in Section 5 leads to a critical voltage V_{1c} given by

$$\varepsilon_a V_{1c}^2/4=k_3\tau^2 l^2+k_1(\pi/2-\bar{\lambda}_{2c})^2, \tag{6.3}$$

where

$$\bar{\lambda}_{2c}=\sin^{-1}\{(D_{1c}^2\varepsilon_a/\varepsilon_\perp^2-k_3\tau^2)/(D_{1c}^2\varepsilon_a/\varepsilon_\perp^2-k_3\tau^2+4A^2/k_1)\}^{\frac{1}{2}},\ D_{1c}^2=V_{1c}^2\varepsilon_\perp^2/4l^2. \tag{6.4}$$

One also finds that

$$\begin{aligned}
&\left\{\frac{4A^2}{k_1M}+\left(\frac{\pi}{2}-\bar{\lambda}_{2c}\right)\right\}\frac{d}{d\beta}\left(\frac{V^2\varepsilon_a}{4l^2}\right)\Bigg|_{\beta=0}\\
&=\frac{M\sin 2\bar{\lambda}_{2c}}{2}\left\{\frac{k_3}{k_1}+3\left(\frac{D_{1c}^2\varepsilon_a^2}{\varepsilon_\perp^3}-k_3m_2\tau^2\right)\Big/\left(\frac{D_{1c}^2\varepsilon_a}{\varepsilon_\perp^2}-k_3\tau^2\right)\right\}\left(\frac{\pi}{4}-\frac{\bar{\lambda}_{2c}}{2}\right)\\
&+\frac{M\sin^2 2\bar{\lambda}_{2c}}{8}\left\{\frac{k_3}{k_1}+\left(\frac{D_{1c}^2\varepsilon_a^2}{\varepsilon_\perp^3}-k_3m_2\tau^2\right)\Big/\left(\frac{D_{1c}^2\varepsilon_a}{\varepsilon_\perp^2}-k_3\tau^2\right)\right\}\\
&+\frac{1}{M^2}\left(k_3m_2\tau^2-\frac{4A^2k_3}{k_1^2}-\frac{D_{1c}^2\varepsilon_a^2}{\varepsilon_\perp^3}\right)\left(\frac{D_{1c}^2\varepsilon_a}{\varepsilon_\perp^2}-k_3\tau^2\right)\\
&+\left(\frac{D_{1c}^2\varepsilon_a^2}{\varepsilon_\perp^3}-k_3m_2\tau^2\right)-\left\{\frac{4A^2}{k_1M}+\left(\frac{\pi}{2}-\bar{\lambda}_{2c}\right)\right\}\frac{D_{1c}^2\varepsilon_a^2}{\varepsilon_\perp^3}\left(1+\frac{\sin 2\bar{\lambda}_{2c}}{(\pi-2\bar{\lambda}_{2c})}\right),
\end{aligned} \tag{6.5}$$

where

$$M\equiv D_{1c}^2\varepsilon_a/\varepsilon_\perp^2-k_3\tau^2+4A^2/k_1. \tag{6.6}$$

The energy difference per unit area is now given by

$$\Delta\tilde{E}=\int_{\bar{\lambda}_2}^{\pi/2} Q_2 P\,d\lambda+2A\sin^2\bar{\theta}, \tag{6.7}$$

where

$$Q_2=k_2\tau^2\left\{\frac{\cos^4\theta_m}{g(\theta_m)}-\frac{2\cos^4\theta}{g(\theta)}+1\right\}-\frac{D^2}{\varepsilon_\perp+\varepsilon_a\beta}+\frac{V^2\varepsilon_\perp}{4l^2} \tag{6.8}$$

and

$$P=\left\{\frac{f(\theta)[(\varepsilon_\perp+\varepsilon_a\beta)(\varepsilon_\perp+\varepsilon_a\sin^2\lambda\beta)(1+m_2\beta)(1+m_2\beta\sin^2\lambda)]}{\cos\theta[D^2\varepsilon_a(1+m_2\beta)(1+m_2\sin^2\lambda\beta)-k_3\tau^2(\varepsilon_\perp+\varepsilon_a\beta)(\varepsilon_\perp+\varepsilon_a\sin^2\lambda\beta)]}\right\}^{\frac{1}{2}}. \tag{6.9}$$

Differentiating (6.7) twice with respect to β now yields $(5.12)_1$ and

$$\left.\frac{d^2(\Delta\tilde{E})}{d\beta^2}\right|_{\beta=0}=\int_{\bar{\lambda}_{2c}}^{\pi/2}\left\{\left(\frac{k_1}{D_{1c}^2\varepsilon_a/\varepsilon_\perp^2-k_3\tau^2}\right)^{\frac{1}{2}}\left.\frac{d^2Q_2}{d\beta^2}\right|_{\beta=0}+2\left.\left(\frac{dQ_1}{d\beta}\frac{dP}{d\beta}\right)\right|_{\beta=0}\right\}d\lambda$$
$$-\left(\frac{2k_1}{D_{1c}^2\varepsilon_a/\varepsilon_\perp^2-k_3\tau^2}\right)\left.\frac{dQ_1}{d\beta}\right|_{\beta=0}\left.\frac{d\bar{\lambda}_2}{d\beta}\right|_{\beta=0}+2A\left.\frac{d^2}{d\beta^2}(\sin^2\bar{\theta})\right|_{\beta=0}. \tag{6.10}$$

We therefore expect a smooth transition to occur as V exceeds V_{1c} provided the right-hand sides of (6.4) and (6.10) are positive and negative, respectively. If V increases monotonically with θ_m and $\Delta\tilde{E}$ remains negative, one can show that the liquid crystal aligns everywhere perpendicular to the plates as V exceeds V_2 defined by

$$l=\frac{2k_3^{\frac{1}{2}}}{(V_2^2\varepsilon_a-4k_2^2\tau^2l^2/k_3)^{\frac{1}{2}}}\cosh^{-1}\left\{\frac{\varepsilon_aV_2^2-4k_2^2\tau^2l^2/k_3}{\varepsilon_aV_2^2-4k_2^2\tau^2l^2/k_3-16A^2l^2/k_3}\right\}^{\frac{1}{2}}. \tag{6.11}$$

Of course a necessary condition for this to occur is

$$\varepsilon_aV_2^2>\frac{4l^2}{k_3}(k_2^2\tau^2+4A^2). \tag{6.12}$$

If

$$k_3\tau^2-\varepsilon_aD^2/\varepsilon_\perp^2-4A^2/k_1>0\quad\text{and}\quad \pi/2\geqq\bar{\theta}\geqq\theta_m\geqq0, \tag{6.13}$$

the condition (5.23) again results and so we disregard this case.

7. Concluding remarks

This paper presents analyses for several bifurcation problems that are of interest in the theory of cholesteric liquid crystals. Although one must admit that only a restricted class of non-parallel distortions are considered, we note that results obtained by similar analyses in the theory of nematic liquid crystals are found to agree well with experimental observations. While an examination of more general types of solution is clearly desirable, such an investigation is beyond the scope of this paper.

In common with other investigations, we employ a static stability criterion to discriminate between several possible equilibrium configurations. Apart from a preliminary analysis in a simpler context by Straughan [13], no attempt has been made to solve such stability problems using the dynamic equations. While it is clearly desirable to have the dynamic results, one would equally want the above results for comparison.

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