CONTINUOUS SUMS OF MEASURES AND CONTINUOUS SPECTRA

by S. SANKARAN

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1. Introduction. Von Neumann's definition of the continuous sum of Hilbert spaces led Segal [3] to define the continuous sum of measures on a measurable space. In this note we employ Segal's definition to investigate the measure structures associated with a self-adjoint transformation of pure point spectrum and a self-adjoint transformation of pure continuous spectrum. While these transformations, as operators on separable Hilbert spaces, are the antithesis of each other we show that in their measure structure one is a particular case of the other.

In Theorem 2 we show that to every self-adjoint transformation T there corresponds a simple self-adjoint transformation A such that T has pure point (resp. pure continuous) spectrum if and only if A has pure point (resp. pure continuous) spectrum. This shows that it is enough to consider simple self-adjoint transformations in the proof of the Main Theorem. This theorem asserts that, if T is a self-adjoint transformation defined in a Hilbert space $\mathbf{H}$, and $E(\lambda)$ the resolution of the identity corresponding to T, then there exists an element z in $\mathbf{H}$ such that a necessary and sufficient condition for T to have pure point (resp. pure continuous) spectrum is that the measure μ defined by the function $\| E(\lambda)z \|^2$ is the discrete (resp. continuous) sum of mutually disjoint measures of point mass.

In this paper, the term " Hilbert space " stands for " complex separable Hilbert space "; if $\mathbf{S}$ is a set of everywhere defined operators in a Hilbert space $\mathbf{H}$, and $w \in \mathbf{H}$, the closed linear manifold generated by the set $(Aw\colon\ A \in \mathbf{S})$ is denoted by $[Aw\colon\ A \in \mathbf{S}]$. If μ and ν are measures on a measure space, we write $\mu \gg \nu$ (or $\nu \ll \mu$) to denote that ν is absolutely continuous with respect to μ.

2. Preliminaries.

DEFINITION 1. Let X be a locally compact Hausdorff space, and $\mathbf{B}$ be the σ-ring generated by the open subsets of X. The members of $\mathbf{B}$ are called the *Borel sets* of X, and the pair $(X, \mathbf{B})$ is called a *Borel space*. A non-negative function μ of the Borel sets of X is called a *measure* of the Borel space if μ has the property $\mu(\bigcup_n B_n) = \sum_n \mu(B_n)$, where $B_n \cap B_m = \emptyset$ if $n \neq m$. $(X, \mathbf{B}, \mu)$ is called a *Borel measure space.*

Let $(X, \mathbf{B})$ be a Borel space, and $(Y, \mathbf{D}, \nu)$ a Borel measure space. Let μ_n $(n = 1, 2, \ldots)$ and μ_y $(y \in Y)$ be measures of $(X, \mathbf{B})$.

DEFINITION 2. A measure μ of $(X, \mathbf{B})$ is said to be the *discrete sum* of the measures μ_n, if, for each $B \in \mathbf{B}$, $\mu(B) = \sum_n \mu_n(B)$. The measure μ is said to be the *continuous sum* of the measures μ_y if

(i) for each $B \in \mathbf{B}$, the function $b(y) = \mu_y(B)$ is integrable with respect to v, and

(ii) $\mu(B) = \int_Y \mu_y(B)\, dv(y)$.

(See [3], Definition 8.1.)

Let $\mathbf{H}$ be a complex separable Hilbert space.

Definition 3. A mapping P of the Borel sets of a Borel space $(X, \mathbf{B})$ into the set of projections of $\mathbf{H}$ is called a *projection-valued measure* if

(i) $P_\emptyset = 0$, $P_X = I$, where I is the identity operator of $\mathbf{H}$,

(ii) $P_{B_1 \cap B_2} = P_{B_1} P_{B_2}$,

(iii) $P_{\bigcup_n B_n} = \sum_n P_{B_n}$, where $B_n \cap B_m = \emptyset$ if $n \neq m$.

If P is a projection-valued measure of a Borel space $(X, \mathbf{B})$, then to each element of $\mathbf{H}$ there corresponds a measure of $(X, \mathbf{B})$; for, if $z \in \mathbf{H}$, then μ_z, where $\mu_z(B) = \| P_B z \|^2$, is a measure of $(X, \mathbf{B})$.

Let T be a self-adjoint transformation defined in $\mathbf{H}$, and let $E(\lambda)$ be the resolution of the identity corresponding to T.

Definition 4. T is said to have *pure point spectrum* if $\mathbf{H}$ contains a complete orthonormal set of characteristic elements of T. T is said to have *pure continuous spectrum* if $\mathbf{H}$ contains no non-zero characteristic element of T. T is said to be *simple* if there exists an element z in $\mathbf{H}$ such that $[E(\lambda)z\colon -\infty \leqq \lambda \leqq \infty] = \mathbf{H}$.

Theorem 1. *Let P be a projection-valued measure of a Borel space $(X, \mathbf{B})$ to a Hilbert space $\mathbf{H}$. There exists an element z in $\mathbf{H}$ with the property that $\mu_z(B) = 0$ if and only if $P_B = 0$, where $\mu_z(B) = \| P_B z \|^2$.*

Proof. Let $\mathfrak{U}$ be the von Neumann algebra (i.e., weakly closed self-adjoint algebra) generated by the set $S = (P_B\colon B \in \mathbf{B})$ of projections in $\mathbf{H}$. It follows from (ii) of Definition 3 that the members of S commute with each other, and therefore the members of $\mathfrak{U}$ commute with each other. We recall the definition of ordered additive decomposition [2] of $\mathbf{H}$ relative to the Abelian von Neumann algebra $\mathfrak{U}$:

(i) $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 + \ldots + \mathbf{H}_n + \ldots$,

where

(ii) $\mathbf{H}_n = [Az_n\colon A \in \mathfrak{U}]$

and

(iii) $\mu_{z_1} \geqq \mu_{z_2} \geqq \cdots \geqq \mu_{z_n} \geqq \cdots$.

Let $z = z_1$. Assume that $\mu_z(B) = 0$. It follows from (iii) that $\| P_B z_n \|^2 = \mu_{z_n}(B) = 0$ for $n = 2, 3, \ldots$. Hence $P_B A z_n = A P_B z_n = 0$ for each n and all $A \in \mathfrak{U}$. That is, $P_B w = 0$ when $w = Az_n$. The set $(w = Az_n\colon A \in \mathfrak{U})$ is dense in $\mathbf{H}_n$. Hence $P_B \mathbf{H}_n = 0$. It follows from (i) that $P_B = 0$.

Conversely, $P_B = 0$ implies that $\mu_z(B) = \| P_B z \|^2 = 0$.
This proves the theorem.

COROLLARY 1. *For every element* $w \in \mathbf{H}$, $\mu_w \ll \mu_z$.
For $\mu_z(B) = 0$ implies that $P_B = 0$; hence $\mu_w(B) = \| P_B w \|^2 = 0$.

THEOREM 2. *Let T be a self-adjoint transformation in* $\mathbf{H}$ *and* $E(\lambda)$ *the resolution of the identity corresponding to T. There exists an element z in* $\mathbf{H}$ *such that the transformation* $A = TE_1$ *is simple and has pure point (resp. pure continuous) spectrum if and only if T has pure point (resp. pure continuous) spectrum, where E_1 is the projection of* $\mathbf{H}$ *on* $[E(\lambda)z\colon -\infty \leqq \lambda \leqq \infty]$.

Proof. Let X be the extended real line $(x\colon -\infty \leqq x \leqq \infty)$ with the usual topology and $\mathbf{B}$ the set of all Borel subsets of X. $\mathbf{B}$ is the σ-ring generated by the bounded semi-closed intervals $[a, b) = (x\colon a \leqq x < b)$. (See [1], § 15, Theorem B).

It is an easy consequence of the spectral theorem that every self-adjoint transformation defines a projection-valued measure on the Borel space $(X, \mathbf{B})$; for, if $B = (x\colon a \leqq x < b)$, let $E(B) = E(b) - E(a)$. From the last paragraph it is obvious that the mapping $B \to E(B)$ can be extended to all members of $\mathbf{B}$, and that the extended mapping $B \to E(B)$ is a projection-valued measure.

We can find an element z in $\mathbf{H}$ with the property that $\mu_z(B) = \| E(B)z \|^2 = 0$ if and only if $E(B) = 0$. Let

$$\mathbf{H}_1 = [E(B)z\colon B \in \mathbf{B}] = [E(\lambda)z\colon -\infty \leqq \lambda \leqq \infty],$$

and let E_1 be the projection of $\mathbf{H}$ onto $\mathbf{H}_1$. Since $E_1 E(\lambda) = E(\lambda)E_1$, for all λ $(-\infty \leqq \lambda \leqq \infty)$, it follows that $TE_1 = E_1 T$. Hence the transformation

$$A = TE_1 \quad (= E_1 T = E_1 T E_1)$$

is self-adjoint in $\mathbf{H}_1$ and its resolution of the identity $F(\lambda)$ is $E(\lambda)E_1$. The transformation A is simple; for $F(\lambda)z = E(\lambda)E_1 z = E(\lambda)z$ (z being an element of $\mathbf{H}_1$) implies that

$$[F(\lambda)z\colon -\infty \leqq \lambda \leqq \infty] = [E(\lambda)z\colon -\infty \leqq \lambda \leqq \infty] = \mathbf{H}_1.$$

Now assume that A has pure point spectrum and that $B = \{\lambda_1, \lambda_2, \ldots\}$ are the points of the point spectrum of A. If $Aw_n = \lambda_n w_n$, with $w_n \neq 0$, it follows from

$$T(E_1 w_n) = TE_1 w_n = Aw_n = \lambda_n w_n = \lambda_n (E_1 w_n)$$

that λ_n is a point of the point spectrum of T. Let $\mathbf{M}_n$ be the characteristic manifold of T for the characteristic value λ_n. We shall show that $\sum_n \oplus \mathbf{M}_n = \mathbf{H}$, which would prove that T has pure point spectrum. Choose $w \in \mathbf{H} \ominus \sum_n \oplus \mathbf{M}_n$. Since

$$\mu_z(X-B) = \| E(X-B)z \|^2 = \| E(X-B)E_1 z \|^2 = \| F(X-B)z \|^2 = 0,$$

it follows from Corollary 1 that $\mu_w(X-B) = 0$. Since w is orthogonal to $\mathbf{M}_n$ and $E(\{\lambda_n\})$ is the projection of $\mathbf{H}$ on $\mathbf{M}_n$, it follows that

$$\mu_w(\{\lambda_n\}) = \| E(\{\lambda_n\})w \| = 0 \quad (n = 1, 2, \ldots, N).$$

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Hence $\mu_w(B) = 0$, and therefore

$$\| w \|^2 = \| E(X)w \|^2 = \mu_w(X) = \mu_w(B) + \mu_w(X-B) = 0.$$

It is easy to verify that, if T has pure point spectrum, then $A = TE_1$ has pure point spectrum.

Finally, we observed earlier that $\mu_z(B) = \| F(B)z \|^2$. Now, A being a simple self-adjoint transformation, a point λ is in the point spectrum of A (resp. T) if and only if $\mu_z(\{\lambda\})$ (resp. $E(\{\lambda\})) \neq 0$. But from the choice of z we know that $\mu_z(\{\lambda\}) \neq 0$ if and only if $E(\{\lambda\}) \neq 0$. Hence λ is in the point spectrum of A if and only if λ is in the point spectrum of T. Equivalently, A has pure continuous spectrum if and only if T has pure continuous spectrum.

This proves the theorem.

Corollary 2. *The self-adjoint transformation A of Theorem 2 is unique up to unitary equivalence.*

For, let A_1, A_2 be two simple self-adjoint transformations satisfying the condition of Theorem 2; let z_1, z_2 be the elements which define A_1, A_2 respectively. It is easy to show that $\mu_{z_1}(B) = 0$ if and only if $\mu_{z_2}(B) = 0$. Hence A_1 and A_2 are unitarily equivalent.

3. Main Theorem. *Let T be a self-adjoint transformation defined in a Hilbert space* **H**, *$E(\lambda)$ the resolution of the identity corresponding to T, and z an element in* **H** *possessing the properties of Theorem 2. Then a necessary and sufficient condition for T to have pure point (resp. pure continuous) spectrum is that the measure μ defined by the function $\| E(\lambda)z \|^2$ is the discrete (resp. continuous) sum of mutually disjoint measures of point mass.*

Proof. As we pointed out in the introduction, there is no loss of generality in assuming that T is a simple self-adjoint transformation and that $[E(\lambda)z\colon -\infty \leqq \lambda \leqq \infty] = \mathbf{H}$. The proof depends on certain known results concerning simple self-adjoint transformations, which we list here.

Let μ be the measure on the Borel subsets of the real line defined by the monotone increasing function $\| E(\lambda)z \|^2$.

(1) ([4], Definition 7.2 and Theorem 7.9.) If λ_0 is a point of the point spectrum of T, then there exists an $\varepsilon > 0$ such that the range of $E(\Delta)$ is a one-dimensional manifold in **H**, where $\Delta = (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$.

(2) ([4], Theorem 7.16.) A non-zero closed linear manifold **M** is an invariant subspace for T if and only if **M** is isometric to a subspace **M**′ of $L^2(\mu)$ consisting of the functions which vanish outside a Borel set B of positive measure; in particular, if **M** is the characteristic manifold corresponding to a characteristic value λ_0, then $B = \{\lambda_0\}$.

(3) T has pure continuous spectrum if and only if μ is absolutely continuous with respect to Lebesgue measure. For, if T has a point λ_0 in the point spectrum, then $\mu(\{\lambda_0\}) > 0$. Since the Lebesgue measure of $\{\lambda_0\}$ is zero, it follows that μ is not absolutely continuous with respect to Lebesgue measure. Conversely, if T has pure continous spectrum, then the monotone increasing function $f(\lambda) = \| E(\lambda)z \|^2$ is continuous. It follows from Lemma 7.1 of [4] that μ is the Lebesgue measure on $0 \leqq y \leqq r = \| z \|^2$.

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Pure point case.

The condition is necessary. Assume that (w_n) is a complete orthonormal set of characteristic elements; let w_n correspond to the characteristic value λ_n. We know from (1) that it is possible to choose $\varepsilon_n > 0$, such that the range of $E(\Delta_n)$ is generated by w_n, where

$$\Delta_n = (\lambda_n - \varepsilon_n, \lambda_n + \varepsilon_n).$$

We may assume that $\Delta_n \cap \Delta_m = \emptyset$ for $n \neq m$. If $\mu(B) > 0$, where $B \subseteq \Delta_n - \{\lambda_n\}$, then it follows from (2) that T has a non-zero invariant subspace contained in the range of $E(\Delta_n)$ and orthogonal to w_n, which is impossible. Hence $\mu(\Delta_n) = \mu(\{\lambda_n\})$. Now suppose that $\Delta \subseteq [-\infty, \infty] - \bigcup_{n=1}^{\infty} \Delta_n$ is a Borel set. Then $E(\Delta)E(\Delta_n) = 0$ for every n. Since (w_n) is complete, it follows that $E(\Delta) = 0$, and therefore $\mu(\Delta) = 0$. Hence, for every Borel set B, we have

$$\mu(B) = \mu\left(\left(\bigcup_{n=1}^{\infty} \Delta_n\right) \cap B\right) = \mu\left(\bigcup_{n=1}^{\infty} (\Delta_n \cap B)\right)$$

$$= \sum_{n=1}^{\infty} \mu(\Delta_n \cap B) = \sum_{n=1}^{\infty} \mu_n(B).$$

It is clear that the μ_n have point mass, and are mutually disjoint.

The condition is sufficient. Assume that $\mu = \mu_n$, where μ_n, has its mass at λ_n. Since $\mu(\{\lambda_n\}) \neq 0$, by a known theorem (see [1], pp. 178–182), the points λ_n are points of discontinuity for the function $\| E(\lambda) z_0 \|^2$. Hence ([4], p. 184) the points λ_n are in the point spectrum of T. Let $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ and let $\mathbf{N} = \mathbf{H} \ominus \mathbf{M}$, $\mathbf{M}$ being the characteristic manifold of T. Since $\mathbf{N}$ is an invariant subspace for T, there is, by (2), a Borel set B which corresponds to it. Since $\mathbf{N}$ is orthogonal to $\mathbf{M}$, B is contained in $[-\infty, \infty] - \Lambda$, and consequently $\mu(B) = \sum \mu_n(B) = 0$. Hence $\mathbf{N} = 0$. That is, $\mathbf{M} = \mathbf{H}$, and T has pure point spectrum.

Pure continuous case.

The condition is necessary. Let the spectrum of T be purely continuous. Since μ is then absolutely continuous with respect to Lebesgue measure, we have, by the Radon-Nikodym theorem,

$$\mu(B) = \int_B f(y)\, dy,$$

where $f(y)$ is a non-negative Lebesgue integrable function. For each real number y, define a function μ_y on the Borel subsets of the real line as follows:

$$\mu_y(B) = f(y)\chi_B(y),$$

where B is an arbitrary Borel subset of the real line, and

$$\chi_B(y) = \begin{cases} 1 & \text{if } y \in B, \\ 0 & \text{if } y \notin B. \end{cases}$$

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It is easy to verify that, for each y, μ_y is a measure on the Borel subsets of the real line, and that the mass of μ_y is at y. Finally,

$$\mu(B) = \int_B f(y)\, dy = \int_{-\infty}^{\infty} f(y)\chi_B(y)\, dy = \int_{-\infty}^{\infty} \mu_y(B)\, dy,$$

and this shows that μ is the continuous sum of the measures μ_y.

The condition is sufficient. Let μ be the continuous sum, with respect to Lebesgue measure, of the measures μ_y, y being real. Assume that μ_y has its mass at y. Since the measures μ_y are mutually disjoint, the measures μ_y and μ_s, for $y \neq s$, do not have their mass at the same point. Consequently, the function $f(y) = \mu_y(X)$, where X is the real line, is well defined. From property (i) of Definition 2, it follows that $f(y)$ is integrable with respect to Lebesgue measure. Finally, observing that

$$\mu_y(X) = \mu_y(\{y\}) = \mu_y(B) \quad \text{if} \quad y \in B,$$

we see that

$$\mu(B) = \int_{-\infty}^{\infty} \mu_y(B)\, dy = \int_B \mu_y(X)\, dy = \int_B f(y)\, dy.$$

Hence μ is absolutely continuous with respect to Lebesgue measure; it follows from (3) that T has pure continuous spectrum.

This completes the proof.

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Queen Elizabeth College
London