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THE NON-ABELIAN TENSOR PRODUCT OF FINITE GROUPS IS FINITE: A HOMOLOGY-FREE PROOF

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Abstract. In this note, we give a homology-free proof that the non-abelian tensor product of two finite groups is finite. In addition, we provide an explicit proof that the non-abelian tensor product of two finite p-groups is a finite p-group.

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1. Introduction. R. Brown and J.-L. Loday introduced the non-abelian tensor product $G \otimes H$ for a pair of groups G and H in [**1**, **2**] in the context of an application in homotopy theory. It is defined for a pair of groups that act on each other provided the actions satisfy the compatibility conditions of Definition 1.1. Note that we write conjugation on the left, so ${}^{g}g' = gg'g^{-1}$ for $g, g' \in G$ and ${}^{g}g' \cdot g'^{-1} = [g, g']$ for the commutator of g and g'.

DEFINITION 1.1. Let G and H be groups that act on themselves by conjugation and each of which acts on the other. The mutual actions are said to be compatible if

$$ {}^{{}^{h}g}h' = {}^{hgh^{-1}}h' \textit{ and } {}^{{}^{g}h}g' = {}^{ghg^{-1}}g' \textit{ for all } g, g' \in G, h, h' \in H. \tag{1.1.1}$$

It is worth noting that the condition,

$$ {}^{{}^{g}g'}h = {}^{gg'g^{-1}}h \textit{ and } {}^{{}^{h}h'}g = {}^{hh'h^{-1}}g \textit{ for all } g, g' \in G, h, h' \in H,$$

always holds. For groups that act compatibly on each other, the non-abelian tensor product is then defined as follows.

DEFINITION 1.2. If G and H are groups that act compatibly on each other, then the non-abelian tensor product $G \otimes H$ is the group generated by the symbols $g \otimes h$ for $g \in G$ and $h \in H$ with relations

$$ gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h), \tag{1.2.1}$$

$$ g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h'), \tag{1.2.2}$$

for all $g, g' \in G$ and $h, h' \in H$.

The special case, where $G = H$ and all actions are given by conjugation, is called the tensor square $G \otimes G$. The tensor square of a group is always defined. The following question arises: let G and H be finite groups acting compatibly on each other. Then, is it true that $G \otimes H$ is finite? Already Brown and Loday in [**3**] established that the tensor square $G \otimes G$ is finite for finite G. In [**5**], Ellis settled the general question affirmatively as follows.

THEOREM 1.3. *If G and H are finite groups acting on each other, and if their actions are compatible, then $G \otimes H$ is finite.*

Ellis in his proof uses part of an exact sequence from homology as given in [**2**] and the fact that the homology of a finite group is finite. In [**7**], L.-C. Kappe mentions that no purely algebraic proof of Theorem 1.3 is known. Already the authors of [**1**], Brown, Johnson and Robertson, ask the question if such a proof can be given. In this paper, we give a purely group theoretic proof of Theorem 1.3. We will show that the non-abelian tensor product of two finite groups satisfies the assumptions of Dietzmann's Lemma [**4**] (for a more accessible reference we refer to [**9**]). Noting that a subset of a group is normal if it contains all conjugates of its elements, Dietzmann's Lemma can be stated as follows.

THEOREM 1.4. *In any group a normal finite subset consisting of elements of finite order generates a finite subgroup.*

In addition, we use the embedding of the non-abelian tensor product $G \otimes H$ in an overgroup as given in [**6**] (see Proposition 2.7 for details). In [**5**], Ellis indicates that the result of Theorem 1.3 remains true if finite is replaced by finite p-group. We give an explicit proof of that result.

THEOREM 1.5. *Let G and H be p-groups. If G and H are finite, then $G \otimes H$ is a finite p-group.*

2. Preparatory results.

In this section, we present several general results on tensor products needed in the sequel. The following result can be found as Proposition 3 of [**1**].

PROPOSITION 2.1. *The following relations hold for all $g, g' \in G$ and $h, h' \in H$:*

$$^{g}(g^{-1} \otimes h) = (g \otimes h)^{-1} = {}^{h}(g \otimes h^{-1}), \tag{2.1.1}$$

$$(g \otimes h)(g' \otimes h')(g \otimes h)^{-1} = {}^{g^h g^{-1}}g' \otimes {}^{g^h g^{-1}}h', \tag{2.1.2}$$

$$(g^h g^{-1}) \otimes h' = (g \otimes h)^{h'}(g \otimes h)^{-1}, \tag{2.1.3}$$

$$g' \otimes ({}^{g}h h^{-1}) = {}^{g'}(g \otimes h)(g \otimes h)^{-1}, \tag{2.1.4}$$

$$[g \otimes h, g' \otimes h'] = (g^h g^{-1}) \otimes ({}^{g'}h' h'^{-1}). \tag{2.1.5}$$

In [**10**], the derivative of one group by another was introduced and its properties were investigated.

DEFINITION 2.2. Let G and H be groups with H acting on G. Then the subgroup $D_H(G) = \langle g \cdot {}^{h}g^{-1} | g \in G, h \in H\rangle$ of G is the derivative of G by H.

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Denoting with U^V the closure of U under the operator group V, we find the following result.

PROPOSITION 2.3. *Let G and H be groups acting compatibly. Then $D_H(G)$ is a normal subgroup of G and $D_H(G)$ is operator invariant under the action of H, that is $D_H(G)^H = D_H(G)$.*

Using Proposition 2.3, we obtain the following expansion formula which we need for the proof of Theorem 1.5.

LEMMA 2.4. *Let G and H be groups which act compatibly and let k be a positive integer. Then there exists $w_k \in D_H(G) \otimes H$ such that*

$$g \otimes h^k = w_k (g \otimes h)^k,$$

for all $g \in G$, $h \in H$.

Proof. We prove our claim by induction on k. For $k = 1$, the statement is obviously true. Assume the result is true for $k - 1$. Expansion using (1.2.2) yields

$$g \otimes h^k = (g \otimes h^{k-1})({}^{h^{k-1}}g \otimes h). \tag{2.4.1}$$

Observe that $g \cdot ({}^{h^{k-1}}g^{-1}) \in D_H(G)$. Hence, $g \equiv {}^{h^{k-1}}g \ (mod\ D_H(G))$. Therefore there exists $s \in D_H(G)$ with ${}^{h^{k-1}}g = gs$. Thus by expansion using (1.2.2), we obtain ${}^{h^{k-1}}g \otimes h = gs \otimes h = ({}^g s \otimes {}^g h)(g \otimes h)$. Substituting this into the right-hand side of (2.4.1) leads to

$$g \otimes h^k = (g \otimes h^{k-1})({}^g s \otimes {}^g h)(g \otimes h).$$

Commuting the first and second factor on the right-hand side yields

$$g \otimes h^k = [g \otimes h^{k-1},\ {}^g s \otimes {}^g h]({}^g s \otimes {}^g h)(g \otimes h^{k-1})(g \otimes h). \tag{2.4.2}$$

By Proposition 2.3, it follows that $D_H(G) \otimes H$ is normal in $G \otimes H$. This fact together with (2.1.5) implies that $[g \otimes h^{k-1},\ {}^g s \otimes {}^g h] = [{}^g s \otimes {}^g h,\ g \otimes h^{k-1}]^{-1} \in D_H(G) \otimes H$. Thus (2.4.2) can be simplified as

$$g \otimes h^k = w(g \otimes h^{k-1})(g \otimes h),$$

where $w = [g \otimes h^{k-1},\ {}^g s \otimes {}^g h]({}^g s \otimes {}^g h)$ with $w \in D_H(G) \otimes H$. Using the induction hypothesis for $k - 1$ leads to

$$g \otimes h^k = w w_{k-1} (g \otimes h)^k.$$

Setting $w w_{k-1} = w_k$ proves our claim. □

Information and presentation of the overgroup $\eta(G, H)$ can be found in **[8]**, where the author notes that $\eta(G, H) \cong ((G \otimes H) \rtimes H) \rtimes G$ by Theorem 2.5. Let $G * H$ be the free product of G and H. The definition of the subgroup J of $G * H$ and the following three results which we need for the proof of our theorems can be found in **[6]**.

THEOREM 2.5. *There is an isomorphism $((G \otimes H) \rtimes H) \rtimes G \cong G * H/J$, where $\rtimes$ denotes a semi-direct product.*

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PROPOSITION 2.6. *The canonical homomorphisms $G \to G * H/J$ and $H \to G * H/J$ are injective.*

For brevity we set $K = \eta(G, H)$. Denoting the normal closures of G and H in K by G^K and H^K, respectively, we obtain a description of the non-abelian tensor product as the intersection of two normal subgroups of K.

PROPOSITION 2.7. *There is an isomorphism $G \otimes H \cong G^K \cap H^K$.*

We need the following two lemmas for the proof of Theorem 1.3 and Theorem 1.5.

LEMMA 2.8. *Let G be a group viewed as a subgroup of K, then the subgroup $D_H(G)$ of G is a K invariant subgroup of K and hence $D_H(G)^K = D_H(G)$.*

Proof. If the mutual actions of G and H are compatible then by Proposition 2.3 we have that $D_H(G)$ is invariant under the action of H and G. So $D_H(G)$ is invariant under the action of $G * H$ and hence under the action of K. Thus the normal closure of $D_H(G)$ in K is $D_H(G)$. □

LEMMA 2.9. *Let G and H be groups acting compatibly on each other. If G and H are finite, then G^K and H^K are finite.*

Proof. By Proposition 2.6, the groups H and G embed isomorphically into K. By the compatibility condition, we have

$$^{h'}(^{g}h) = {}^{^{h'}g}(^{h'}h) \tag{2.9.1}$$

for all $h', h \in H$ and $g \in G$. The set $S = \{h, {}^{g}h | g \in G, h \in H\}$, where H and G are considered as subgroups of K, is a finite normal set in K. The finiteness of S follows from the finiteness of G and H. Let $z \in G * H$. The proof is by induction on the length of the word z. Suppose for any z in $G * H$ of length n we have ${}^{z}h$ and ${}^{z}(^{g}h)$ are elements of S. Let zh' be an element of $G * H$ of length $n+1$ with $h' \in H$. Then ${}^{zh'}h = {}^{z}(^{h'}h)$ and ${}^{zh'}(^{g}h) = {}^{z}(^{h'}(^{g}h)) = {}^{z}(^{g'}(^{h'}h))$ for some $g' \in G$ by (2.9.1). By induction, ${}^{z}(^{h'}h)$ and ${}^{z}(^{g'}(^{h'}h))$ are in S. Let $zg'' \in G * H$ of length $n+1$ with $g'' \in G$. Then ${}^{zg''}h = {}^{z}(^{g''}h)$ and ${}^{zg''}(^{g}h) = {}^{z}(^{g''}(^{g}h)) = {}^{z}(^{g'''}h)$ for some $g''' \in G$. By induction, ${}^{z}(^{g''}h)$ and ${}^{z}(^{g'''}h)$ are elements of S. Hence S is a normal set. The elements of S have finite order because H is a finite subgroup of K and the order is preserved by conjugation. By Theorem 1.4, the set S generates a finite normal subgroup of K containing H. Hence the normal closure of H in K is finite. By a similar argument we can show that the normal closure of G in K is finite. □

3. Proof of the main theorems.

Now we are ready to prove the main theorems.

Proof of Theorem 1.3. By Proposition 2.7, we have $G \otimes H \cong G^K \cap H^K$, where G and H are considered as subgroups of K. However by Lemma 2.9, it follows that G^K and H^K are finite and hence their intersection is finite. Thus $G \otimes H$ is finite. □

To prove Theorem 1.5, we let $\eta(D_H(G), H) = K_1$, where K_1 is the group found in Theorem 2.5 for the pair of groups $D_H(G)$ and H.

Proof of Theorem 1.5. As a consequence of Lemma 2.8, the normal closure of $D_H(G)$ in K_1 is $D_H(G)$. Furthermore, $D_H(G)$ is a finite p-group as it is a subgroup of G. By Proposition 2.7, it follows that $D_H(G) \otimes H \cong D_H(G)^{K_1} \cap H^{K_1}$. Thus $D_H(G) \otimes H$ is a finite p-group. Now we will show that $g \otimes h$ has p-power order for all $g \in G$, $h \in H$. Since H is a finite p-group, we have $h^{p^{\alpha}} = 1$, $h \in H$ and for some positive integer α. By Lemma 2.4, where $k = p^{\alpha}$ we obtain that $1_{\otimes} = g \otimes 1 = g \otimes h^{p^{\alpha}} = w_{p^{\alpha}}(g \otimes h)^{p^{\alpha}}$ for some $w_{p^{\alpha}} \in D_H(G) \otimes H$, $g \in G$, $h \in H$. Therefore $w_{p^{\alpha}}$ has p-power order. Since $w_{p^{\alpha}}$ has p-power order, $g \otimes h$ has p power order for all $g \in G$, $h \in H$. By Theorem 3.4(i) of [**10**], $G \otimes H$ is a nilpotent group and hence a direct product of its Sylow subgroups. Therefore, $Y = \{g \otimes h \mid g \in G, h \in H\}$ is contained in the Sylow p-subgroup of $G \otimes H$. Since $G \otimes H =< Y >$, we conclude that $G \otimes H$ is a finite p-group. □

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