THE DISTRIBUTION OF SEQUENCES MODULO 1

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Introduction. In a recent paper **(2)**, Helson and Kahane consider the problem of the existence of real numbers x such that the sequence $(\lambda_n x)$ (when reduced modulo 1) is not summable by a given regular Toeplitz method, where (λ_n) is a lacunary sequence of positive real numbers. Thus, as an example, they show the existence of uncountably many x such that the sequence $(\theta^n x)$ does not have a distribution function modulo 1, where θ is some fixed number >1.

The purpose of this paper is twofold. The first is to exhibit a large class of sequences of functions, characterized by certain growth properties, which can easily be shown to have the above non-summability property for uncountably many values of the argument. This class of sequences includes, in particular, the two classes most commonly studied, namely, lacunary and exponential sequences. The second question is to determine the distribution functions that can arise as the distribution functions modulo 1 of sequences generated in a natural way by sequences of functions. Under the assumption of a stronger growth property, we shall show that all possible distribution functions do in fact arise.

We might mention briefly some reasons for considering growth properties. Slowly increasing sequences such as $(\log n)$ cannot have distribution functions modulo 1 because their values "bunch up." On the other hand, functions such as polynomials are too well behaved to exhibit many anomalies. Thus, if p is a polynomial with at least one irrational coefficient (other than the constant term), then the sequence $(p(n))$ is uniformly distributed (mod 1). If all the coefficients are rational, then the sequence $(p(n))$ takes on only a finite number of values modulo 1. Thus the most natural sequences from our point of view are the ones generated by rapidly increasing sequences of functions. For a discussion of the above-mentioned results, the reader is referred to the expository paper by Cigler and Helmberg **(1)**, which contains a thorough bibliography through about 1960.

1. Definitions. We shall be concerned entirely with sequences modulo 1; i.e., when we write a sequence (x_n) we shall only be concerned with the corresponding sequence of fractional parts. A *distribution function* will be any non-decreasing function mapping the unit interval I into itself and the end points onto the end points. If (x_n) is a sequence and d is a distribution

Received March 15, 1966.

https://doi.org/10.4153/CJM-1967-064-0 Published online by Cambridge University Press

function, then we say that (x_n) has d as its distribution function, if, for every number $a \in I$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi(x_k) = d(a),$$

where χ is the characteristic function of the interval $[0, a]$. If the indicated limit fails to exist for some a, then the sequence (x_n) does not have a distribution function. If the sequence (x_n) has the function $d(x) = x$ as its distribution function, then it is said to be *uniformly distributed.*

Property A. Let (n_k) be an increasing sequence of positive numbers tending to infinity, and let f be a continuous function. Then we say that f has Property A (relative to (n_k)) if the following condition holds: Given any $\delta > 0$ there exists a $p > 1$ such that if k is any positive integer and α and β are any two real numbers satisfying

(i) $\beta > \alpha > p^{n_k}$,

(ii) $f([\alpha, \beta])$ contains an interval of length 1,

then

$$f([\alpha^{n_{k+1}/n_k}, \beta^{n_{k+1}/n_k}]) \tag{1}$$

contains an interval of length $\delta/2$.

Property B. Let g be any continuous function and let (f_n) be a sequence of strictly increasing continuous functions each tending to infinity. Suppose further that, given any $\delta > 0$, there exists a number p, with $\lim_{k\to\infty} f_k(p) = \infty$, such that if n is any positive integer and α and β are any two numbers satisfying

(i) $\beta > \alpha > f_k(p)$,

(ii) $g([\alpha, \beta])$ contains an interval of length $\delta/2$,

then

(2) $g(f_{n+1}(f_n^{-1}([\alpha, \beta])))$ contains an interval of length 1.

Then we say that the pair $\{g, (f_n)\}$ has Property B.

Of course, Property B is a generalization of Property A. In the next section we shall see that it is very easy to prove that sequences satisfying the above conditions—and one additional trivial condition—will fail to have distribution functions for certain values of their arguments.

2. THEOREM 2.1. *Suppose (n_k) is an increasing sequence of positive numbers tending to infinity and that f is a continuous function having Property* A (relative to (n_k)), *and suppose that for every positive integer n there exists a number N_n such that*

(3) *$f([n, N_n])$ contains an interval of length l.*

Then, given any sequence (I_n) of closed subintervals of I, each of whose lengths is at least $\delta > 0$ (where δ is an arbitrary positive number), there exist uncountably many x such that, for every integer $k \geqslant 1$, $f(x^{n_k}) \in I_k$, of course (mod 1).

Proof. It is trivial to see that if for each sequence (I_n) as described above we may find an appropriate x, then we may in fact find uncountably many such x for each (I_n); cf. **(6)**. Hence we need only demonstrate the existence of one such x.

Now, let (I_n) be any sequence of closed intervals with $\inf_n$ (length of $I_n) = \delta > 0$. We may assume that δ is small and that in fact the length of each interval I_n is equal to δ. We may also assume that $1 = n_1$ ($< n_2 < \ldots$, of course). Finally, note that if $f([a, b])$ contains an interval of length 1, then we may find $[c, d] \subset [a, b]$ such that $f([c, d]) \subset I_n \pmod 1$ and $f([c, d])$ contains an interval of length $\delta/2 \pmod 1$.

Given the δ, we can find p corresponding to it as in Property A. By (3), we can find an interval $[\alpha_1, \beta_1]$, with $\alpha_1 > p$, such that $f([\alpha_1, \beta_1]) \subset I_1 \pmod 1$ and $f([\alpha_1, \beta_1])$ contains an interval of length $\delta/2$. By **(1)**, it now follows that $f([\alpha_1^{n_2}, \beta_1^{n_2}])$ contains an interval of length 1; so we may find $[\alpha_2, \beta_2]$ such that $f([\alpha_2, \beta_2]) \subset I_2 \pmod 1$, $f([\alpha_2, \beta_2])$ contains an interval of length $\delta/2$, and

$$[\alpha_2, \beta_2] \subset [\alpha_1^{n_2}, \beta_1^{n_2}].$$

Then $\alpha_2 \geqslant \alpha_1^{n_2} > p^{n_2}$. By (1), it again follows that $f([\alpha_2^{n_3/n_2}, \beta_2^{n_3/n_2}])$ contains an interval of length $\delta/2$; so we may find $[\alpha_3, \beta_3] \subset [\alpha_2^{n_3/n_2}, \beta_2^{n_3/n_2}]$ such that $f([\alpha_3, \beta_3)] \subset I_3 \pmod 1$ and $f([\alpha_3, \beta_3])$ contains an interval of length $\delta/2$. Again, $\alpha_3 \geqslant \alpha_2^{n_3/n_2} > p^{n_3}$. We repeat this process inductively. Having found $[\alpha_1, \beta_1], \ldots, [\alpha_{k-1}, \beta_{k-1}]$ such that

(i) $f([\alpha_i, \beta_i]) \subset I_i \pmod 1$ $(i = 1, 2, \ldots, k-1)$,

(ii) $f[(\alpha_i, \beta_i])$ contains an interval of length $\delta/2$,

(iii) $\alpha_i > p^{n_i}$,

we can find $[\alpha_k, \beta_k] \subset [\alpha_{k-1}^{n_k/n_{k-1}}, \beta_{k-1}^{n_k/n_{k-1}}]$ such that conditions (i)–(iii) hold for $i = k$. We thus obtain a nested sequence of non-empty closed intervals

$$[\alpha_1, \beta_1] \supset \ldots \supset [\alpha_k^{1/n_k}, \beta_k^{1/n_k}] \supset \ldots$$

and, by compactness, we can find a number θ in their intersection. But then $\theta^{n_k} \in [\alpha_k, \beta_k]$ and hence $f(\theta^{n_k}) \in I_k \pmod 1$.

THEOREM 2.2. *Suppose that $\{g, (f_n)\}$ has Property* B *and further that for each integer m there exists an integer $N_m > m$ such that*

(4) $$g([m, N_m]) \text{ contains an interval of length } 1.$$

Then, given any sequence of closed intervals (I_n), with the infimum of their lengths positive, there exist uncountably many numbers θ such that

$$g(f_k(\theta)) \in I_k \pmod 1 \qquad (k = 1, 2, 3, \ldots).$$

The proof of this theorem parallels that of 2.1 and will be omitted.

COROLLARY 2.3. *Suppose (n_k) is any sequence of positive numbers with $\inf_k (n_{k+1} - n_k) = \epsilon > 0$. Then the conclusions of Theorem* 2.1 *are valid with f being any one of the following:*

(i) *f a function whose derivative f' is positive and non-decreasing for x sufficiently large, say $x \geqslant M$;*

(ii) *f any non-constant polynomial;*

(iii) $f(x) = x$ **(6)**;

(iv) *$f(x) = Ax^\sigma$, where $A \neq 0$ and σ is positive;*

(v) *$f(x) = (p(x))^\sigma$, where σ is any positive number and p is any non-constant polynomial which is eventually non-negative;*

(vi) *f any periodic, continuous function with range containing an interval of length* 1.

Proof. In all cases condition (4) is satisfied. We thus only need to show that the indicated functions have Property A, or deduce the conclusion from another part of the corollary.

(i) Given any $\delta > 0$ (we may assume δ is small) let

$$p = \max((2/\delta)^{1/\epsilon}, M).$$

Suppose k is any positive integer and α and β are as in Property A. Then

$$f(\beta^{n_{k+1}/n_k}) - f(\alpha^{n_{k+1}/n_k}) = (\beta^{n_{k+1}/n_k} - \alpha^{n_{k+1}/n_k})f'(\xi)$$

for some $\xi \in [\alpha^{n_{k+1}/n_k}, \beta^{n_{k+1}/n_k}]$. Hence this is

$$\begin{aligned}
&\geqslant (\beta - \alpha)\alpha^{(n_{k+1}-n_k)/n_k}f'(\xi) \geqslant (\beta - \alpha)\alpha^{\epsilon/n_k}f'(\xi)\\
&\geqslant (\beta - \alpha)p^\epsilon f'(\xi) \geqslant (\beta - \alpha)(2/\delta)f'(\xi)\\
&\geqslant (\beta - \alpha)f'(\eta)(2/\delta) \geqslant (f(\beta) - f(\alpha))(2/\delta) \geqslant 1
\end{aligned}$$

where $f(\beta) - f(\alpha) = f'(\eta)(\beta - \alpha)$, $\eta \in [\alpha, \beta]$. We used the fact that f' was non-decreasing.

(ii) and (iii) are trivial. (iv) follows from (i) by noting that if $A\theta^{n_k} \in I_k$ (mod 1) and we set $\phi = \theta^{1/\alpha}$, then $f(\phi^{n_k})$ is in I_k (mod 1), where f is the function of part (iv).

(v) If $p(x) = p_n x^n + \ldots + p_0$ (where $p_n \neq 0$), then if $n\alpha \geqslant 1$, the result follows from (i). If $0 < \alpha n < 1$, then we observe that

$$\lim_{x\to\infty}(p(x)^\alpha - (p_n x^n)^\alpha) = 0$$

and the result follows easily from (iv).

(vi) We show that f has Property A. Let $\delta > 0$ be arbitrary. Then there exists, by uniform continuity, a number $r > 0$ such that if $f([\alpha, \beta])$ contains an interval of length δ, then $\beta - \alpha \geqslant r$. Let P be the period of f, and let

$$p = \max\,(1, (P/r)^{1/\epsilon}).$$

Suppose that k is any positive integer and $\beta > \alpha > p^{n_k}$ and $f([\alpha, \beta])$ contains an interval of length $\delta/2$. Then $\beta - \alpha > r$, so that

$$\beta^{n_{k+1}/n_k} - \alpha^{n_{k+1}/n_k} \geqslant (\beta - \alpha \mid \alpha^{\epsilon/n_k} \geqslant rp^\epsilon \geqslant P.$$

The result follows.

COROLLARY 2.4. *Let $f_n(x) = h(n)x^{l_n}$, where (l_n) is a strictly increasing sequence of positive numbers,*

$$(h(n+1))^{l_n} \geqslant (h(n))^{l_{n+1}} > 0 \qquad \textit{for all } n$$

and

$$\lim_{n\to\infty}(h(n+1)/h(n)^{l_{n+1}/l_n}) = \infty.$$

Then, given any sequence (I_n) of closed intervals the infimum of whose lengths is positive, there exist uncountably many θ such that

$$f_n(\theta) \in I_n \pmod 1 \qquad (n = 1, 2, 3, \ldots).$$

Proof. Let $g(x) = x$. Then condition (4) is certainly satisfied. Hence we need only show that the pair $\{g, (f_n)\}$ has Property B. To do this, let $\delta > 0$. Let N be an integer such that for $n \geqslant N$

$$h(n+1)/h(n)^{l_{n+1}/l_n} \geqslant 2/\delta.$$

Let p be a number such that $x \geqslant p$ implies that

$$x^{l_{n+1}-l_n} \geqslant 2/\delta \qquad \text{for } m = 1, 2, \ldots, N.$$

Now let m be any positive integer and α and β any two numbers such that $\beta - \delta/2 \geqslant \alpha > p^m$. Then we have

$$\begin{aligned} f_{m+1}(f_m^{-1}(\beta)) - f_{m+1}(f_m^{-1}(\alpha)) &= f_{m+1}((\beta/h(m))^{1/l_n}) - f_{m+1}((\alpha/h(m))^{1/l_n}) \\ &= \frac{h(m+1)}{(h(m))^{l_{m+1}/l_m}}(\beta^{l_{m+1}/l_m} - \alpha^{l_{m+1}/l_m}) \geqslant \frac{h(m+1)}{(h(m))^{l_{m+1}/l_m}} \\ &\times (\beta-\alpha)\alpha^{(l_{m+1}-l_m)/l_m} \geqslant (h(m+1)/h(m)^{l_{m+1}/l_m})(\delta/2)p^{(l_{m+1}-l_m)}. \end{aligned}$$

If $m \geqslant N$, then $h(m+1)/h(m)^{l_{m+1}/l_m} \geqslant 2/\delta$, while if $m < N$, then

$$p^{(l_{m+1}-l_m)} \geqslant 2/\delta.$$

In either case, both numbers are $\geqslant 1$ and hence

$$f_{m+1}(f_m^{-1}(\beta) - f_{m+1}(f_m^{-1}(\alpha)) \geqslant 1$$

and the result follows.

An example of such a sequence of functions (f_n) is given by

$$f_n(x) = n!\,x^{1-(1/n)}.$$

The sequence (l_n) is needed only to provide for the required approximations when n is small. An examination of the previous proof yields the following result:

COROLLARY 2.5. *Suppose* $\lim_{n\to\infty} h(n+1)/h(n) = \infty$. *Then, with (I_n) as usual, we may find x such that for all n sufficiently large*

$$h(n)x \in I_n \pmod 1.$$

THEOREM 2.6. *Let* (h_n) *be a lacunary sequence of numbers, i.e., there exists a number* $r > 1$ *such that* $h_{n+1}/h_n \geqslant r$ *for all* n. *Suppose* (I_n) *is a sequence of closed intervals with the corresponding lengths at least* $2/r$, r *as indicated. Then there exist uncountably many* x *such that* $h_n x \in I_n \pmod 1$, $n = 1, 2, 3, \ldots$.

Proof. The proof follows the lines already indicated. At the nth step we obtain an interval U_n of length at least $1/r$ which is a subinterval of $I_n \pmod 1$. Its image under the map $x \to (h_{n+1}/h_n)x$ is of length at least 1 and the argument can be repeated.

The following result is now trivial.

THEOREM 2.7. *Let* (s_n) *be a sequence of functions satisfying any of the conditions listed in* 2.1–2.5. *Then there exist uncountably many* x *such that the sequence* $(s_n(x))$ *does not have a distribution function* (mod 1).

Proof. Let (a_n) be any sequence of 0's and 1's for which

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{\substack{k\leqslant n\\ a_k=1}} 1 = \limsup_{n\to\infty} \frac{1}{n} \sum_{\substack{k\leqslant n\\ a_k=0}} 1 = 1.$$

Let $I_n = [0, 1/4]$ if $a_n = 1$ and $I_n = [1/2, 3/4]$ if $a_n = 0$. Then we may find uncountably many x such that, for n sufficiently large, $s_n(x) \in I_n \pmod 1$, and this sequence $(s_n(x))$ obviously cannot have a distribution function.

Note that this theorem does not include lacunary sequences. However, in this case we can make an even stronger statement.

THEOREM 2.8. *Let* (f_n) *be a sequence of continuous functions satisfying either the conditions of Corollary* 2.3 *or the following:* $f_n(x) = \lambda_n x$, *where* $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$ *is an increasing sequence of positive numbers such that there exists a* $\delta > 0$ *and a subsequence* $\lambda_{n_1} < \lambda_{n_2} < \ldots < \lambda_{n_k} < \ldots$ *such that*

(i) $\lambda_{n_{k+1}} \geqslant (1 + \delta)\lambda_{n_k}$ $(k = 1, 2, 3, \ldots)$,

(ii) $\liminf_{k\to\infty} k/n_k > 0$.

Then, in either case, the set of x *for which the sequence* $(f_n(x))$ *does not have a distribution function* (mod 1) *is dense and uncountable* (in the first case we restrict ourselves to the interval $[M, \infty)$, M as in the corollary).

Proof. We shall just prove the density result; uncountability follows in the usual manner. For simplicity, assume that $M = 0$. Let (a, b) be any subinterval of $(0, \infty)$. We wish to find $x \in (a, b)$ for which $(s_n(x))$ does not have a distribution function.

If we refer to the proof of Corollary 2.3(i), or to Theorem 2.6 applied to the sequence (λ_{n_k}), using the exponential rate of growth of these sequences, we arrive at the following fact: if N is a sufficiently large integer, and (I_n) is any sequence of closed intervals of lengths at least $1/(2N)$, then it is possible to find a sequence of integers (n_k) of density at least $1/N$, i.e.,

$$\liminf_{n\to\infty} \frac{1}{n} \cdot \sum_{n_k\leqslant n} 1 \geqslant 1/N,$$

such that there exists an x in (a, b) such that

(5) $$s_{n_k}(x) \in I_k \pmod 1 \qquad \text{for } k = 1, 2, 3, \ldots .$$

We can now prove our theorem in the usual manner: let (a_i) be any sequence of integers, $0 \leqslant a_i \leqslant 2N - 1$, such that, for each integer k,

$$0 \leqslant k \leqslant 2N - 1,$$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{\substack{i\leqslant n\\ a_i=k}} 1 = 1.$$

We now choose

$$I_k[a_k/(2N),\ (a_k + 1)/(2N)], \qquad k = 1, 2, 3, \ldots .$$

If now $x \in (a, b)$ such that (5) holds, then the sequence $(s_n(x))$ cannot have a distribution function.

The class of functions included in Theorem 2.8 contains all lacunary sequences as well as polynomials in x^n. We thus obtain as particular corollaries the result of Helson and Kahane as well as a generalization of Vijayaraghavan's result on the sequence (x^n); see **(6)**. Under certain circumstances it is possible to make even stronger statements about the exceptional sets, but we shall not go into that now. The reader should also bear in mind the facts that under the conditions of Corollary 2.3(i) a well-known result of Koksma **(3)** will give the result that the sequence $(f_n(x))$ is uniformly distributed (mod 1) for almost all $x > 1$ (in the sense of Lebesgue measure), and a result of Weyl yields the same information about lacunary sequences, see **(7)**.

In addition to giving us information about the distribution functions of certain sequences, the above theorems are translatable into the language of diophantine approximation where many of them state that given any sequence (a_n) and any $\epsilon > 0$ we may find a number x such that, for every n,

$$|s_n(x) - a_n| < \epsilon \pmod 1.$$

We shall not discuss this matter further, but turn our attention instead to another distribution question in the next section.

3. Our principal objective in this section will be to prove the following theorem:

THEOREM 3.1. *Let d be an arbitrary distribution function and let (n_k) be any sequence of real numbers satisfying*

$$\lim_{k\to\infty}(n_{k+1} - n_k) = \infty.$$

Then there exists a number θ such that the sequence (θ^{n_k}) has d as its distribution function.

https://doi.org/10.4153/CJM-1967-064-0 Published online by Cambridge University Press

I should mention that the conditions of this theorem are more stringent than earlier growth restrictions. Whether these conditions could be weakened I do not know. We should also remark that if the given distribution function d is continuous, then the theorem is easy to prove. The difficulties arise only when d is not continuous, because in this latter case if a sequence (x_n) has d as its distribution function and a sequence (y_n) is given, with

$$\lim_{n\to\infty}(y_n - x_n) \equiv 0 \pmod 1,$$

then it does *not* follow that (y_n) has d (or even any function) as its distribution function.

LEMMA 3.2. *Let* (y_n), $0 \leqslant y_n \leqslant 1$, *be any sequence that is uniformly distributed* (mod 1). *Let d be any distribution function and let*

$$d'(w) = d(w+0) = \lim_{t\to w+0} d(t).$$

Let

$$g(x) = \inf\{w \mid d'(w) \geqslant x\}.$$

Finally, let $z_n = g(y_n)$, $n = 1, 2, 3, \ldots$. *Then the sequence* (z_n) *has* d' *as its distribution function.*

Proof. Since

$$d'(a+0) = \lim_{t\to a+0} d'(t) = \lim_{t\to a+0}\ \lim_{w\to t+0} d(w) = d(a+0) = d'(a),$$

d' is continuous from the right. Now, if $g(x) \leqslant a$, then

$$\inf\{w \mid d'(w) \geqslant x\} \leqslant a$$

so that $d'(w) > x$ for $w > a$. Hence

$$\begin{aligned}\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi(z_n) &= \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi(g(y_n))\\ &\leqslant \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \psi(y_n) = d'(w)\end{aligned}$$

(where χ is the characteristic function of $[0, a]$ and ψ is the characteristic function of $[0, d'(w)]$) for every $w > a$. On the other hand, if $g(x) > a$, then $d'(a) < x$, so

$$\begin{aligned}\limsup_{N\to\infty} \frac{1}{N}\left(\sum_{n=1}^{N} \phi(z_n)\right) &\leqslant \limsup_{N\to\infty} \frac{1}{N}\left(\sum_{n=1}^{N} \Phi(y_n)\right)\\ &= 1 - d'(a) = 1 - d'(a+0)\end{aligned}$$

where ϕ and Φ are the characteristic functions of $(a, 1]$ and $(d'(a), 1]$, respectively. Thus (z_n) does indeed have d' as its distribution function.

For future reference we also note some further properties of d'. In the first place, $d'(w) = d(w)$ at those w at which d is right continuous. Furthermore, if d' is discontinuous at a, then g has a period of constancy at $d'(a)$, namely, g is constant on the interval $(d'(a-0), d'(a+0)]$. To see this, we note that on the one hand

$$\inf\{w \mid d'(w) \geqslant d'(a+0)\} = a$$

so that $g(d'(a+0)) = a$. On the other hand, if $t > d'(a-0)$, then

$$\inf\{w \mid d'(w) \geqslant t\} \geqslant a$$

so that $g(t) \geqslant a$, proving the last assertion. Finally,

$$d'(a-0) = \lim_{w\to a-0} d'(w) = \lim_{w\to a-0} \lim_{t\to w+0} d(t) = \lim_{t\to a-0} d(t).$$

Hence, $d'(a-0) = d(a-0)$.

Now, let (ϵ_n) be any sequence of positive numbers tending to 0, and let (z_n) be as above. We now define a new sequence (x_n) as follows:

(i) If d is right continuous at z_n, let $x_n = z_n - \epsilon_n$ (or let $x_n = \epsilon_n$ if $z_n - \epsilon_n \leqslant 0$).

(ii) If d is left continuous at z_n, but not right continuous, let $x_n = z_n + \epsilon_n$ (or $x_n = 1 - \epsilon_n$ if $z_n + \epsilon_n \geqslant 1$).

(iii) Suppose that $d(z_n - 0) = \alpha < d(z_n) = \beta < d(z_n + 0) = \gamma$. Let $n_1 < n_2 < \ldots < n_k$ be the sequence of all of those indices for which $z_{n_k} = z_n$, and let $m_1 < \ldots < m_k < \ldots$ be any subsequence of (n_k) for which

$$\lim_{k\to\infty} \frac{1}{k} \sum_{m_l \leqslant n_k} 1 = \frac{\beta - \alpha}{\gamma - \alpha}.$$

Now, let $x_n = z_n - \epsilon_n$ if $n = m_i$ for some i (or $x_n = \epsilon_n$ if $z_n - \epsilon_n \leqslant 0$), and let $x_n = z_n + \epsilon_n$ (or $1 - \epsilon_n$ if $z_n + \epsilon_n \geqslant 1$) if $n = n_k$ for some k, but $n \neq m_i$ for any i.

LEMMA 3.3. *The sequence* (x_n) *constructed above has* d *as its distribution function.*

Proof. We should first note that we have not really altered the distribution function by our redefinition when $z_n \pm \epsilon_n$ is either $\geqslant 1$ or $\leqslant 0$, because in both cases we have sequences tending to 0 or 1.

By an earlier remark,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \phi(z_n) = d(a+0) - d(a-0)$$

where ϕ is the characteristic function of the point set $\{a\}$. Furthermore, suppose that $b < a < c$. Then, for n large, $z_n \in [0, b]$ implies that $x_n \in [0, a]$, and $x_n \in [0, a]$ implies that $z_n \in [0, c]$. Hence,

$$d'(b) \leqslant \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi(x_n) \leqslant \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi(x_n) \leqslant d'(c)$$

where χ is the characteristic function of $[0, a]$. Hence

$$d(a-0) \leqslant \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi(x_n)$$
$$\leqslant \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi(x_n) \leqslant d(a+0).$$

If d is continuous at a, then the limit in question does exist and is equal to $d(a)$. Suppose then that d is not continuous at a. We must then consider three cases:

(i) Suppose $d(a+0) = d(a)$. Then, if $z_n = a$, $x_n > b$ for n large. Thus

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi(x_n) \geqslant d'(b) + (d'(a) - d'(a-0)).$$

This means that

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi(x_n) \geqslant d'(a-0) + d'(a) - d'(a-0) = d(a).$$

(ii) Suppose that $d(a-0) = d(a)$. If $z_n = a$, then $x_n < c$ for n large. Hence

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi(x_n) \leqslant d'(c) - (d'(a+0) - d'(a-0))$$

so that

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi(x_n) \leqslant d'(a+0)) - d'(a+0)$$
$$+ d'(a-0) = d'(a-0) = d(a-0) = d(a).$$

(iii) Suppose that $d(a-0) < d(a) < d(a+0)$. Then, if $z_n = a$, either $x_n < a$ or $x_n > a$, the former occurring with frequency

$$\frac{d(a) - d(a-0)}{d(a+0) - d(a-0)}.$$

Then

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi(x_n) \geqslant d'(a-0) + \frac{d(a) - d(a-0)}{d(a+0) - d(a-0)} \lim_{N\to\infty} \sum_{n=1}^{N} \phi(z_n)$$
$$= d'(a-0) + (d'(a+0) - d'(a-0))\left(\frac{d(a) - d(a-0)}{d(a+0) - d(a-0)}\right)$$
$$= d(a).$$

Similarly we can show that

$$\limsup_{N\to\infty} \frac{1}{N}\sum \chi(x_n) \leqslant d(a).$$

Thus, in any one of the possible cases the appropriate limit relations hold and it follows that (x_n) has d as its distribution function.

We are now in a position to prove our Theorem 3.1.

Proof of Theorem 3.1. We may assume, without loss of generality, that $\delta_k = n_{k+1} - n_k > 0$ for all k. Let $\delta = \min_k \delta_k$, and let $h_k = \min_{n \geqslant k} \delta_n$. Finally, let $\epsilon_1 = 1$ and $\epsilon_k = 2e^{-h_{k-1}}$ for $k > 1$. Note that $\epsilon_k > 0$ and that $\lim_{k\to\infty} \epsilon_k = 0$. Define the interval I_n, for each integer n, as follows:

(i) If d is right continuous at z_n, let $I_n = [z_n - \epsilon_n, z_n - \epsilon_n/2]$ (or $[0, \epsilon_n/2]$ if $z_n - \epsilon_n \leqslant 0$).

(ii) If d is left continuous, but not right continuous, at z_n, let

$$I_n = [z_n + \epsilon_n/2, z_n + \epsilon_n] \quad (\text{or } [1 - \epsilon_n/2, 1] \text{ if } z_n + \epsilon_n \geqslant 1).$$

(iii) If d is neither left nor right continuous at z_n, we follow the construction prior to Lemma 3.3 and let

$$I_n = [z_n - \epsilon_n, z_n - \epsilon_n/2] \quad (\text{or } [0, \epsilon_n/2] \text{ if } z_n - \epsilon_n \leqslant 0),$$

or

$$I_n = [z_n + \epsilon_n/2, z_n + \epsilon_n] \quad (\text{or } [1 - \epsilon_n/2, 1] \text{ if } z_n + \epsilon_n \geqslant 1).$$

In each case, the length of I_n is $\frac{1}{2}\epsilon_n$. By Lemma 3.3, we need only demonstrate the existence of a θ with

$$\theta^{n_k} \in I_k \pmod 1, \qquad k = 1, 2, 3, \ldots ,$$

to complete the proof of the theorem.

Let A be any number such that

(i) $Ah_k - h_{k-1} \geqslant \log 2$ for all $k > 1$,

(ii) $Ah_1/n_1 \geqslant \log 2$.

Such a choice of A is possible since the sequence (h_k) is non-decreasing. Let $[\gamma_1, \mu_1] \equiv I_1 \pmod 1$, with $\gamma_1 \geqslant e^A$, and let $\alpha_1 = \gamma_1^{1/n_1}$, $\beta_1 = \mu_1^{1/n_1}$. Then $\theta \in [\alpha, \beta]$ implies that $\theta^{n_1} \in I_1 \pmod 1$. Let $[u_1, v_1] \equiv [\alpha_1^{n_2}, \beta_1^{n_2}]$. Then

$$v_1 - u_1 \geqslant (\beta_1 - \alpha_1)\alpha_1^{\delta_1} \geqslant \gamma_1^{\delta_1/n_1} \geqslant 2.$$

Hence we may find $[\gamma_2, \mu_2] \equiv I_2 \pmod 1$, $[\gamma_2, \mu_2] \subset [u_1, v_1]$. Let $\alpha_2 = \gamma_2^{1/n_2}$, $\beta_2 = \mu_2^{1/n_2}$. Then $\alpha_2 \geqslant \mu_1^{1/n_2} = \alpha_1$, $\beta_2 \leqslant v_1^{1/n_2} = \beta_1$, so $[\alpha_2, \beta_2] \subset [\alpha_1, \beta_1]$ and $\theta \in [\alpha_2, \beta_2]$ implies that $\theta^{n_2} \in I_2 \pmod 1$. We continue this process inductively. Having chosen $[\alpha_k, \beta_k] \subset [\alpha_{k-1}, \beta_{k-1}]$ so that $\theta \in [\alpha_k, \beta_k]$ implies $\theta^{n_k} \in \mathrm{I}_k \pmod 1$, we let $[u_k, v_k] = [\alpha_k^{n_{k+1}}, \beta_k^{n_{k+1}}]$. Then

$$\begin{aligned} v_k - u_k = \mu_k^{n_{k+1}/n_k} - \gamma_k^{n_{k+1}/n_k} \geqslant (\mu_k - \gamma_k)\gamma_k^{\delta_k/n_k} \geqslant \tfrac{1}{2}\epsilon_k\, \gamma_1^{\delta_k} \\ = e^{-h_{k-1}}\, e^{A\delta_k} \geqslant e^{Ah_k - h_{k-1}} \geqslant 2 \end{aligned}$$

so we may find $[\gamma_{k+1}, \mu_{k+1}] \equiv I_{k+1} \pmod 1$, $[\gamma_{k+1}, \mu_{k+1}] \subset [u_k, v_k]$. Let

$$\alpha_{k+1} = \gamma_{k+1}^{1/n_{k+1}}, \beta_{k+1} = \mu_{k+1}^{1/n_{k+1}}.$$

Then $\alpha_{k+1} \geqslant \mu_k^{1/n_{k+1}} = \alpha_k$ and $\beta_{k+1} \leqslant \beta_k$, so $[\alpha_{k+1}, \beta_{k+1}] \subset [\alpha_k, \beta_k]$ and if $\theta \in [\alpha_{k+1}, \beta_{k+1}]$, then $\theta^{n_{k+1}} \in I_{k+1}$ (mod 1). We thereby obtain a nested sequence of closed intervals

$$[\alpha_1, \beta_1] \supset [\alpha_2, \beta_2] \supset \ldots \supset [\alpha_k, \beta_k] \supset \ldots$$

and the point θ in their intersection has the property that

$$\theta^{n_k} \in I_k \pmod 1, \qquad k = 1, 2, 3, \ldots .$$

We can generalize this result in a fairly obvious manner.

Property C. Let h be a continuous function and let (f_n) be a sequence of continuous, increasing functions with $\lim_{x\to\infty} f_n(x) = \infty$ for every n. Suppose further that there exists a number p with $\lim_{n\to\infty} f_n(p) = \infty$ and a sequence (ϵ_n) of positive numbers tending to 0, such that if m is any positive integer and α and β are any two numbers for which

(i) $\beta > \alpha > f_m(p)$,

(ii) $h([\alpha, \beta])$ contains an interval of length $\epsilon_m/2$, then it follows that $h(f_{m+1}(f_m^{-1}([\alpha, \beta])))$ contains an interval of length 1.

Then we say that the pair $\{h, (f_n)\}$ has Property C.

THEOREM 3.4. *Suppose $\{h, (f_n)\}$ has Property* C *and that for each positive integer m there exists an integer N_m such that $h([m, N_m])$ contains an interval of length* 1. *Then, if d is any distribution function, there exists a real number θ such that the sequence $(h(f_n(\theta)))$ has d as its distribution function.*

COROLLARY 3.5. *Let h be a function with a positive, non-decreasing derivative for $x \geqslant M$. Let (n_k) be an increasing sequence of numbers with*

$$\lim_{k\to\infty}(n_{k+1} - n_k) = \infty .$$

Then, if d is any distribution function, there exists a θ such that the sequence $(h(\theta^{n_k}))$ has d as its distribution function.

COROLLARY 3.6. *Let (λ_k) be any sequence of positive numbers with*

$$\lim_{k\to\infty} (\lambda_{k+1}/\lambda_k) = \infty .$$

Let d be an arbitrary distribution function. Then there exists a θ such that the sequence $(\lambda_k \theta)$ has d as its distribution function.

An example of a sequence of this latter type is $(n!)$.

For some conditions which could be used instead of Property C, cf. **(4)**. For the question of the distribution functions of the sequence (xa^n), when a is an integer greater than 1, see **(5)**. It would be interesting to investigate sequences of this type when a is not an integer, but in this case we have neither the approximation technique used in the second half of this paper nor the ergodic properties of the transformation $x \to ax$ at our disposal.

https://doi.org/10.4153/CJM-1967-064-0 Published online by Cambridge University Press

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