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A FAMILY OF TWO-DIMENSIONAL LAGUERRE PLANES OF KLEINEWILLINGHÖFER TYPE II.A.2

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Abstract

Kleinewillinghöfer classified Laguerre planes with respect to linearly transitive groups of central automorphisms. Polster and Steinke investigated two-dimensional Laguerre planes and their so-called Kleinewillinghöfer types. For some of the feasible types the existence question remained open. We provide examples of such planes of type II.A.2, which are based on certain two-dimensional Laguerre planes of translation type. With these models only one type, I.A.2, is left for which no two-dimensional Laguerre planes are known yet.

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1. Introduction

Similar to the Lenz-Barlotti classification of projective planes, compare [10, Anhang, Section 6], Kleinewillinghöfer [5] classified Laguerre planes with respect to *central automorphisms*, that is, permutations of the point set of the Laguerre plane such that generators are mapped to generators and circles are mapped to circles and such that at least one point is fixed and central collineations are induced in the derived projective plane at one of the fixed points. In [14] and [21], two-dimensional Laguerre planes were considered and their so-called Kleinewillinghöfer types were investigated, that is, the Kleinewillinghöfer types of the (full) automorphism groups. In particular, all feasible types of two-dimensional Laguerre planes with respect to Laguerre translations, were completely determined in [14], the case of Laguerre homotheties was dealt with in [21] and Laguerre homologies are covered in [14, 17, 22]; see Section 3 for definitions of these kinds of central Laguerre plane automorphisms. Examples for some of the feasible combined Kleinewillinghöfer types of two-dimensional Laguerre planes (that is, with respect to all three types of central automorphisms Kleinewillinghöfer used in her classification) can be found in [14, Section 6], [9] and [20].

Sections 2 and 3 give brief reviews of some basic facts about two-dimensional Laguerre planes and their Kleinewillinghöfer types, respectively. In Section 4 we provide models for two-dimensional Laguerre planes of Kleinewillinghöfer type II.A.2. Hence, for two-dimensional Laguerre planes only the existence of one combined Kleinewillinghöfer type, type I.A.2, remains open.

2. Two-dimensional Laguerre planes

A *two-dimensional* or *flat Laguerre plane* $\mathcal{L} = (Z, C)$ is an incidence structure of points and circles whose point set is the cylinder $Z = \mathbb{S}^1 \times \mathbb{R}$ (where $\mathbb{S}^1$ is usually represented as $\mathbb{R} \cup \{\infty\}$), whose circles, elements in C, are graphs of continuous functions $\mathbb{S}^1 \to \mathbb{R}$ such that any three points, no two of which are on the same generator $\{c\} \times \mathbb{R}$ of the cylinder, can be joined by a unique circle and such that the circles which touch a fixed circle C at $p \in C$, that is, C itself and the circles which have only p in common with C, partition the complement in Z of the generator $[p]$ that contains p. For more information on two-dimensional Laguerre planes we refer to [1] and [3] or [13, Ch. 5]. We say that two points of Z are parallel if they are on the same generator of Z. This defines an equivalence relation $\|$ on Z whose equivalence classes, also called parallel classes, are the generators of Z.

It readily follows that for each point p of $\mathcal{L}$ the incidence structure $\mathcal{A}_p = (A_p, \mathcal{L}_p)$ whose point set A_p consists of all points of $\mathcal{L}$ that are not on the generator $[p]$ and whose line set $\mathcal{L}_p$ consists of all restrictions to A_p of circles of $\mathcal{L}$ passing through p and of all generators not containing p is an affine plane, which extends to a projective plane $\mathcal{P}_p$. We call $\mathcal{A}_p$ and $\mathcal{P}_p$ the *derived affine* and *projective plane at* p, respectively. In fact, the geometric axioms of a Laguerre plane are equivalent to all derived incidence structures $\mathcal{A}_p$ as defined above being affine planes.

The *classical two-dimensional Laguerre plane* is obtained as the geometry of nontrivial plane sections of a cylinder in $\mathbb{R}^3$ with an ellipse in $\mathbb{R}^2$ as the base, or equivalently, as the geometry of nontrivial plane sections of an elliptic cone, in real three-dimensional projective space, with its vertex removed. The parallel classes are the generators of the cylinder or cone. By replacing the ellipse in the construction of the classical two-dimensional Laguerre plane by arbitrary ovals in $\mathbb{R}^2$, that is, convex, differentiable simply closed curves, we also obtain two-dimensional Laguerre planes. These are the so-called *ovoidal two-dimensional Laguerre planes*.

Every automorphism of a two-dimensional Laguerre plane is continuous and thus a homeomorphism of Z. The collection of all automorphisms of a two-dimensional Laguerre plane $\mathcal{L}$ forms a group with respect to composition, the automorphism group Γ of $\mathcal{L}$. This group is a Lie group of dimension at most 7 with respect to the compact-open topology; see [18]. We call the dimension of Γ the *group dimension* of $\mathcal{L}$. The maximum dimension is attained precisely in the classical two-dimensional Laguerre plane. In fact, group dimension of at least 6 implies classical. Furthermore, two-dimensional Laguerre planes of group dimension 5 must be special ovoidal Laguerre planes; see [8, Theorem 1].

The collection of all automorphisms of $\mathcal{L}$ that fix each generator of $\mathcal{L}$ is a closed normal subgroup of Γ, called the *kernel* of $\mathcal{L}$. The kernel of a two-dimensional Laguerre plane has dimension of at most 4. Furthermore, a kernel of dimension 4 characterizes the ovoidal Laguerre planes among two-dimensional Laguerre planes, that is, a two-dimensional Laguerre plane $\mathcal{L}$ is ovoidal if and only if its kernel is four-dimensional; see [2].

2.1. Two-dimensional Laguerre planes of translation type. One particular family of two-dimensional Laguerre planes we will be referring to, and on which our construction is ultimately based, is the class of *two-dimensional Laguerre planes of translation type* introduced in [7]. These planes depend on two strongly parabolic functions; see [7, 1.3] for a precise definition. Since the two-dimensional Laguerre planes of translation type we are interested in are of a special kind we only present those planes instead of giving the full and lengthy definition for the general case.

Let $f_m : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f_m(x) = |x|^m$$

where $m > 0$. Note that f_m is multiplicative and twice continuously differentiable for all $x \neq 0$. One has

$$f_m'(x) = mx|x|^{m-2} \quad \text{and} \quad f_m''(x) = m(m-1)|x|^{m-2}.$$

In particular, when $m > 2$, the second derivative is always nonnegative so that f_m is strictly convex.

Now $\mathcal{L}(f_m)$, $m > 1$, is the two-dimensional Laguerre plane on the cylinder Z whose circles are the sets

$$D_{0,b,c} = \{(x, bx + c) \mid x \in \mathbb{R}\} \cup \{(\infty, 0)\},$$

where $b, c \in \mathbb{R}$ and

$$D_{a,b,c} = \{(x, af_m(x - b) + c) \mid x \in \mathbb{R}\} \cup \{(\infty, a)\},$$

where $a, b, c \in \mathbb{R}, a \neq 0$. (These are the planes $\mathcal{E}_\Lambda(f_m, f_m)$ in the notation of [7].) Note that the derived affine plane at the point $(\infty, 0)$ is the Euclidean plane. Moreover, circles that touch $D_{0,0,0}$ are precisely the circles $D_{a,b,c}$ where $a = b = 0$ or $c = 0$, and that circles that meet $D_{0,0,0}$ in two points are precisely the circles $D_{a,b,c}$ where $a = 0$, $b \neq 0$ or $a \neq 0$, $c < 0$.

Löwen and Pfüller [7, Theorems 5 and 9] prove the following result.

Proposition 2.1. *A plane $\mathcal{L}(f_m)$, $m > 1$, is ovoidal if and only if $m = 2$. In this case the plane is classical.*

$\mathcal{L}(f_m)$, $m > 1$, has group dimension 4. The (full) automorphism group fixes the point $(\infty, 0)$ and consists of the transformations

$$(x, y) \mapsto \begin{cases} (rx + u, sy + v) & \text{if } x \in \mathbb{R}, \\ (\infty, sy/f_m(r)) & \text{if } x = \infty, \end{cases}$$

where $r, s, u, v \in \mathbb{R}$, $r, s \neq 0$.

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2.2. Cut-and-paste. There are many ways to combine different two-dimensional Laguerre planes into new two-dimensional Laguerre planes; see for example, [12, Section 4] or [13, Section 5.3]. One construction, which we will be using in order to obtain our models of two-dimensional Laguerre planes of type II.A.2 is as follows. Let $\mathcal{L} = (Z, \mathcal{C})$ be a two-dimensional Laguerre plane and let $C_0 \in \mathcal{C}$ be a circle. Consider the collection $\mathcal{C}^1$ of all circles that touch C_0. (Note that this includes the circle C_0.) The circle C_0 separates the cylinder Z into two connected components, the open upper half-cylinder Z^+ and the open lower half-cylinder Z^-. We define $\mathcal{C}^{\pm}$ to be the collection of all circles that are completely contained in $Z^{\pm}$. Finally, let $\mathcal{C}^2$ be the set of all circles that intersect C_0 in precisely two points. Obviously, $\mathcal{C}^1 \cup \mathcal{C}^2 \cup \mathcal{C}^+ \cup \mathcal{C}^-$ is a partition of the circle set. With this notation, the following result was obtained in [12, Proposition 6].

Proposition 2.2. *Let $\mathcal{L}_i = (Z, \mathcal{C}_i)$, $i = 1, 2, 3$ be three two-dimensional Laguerre planes. Suppose that $(\mathcal{C}_1)^1 = (\mathcal{C}_2)^1 = (\mathcal{C}_3)^1 =: \mathcal{C}^1$ for some circle C_0. Then $\mathcal{L} := (Z, \mathcal{C}^1 \cup \mathcal{C}_1^2 \cup \mathcal{C}_2^+ \cup \mathcal{C}_3^-)$ is a two-dimensional Laguerre plane.*

3. Kleinewillinghöfer types of two-dimensional Laguerre planes

Central automorphisms are automorphisms that have at least one fixed point and induce central collineations in the derived projective plane at this fixed point. Kleinewillinghöfer considered four kinds of central automorphisms: C-homologies, G-translations, $(G, B(q, C))$-translations and (p, q)-homotheties; see the following for definitions. The four different kinds of central automorphisms above are distinguished according to the relative position of the centre and axis and whether or not the axis is the line at infinity of the derived affine plane at one of its fixed points. The notions of translation, homothety and homology describe the sort of central collineation one sees in this derived affine plane.

A subgroup of central automorphisms that have the same 'centre' and 'axis' is *linearly transitive* if the induced group of central collineations in a derived projective plane at one of the fixed points is transitive on each central line except for the obvious fixed points, the centre and the point of intersection with the axis. Kleinewillinghöfer considered the automorphism groups of Laguerre planes and determined their types according to linearly transitive subgroups of central automorphisms contained in them. We say that the Laguerre plane is of type X if the (full) automorphism group Γ has Kleinewillinghöfer type X.

A *Laguerre homology* of a Laguerre plane $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that is either the identity or fixes precisely the points of one circle. One speaks of a *C-homology* if C is the circle that is fixed pointwise. A C-homology induces a homology of the derived projective plane $\mathcal{P}_q$ at each $q \in C$ with the centre being the point ω at infinity of lines coming from generators of $\mathcal{L}$ and the axis, the line induced by C. With respect to Laguerre homologies, Kleinewillinghöfer obtained seven types of Laguerre planes, labelled I, II, III, IV, V, VI and VII; see [5, Satz 3.1]. Of these types, type VI cannot occur in two-dimensional Laguerre planes; see [14, Proposition 3.4].

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A *Laguerre translation* of $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that is either the identity or fixes precisely the points of one generator and induces a translation in the derived affine plane at one of its fixed points. Laguerre translations come in two different varieties. Firstly, a nonidentity *G-translation* of $\mathcal{L}$ is a Laguerre translation that fixes precisely the points of the generator G and furthermore fixes each generator globally. For the second variety of Laguerre translations one uses a tangent bundle $B(p, C)$, that is, all circles that touch the circle C at the point p. In the derived affine plane at p, the tangent bundle represents a parallel class of lines, and we can look at translations in this direction. Then a *$(G, B(p, C))$-translation* of $\mathcal{L}$ is a Laguerre translation that fixes C (and each circle in $B(p, C)$) globally. With respect to Laguerre translations Kleinewillinghöfer obtained 11 types of Laguerre planes, labelled A through to K; see [5, Satz 3.3], or [6, Satz 2]. Of these types, types F, I and J cannot occur in two-dimensional Laguerre planes; see [14, Proposition 4.8].

Finally, a *Laguerre homothety* of $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that is either the identity or fixes precisely two nonparallel points and induces a homothety in the derived affine plane at each of these two fixed points. One speaks of a *$\{p, q\}$-homothety* if p, q are the two fixed points. With respect to Laguerre homotheties Kleinewillinghöfer obtained 13 types of Laguerre planes, labelled 1 through to 13; see [5, Satz 3.2] or [6, Satz 1]. Types 5, 6, 7, 9, 10 and 12 cannot occur in two-dimensional Laguerre planes; see [14, Proposition 5.6] and [21].

Combining all three classifications, Kleinewillinghöfer obtained a total of 46 combined types. In two-dimensional Laguerre planes, 21 of these 46 combined types cannot occur. There are models of two-dimensional Laguerre planes of types I.A.1, I.B.1, I.B.3, I.C.1, I.E.1, I.E.4, I.G.1, I.H.1, I.H.11, II.A.1, II.E.1, II.E.4, II.G.1, III,B.1, III.B.3, III.H.1, III.H.11, IV.A.1, IV.A.2, V.A.1, VII.D.1, VII.D.8 and VII.K.13; see [14, Section 6], [9, 16, 17, 20–22]. Here a combined type just refers to the respective single types. For example, type III.B.3 refers to type III with respect to Laguerre homologies, type B with respect to Laguerre translations, and type 3 with respect to Laguerre homotheties. (This notation for combined types is different from the one used in [5] but more consistent.) Note that there is a two-dimensional Laguerre plane of each of the single Kleinewillinghöfer types not excluded for two-dimensional Laguerre planes.

In particular, the Kleinewillinghöfer type II.A.2 we are interested in in this paper is defined as follows.

- In type II (with respect to Laguerre homologies) there is a single circle C for which the group of Laguerre homologies is linearly transitive.
- In type A (with respect to Laguerre translations) there is neither a tangent bundle nor a generator for which the group of Laguerre translations is linearly transitive.
- In type 2 (with respect to Laguerre homotheties) there is a single unordered pair $\{p, q\}$ for which the group of Laguerre homotheties is linearly transitive.

Proposition 3.1. *A two-dimensional Laguerre plane $\mathcal{L}$ of Kleinewillinghöfer type II.A.2 with distinguished points p and q and distinguished circle C has group dimension 2. Furthermore, the group H generated by all C-homologies and all $\{p, q\}$-homologies has index of at most 2 in the automorphism group Γ of $\mathcal{L}$.*

Proof. Note that in type II.A.2 the distinguished points p and q must be on the distinguished circle C, and C and the pair $\{p, q\}$ must be fixed by the full automorphism group. Indeed, if γ is a C-homology and α any automorphism of $\mathcal{L}$, then $\alpha\gamma\alpha^{-1}$ is an $\alpha(C)$-homology, and similarly for $\{p, q\}$-homotheties. Hence, the automorphism group Γ of such a plane is two-dimensional. This follows by repeated application of the dimension formula $\dim G = \dim G_x + \dim G(x)$, which relates the dimensions of a Lie group G acting on Z, the stabilizer G_x of a point $x \in Z$ and its orbit (see [4]) and the stabilizer lemma, which says that the stabilizer of three points on a circle and one point off this circle is trivial (see [19, Lemma 2.10]). Hence,

$$\dim \Gamma = \dim \Gamma_{p,q} = \dim \Gamma_{p,q,c} + 1 = \dim \Gamma_{p,q,c,r} + 2 = 2$$

where $c \in C \setminus \{p, q\}$ and $r \in [p] \setminus \{p\}$. Furthermore, the stabilizer lemma and transitivity properties of H show that H has index of at most 2 in Γ. □

Since the commutator of a C-homology and a $\{p, q\}$-homothety with $p, q \in C$ is a C-homology and a $\{p, q\}$-homothety, it must be the identity. Hence these central automorphisms commute. By [15, Proposition 44.8b] the group H is isomorphic to the direct product of multiplicative loops of some ternary fields coordinatizing the derived projective plane at p (or q).

Due to the small dimension of Γ only limited information is available to construct such a Laguerre plane. However, there are examples of two-dimensional Laguerre planes of small group dimension that admit subgroups of automorphisms of type II.A.2. One possible strategy therefore is to start with such a plane of 'higher' type and distort circles to destroy central automorphisms that do not belong to type II.A.2. This method was, for example, applied in [22] to find two-dimensional Laguerre planes of type IV.A.2 from ovoidal Laguerre planes (of type VII.D.8).

Another way is to take two suitable two-dimensional Laguerre planes of higher type and exchange circles (like according to Proposition 2.2) to remove unwanted central automorphisms. For example, the latter method has been successfully employed to obtain two-dimensional Laguerre planes of Kleinewillinghöfer type E (with respect to Laguerre translations) from two two-dimensional Laguerre planes of translation type as in Section 2.1; see [20] for details.

4. The models

In this section we present models for two-dimensional Laguerre planes of Kleinewillinghöfer type II.A.2. They depend on one real parameter $k > 2$. The strategy is to use the circles of two isomorphic models of two-dimensional Laguerre planes of translation type over the multiplicative parabolic function f_k introduced in Section 2.1

and exchange certain circles. The two distinguished points (as in type 2 with respect to Laguerre homotheties) will be $(\infty, 0)$ and $(0, 0)$. The distinguished circle (as in type II with respect to Laguerre homologies) will be $\mathbb{S}^1 \times \{0\}$. The condition on k can be weakened a bit further, but with the above constraint the arguments we shall be using become shorter.

Description of the models $\mathcal{L}(k)$. Fix $k > 2$. For all $a, b, c \in \mathbb{R}$ we define a function $f_{a,b,c}$, that describes a circle, by

$$f_{a,b,c}(x) = \begin{cases} bx + c & \text{if } a = 0, \\ a|x - b|^k + c & \text{if } ac \leq 0, a \neq 0, \\ \dfrac{a}{a|b|^k + c}(a|bx - 1|^k + c|x|^k) & \text{if } ac > 0. \end{cases}$$

Then $\mathcal{L}(k)$ has point set $Z = (\mathbb{R} \cup \{\infty\}) \times \mathbb{R}$. Two points $(x_1, y_1), (x_2, y_2) \in Z$ are parallel if and only if $x_1 = x_2$, and generators in $\mathcal{L}$ are the sets $\Pi_u = \{u\} \times \mathbb{R}$ where $u \in \mathbb{R} \cup \{\infty\}$. The sets

$$C_{a,b,c} = \{(x, f_{a,b,c,}(x)) \mid x \in \mathbb{R}\} \cup \{(\infty, a)\}$$

are the circles of $\mathcal{L}(k)$, that is,

$$C(k) = \{C_{a,b,c,} \mid a, b, c \in \mathbb{R}\}.$$

The topology of Z induces the Euclidean topology on $\mathbb{R}^2$. The neighbourhoods of points (∞, a) at infinity consist of all points (∞, a') such that $|a' - a|$ is sufficiently small and all points (x, y) of $\mathbb{R}^2$ such that $|x|$ is sufficiently large and $|a - y/|x|^k|$ is sufficiently small. According to [11, Proposition 2], this is the unique topology on Z that makes the Laguerre plane into a two-dimensional Laguerre plane.

We claim that $\mathcal{L}(k) = (Z, C(k))$ is a two-dimensional Laguerre plane. Note that circles $C_{a,b,c}$ such that $ac \leq 0$ are circles of the two-dimensional Laguerre plane $\mathcal{L}(f_k)$ of translation type. Moreover, these circles comprise all circles of $\mathcal{L}(k)$ that meet the circle $C_{0,0,0}$ in at least one point. When $k = 2$ one obtains the classical two-dimensional Laguerre plane.

For each $r, s \in \mathbb{R}$, $r, s \neq 0$, let $\gamma_{r,s} : Z \to Z$ be the permutation of Z defined by

$$\gamma_{r,s} : (x, y) \mapsto \begin{cases} (rx, sy) & \text{if } x \in \mathbb{R}, \\ (\infty, sy/|r|^k) & \text{if } x = \infty. \end{cases}$$

It is readily verified that $\gamma_{r,s}$ is an automorphism of $\mathcal{L}(k)$ and that

$$\gamma_{r,s}(C_{a,b,c}) = \begin{cases} C_{0,bs/r,cs} & \text{if } a = 0, \\ C_{as/|r|^k,br,cs} & \text{if } ac \leq 0, a \neq 0, \\ C_{as/|r|^k,b/r,cs/|r|^{2k}} & \text{if } ac > 0, \end{cases}$$

where $r, s \in \mathbb{R}$, $r, s \neq 0$. Let

$$G = \{\gamma_{r,s} \mid r, s \in \mathbb{R} \setminus \{0\}\}.$$

This is a group of automorphisms of $\mathcal{L}(k)$. Furthermore G has six orbits on the cylinder Z, namely $\{(\infty, 0)\}$, $\Pi_\infty \setminus \{(\infty, 0)\}$, $\{(0, 0)\}$, $\Pi_0 \setminus \{(0, 0)\}$, $\{(x, 0) \mid x \in \mathbb{R}, x \neq 0\}$ and $\{(x, y) \in \mathbb{R}^2 \mid x, y \neq 0\}$.

Theorem 4.1. *$\mathcal{L}(k)$, $k > 2$, is a two-dimensional Laguerre plane.*

Proof. We consider the two-dimensional Laguerre plane $\mathcal{L}(f_k)$ of translation type from Section 2.1. We apply the transformation $\sigma : Z \to Z$ given by

$$\sigma(x, y) = \begin{cases} (1/x, y/|x|^k) & \text{if } x \in \mathbb{R}, x \neq 0, \\ (\infty, y) & \text{if } x = 0, \\ (0, y) & \text{if } x = \infty. \end{cases}$$

This gives us a two-dimensional Laguerre plane $\mathcal{L}'_k = \sigma(\mathcal{L}(f_k))$ that is isomorphic to $\mathcal{L}(f_k)$. The circles of $\mathcal{L}'_k$ are the sets $D'_{a,b,c} = \sigma(D_{a,b,c})$ where $a, b, c \in \mathbb{R}$ and

$$D'_{a,b,c} = \begin{cases} \{(x, bx|x|^{k-2} + c|x|^k) \mid x \in \mathbb{R}\} \cup \{(\infty, c)\} & \text{if } a = 0, \\ \{(x, a|1 - bx|^k + c|x|^k) \mid x \in \mathbb{R}\} \cup \{(\infty, a|b|^k + c)\} & \text{if } a \neq 0. \end{cases}$$

From this list we see that $D'_{0,0,c} = D_{c,0,0}$, $D'_{a,b,0} = D_{a|b|^k,1/b,0}$ where $b \neq 0$ and $D'_{a,0,0} = D_{0,0,a}$. This shows that the circles in $\mathcal{L}'_k$ that touch $D'_{0,0,0} = D_{0,0,0}$ are the same as the circles in $\mathcal{L}(f_k)$ that touch $D_{0,0,0} = D'_{0,0,0}$. Hence the assumptions of Proposition 2.2 are satisfied. We can therefore interchange the circles in $\mathcal{L}(f_k)$ that do not meet $D_{0,0,0}$, that is, the circles $D_{a,b,c}$ where $ac > 0$, with the circles in $\mathcal{L}'_k$ that do not meet $D'_{0,0,0}$, that is, the circles $D'_{a,b,c}$ where $ac > 0$, and obtain a new two-dimensional Laguerre plane. A straightforward algebraic manipulation then yields precisely the circles of the form $C_{a,b,c}$ of $\mathcal{L}(k)$. □

It turns out that there is a unique point at which the derived affine plane of $\mathcal{L}(k)$ is Desarguesian. We only prove a weaker result, which will be sufficient to deduce that this point must be fixed under any automorphism of $\mathcal{L}(k)$.

Proposition 4.2. *The derived affine plane $\mathcal{A}_{(\infty,0)}$ of $\mathcal{L}(k)$ at $(\infty, 0)$ is Desarguesian whereas the derived affine planes $\mathcal{A}_{(\infty,1)}$ and $\mathcal{A}_{(0,0)}$ at $(\infty, 0)$ and $(0, 0)$, respectively, are non-Desarguesian.*

Proof. *The affine plane $\mathcal{A}_{(\infty,0)}$:* Since circles through $(\infty, 0)$ are the same as in the classical two-dimensional Laguerre plane, we immediately obtain that the derived incidence structure $\mathcal{A}_{(\infty,0)}$ of $\mathcal{L}(k)$ at $(\infty, 0)$ is the real Desarguesian affine plane.

The affine plane $\mathcal{A}_1 = \mathcal{A}_{(\infty,1)}$: The nonvertical lines of $\mathcal{A}_1$ are the traces of circles $C_{1,b,c}$ where $b, c \in \mathbb{R}$. Thus nonvertical lines of $\mathcal{A}_1$ are of the form

$$L_{b,c} = \{(x, |x - b|^k + c) \mid x \in \mathbb{R}\}$$

where $b, c \in \mathbb{R}$, $c \leq 0$, and

$$L_{b,c} = \left\{ \left(x, \frac{|bx - 1|^k + c|x|^k}{|b|^k + c} \right) \,\middle|\, x \in \mathbb{R} \right\}$$

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where $b, c \in \mathbb{R}$, $c > 0$. Let $H_c = L_{0,c}$ when $c \leq 0$ and $H_c = L_{0,1/c}$ when $c > 0$. Then

$$H_c = \{(x, |x|^k + c) \mid x \in \mathbb{R}\}$$

for all $c \in \mathbb{R}$. It is obvious that the H_c form a bundle $\mathcal{H}$ of parallel lines in $\mathcal{A}_1$.

Suppose that $\mathcal{A}_1$ is Desarguesian. Then this plane admits a central collineation φ of order 2 with axis H_0 and centre the point at infinity of vertical lines. Since φ permutes the lines in $\mathcal{H}$, there is an involution ψ such that $\varphi(H_c) = H_{\psi(c)}$ for all $c \in \mathbb{R}$. From this,

$$\varphi(x, y) = (x, |x|^k + \psi(y - |x|^k))$$

for all $(x, y) \in \mathbb{R}^2$. The homology φ also permutes the lines $L_{b,-|b|^k}$ through $(0, 0) \in H_0$. Hence for each $b \in \mathbb{R}$ there exists a $\beta \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$|x - \beta|^k - |\beta|^k - |x|^k = \psi(|x - b|^k - |b|^k - |x|^k). \tag{4.1}$$

Substitution of $x = \beta$ and $x = b$ in (4.1) yields

$$-2|\beta|^k = \psi(|\beta - b|^k - |b|^k - |\beta|^k) = \psi^2(-2|b|^k) = -2|b|^k.$$

Hence $|\beta| = |b|$. Since $L_{b,-|b|^k}$ neither is the axis H_0 of φ when $b \neq 0$ nor passes through the centre of φ, we see that $\beta \neq b$. Thus $\beta = -b$ and Equation (4.1) becomes

$$|x + b|^k - |b|^k - |x|^k = \psi(|x - b|^k - |b|^k - |x|^k) \tag{4.2}$$

for all $b, x \in \mathbb{R}$. When $x = b$ in (4.2), then $(2^k - 2)|b|^k = \psi(-2|b|^k)$ for all $b \in \mathbb{R}$. Therefore

$$\psi(z) = (1 - 2^{k-1})z$$

for all $z \leq 0$. We now substitute $b = 2$, $x = 1$ in (4.2) to find with the above formula for ψ that

$$3^k - 2^k - 1 = \psi(-2^k) = (2^{k-1} - 1)2^k.$$

Hence $3^k - 1 = 2^{2k-1}$, which is impossible because $k > 2$. This shows that $\mathcal{A}_1$ is not Desarguesian.

The affine plane $\mathcal{A}_0 = \mathcal{A}_{(0,0)}$: The nonvertical lines of $\mathcal{A}_0$ are the traces of circles $C_{a,b,-a|b|^k}$ where $a, b \in \mathbb{R}$, $a \neq 0$, and $C_{0,b,0}$, $b \in \mathbb{R}$. We make the coordinate transformation

$$(x, y) \mapsto \begin{cases} (1/x, y/|x|^k) & \text{if } x \in \mathbb{R}, x \neq 0, \\ (\infty, y) & \text{if } x = 0, \\ (0, y) & \text{if } x = \infty. \end{cases}$$

In the new coordinates nonvertical lines of $\mathcal{A}_0$ are of the form

$$L_{a,b} = \{(x, a(|bx - 1|^k - |bx|^k)) \mid x \in \mathbb{R}\}$$

where $a, b \in \mathbb{R}$, $a \neq 0$,

$$L_{0,b} = \{(x, bx|x|^{k-2}) \mid x \in \mathbb{R}\}$$

where $b \in \mathbb{R}$.

Assume that $\mathcal{A}_0$ is Desarguesian. Then this plane admits all central collineations φ with the axis the line at infinity and the centre the point at infinity of vertical lines. (That is, φ is a translation in the vertical direction.) In particular, there is one such translation that takes the point $(0,0)$ to the point $(0,1/k)$. Since φ permutes the horizontal lines $L_{a,0}$, $a \in \mathbb{R}$, there is a permutation ψ such that $\varphi(L_{a,0}) = L_{\psi(a),0}$ for all $a \in \mathbb{R}$. From this we obtain that $\varphi(x,y) = (x,\psi(y))$ for all $(x,y) \in \mathbb{R}^2$. Furthermore, φ takes $L_{0,1}$ to a line through $(0,1/k)$ so that there exists a $b \in \mathbb{R}$ such that $\varphi(L_{0,1}) = L_{1/k,b}$. But $L_{1/k,b}$ must be parallel to $L_{0,1}$. This means that the circles $C_{1/k,b,-|b|^k/k}$ and $C_{0,1,0}$ touch each other at the point $(0,0)$, which implies that the functions $f_{1/k,b,-|b|^k/k}$ and $f_{0,1,0}$ that describe these circles have the same derivative at $x = 0$. Thus we obtain that $b = -1$. Hence for all $x \in \mathbb{R}$

$$\psi(x|x|^{k-2}) = \frac{1}{k}(|x+1|^k - |x|^k).$$

In particular we see that

$$\psi(-1/2^{k-1}) = \frac{1}{k}\left(\left|-\frac{1}{2}+1\right|^k - \left|\frac{1}{2}\right|^k\right) = 0 \quad \text{and that}$$

$$\psi(1/2^{k-1}) = \frac{1}{k}\left(\left|\frac{1}{2}+1\right|^k - \left|\frac{1}{2}\right|^k\right) = \frac{3^k-1}{2^k k}.$$

We now consider the line $L_{-2^{1-k},1}$ through $(0,-2^{1-k})$. This line is taken under φ to a line through $(0,0)$ and thus is of the form $L_{0,m}$. Since the point $(1/2,0)$ belongs to $L_{-2^{1-k},1}$ and $\psi(0) = 1/k$, the line $L_{0,m}$ must pass through $(1/2,1/k)$. Thus $m = 2^{k-1}/k$ and

$$\psi(-2^{1-k}(|x-1|^k - |x|^k)) = \frac{2^{k-1}}{k} x|x|^{k-2}$$

for all $x \in \mathbb{R}$. Substitution of $x = 1$ gives us $\psi(2^{1-k}) = 2^{k-1}/k$. Comparison with the value we found earlier yields that $2^{k-1}/k = (3^k-1)/2^k k$ so that $3^k - 1 = 2^{2k-1}$, which is impossible because $k > 2$. This shows that $\mathcal{A}_0$ is not Desarguesian. □

Corollary 4.3. *The derived affine plane $\mathcal{A}_p$ of $\mathcal{L}(k)$ at a point p of the generator $[(\infty,0)]$ through $(\infty,0)$ or a point on $C_{0,,0}$ is Desarguesian if and only if $p = (\infty,0)$.*

Proof. Note that for arbitrary $u \in \mathbb{R}$ the nonvertical lines of $\mathcal{A}_{(u,0)}$ come from circles that do meet $C_{0,0,0}$. Thus they are the same as in the two-dimensional Laguerre plane $\mathcal{L}(f_k)$ of translation type introduced in Section 2.1. Since the automorphism group of $\mathcal{L}(f_k)$ is transitive on $C_{0,0,0} \setminus \{(\infty,0)\}$, we conclude that $\mathcal{A}_{(u,0)}$ is isomorphic to $\mathcal{A}_{(0,0)}$. Since the latter affine plane is non-Desarguesian by Proposition 4.2, so is $\mathcal{A}_{(u,0)}$.

Since G is transitive on $[(\infty,0)] \setminus \{(\infty,0)\}$, a derived plane $\mathcal{A}_{(\infty,v)}$ where $v \neq 0$ is isomorphic to the derived plane at $(\infty,1)$, which is non-Desarguesian. □

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Proposition 4.4. *Every automorphism α of $\mathcal{L}(k)$ fixes the circle $C_{0,0,0}$.*

Proof. Let α be an automorphism of $\mathcal{L}(k)$ and consider the circle $C = \alpha(C_{0,0,0})$. Assume that $C \neq C_{0,0,0}$. It is readily verified that

$$\{\gamma_{1,s} \mid s \in \mathbb{R}, s \neq 0\}$$

is a linearly transitive group of $C_{0,0,0}$-homologies. Conjugation by α then yields a linearly transitive group of C-homologies. Any nonidentity C-homology will also move $C_{0,0,0}$. We may therefore, without loss of generality, assume that α belongs to the kernel Δ of $\mathcal{L}(k)$. In this case, when $p \in C \setminus C_{0,0,0}$, then Δ is transitive on the generator $[p]$. From Corollary 4.3 it follows that $(\infty, 0)$ must belong to both C and $C_{0,0,0}$. Therefore $(\infty, 0)$ is fixed by α. But then α induces a central collineation α' of $\mathcal{A}_{(\infty,0)}$ with the centre the point ω at infinity of vertical lines. Thus α' is of the form

$$\alpha' : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x, sy + mx + t)$$

where $m, s, t \in \mathbb{R}$, $s \neq 0$. By applying $\gamma_{1,s}^{-1}$ we may without loss of generality assume that $s = 1$.

Now $\alpha(C_{1,0,0}) = \{(x, |x|^k + mx + t) \mid x \in \mathbb{R}\}$ augmented by a point at infinity. In fact, dividing by $|x|^k$ and considering the limit as x tends to ∞, one sees that the point at infinity is $(\infty, 1)$. This set has to be a circle $C_{1,b,c}$ of $\mathcal{L}(k)$ where $b, c \in \mathbb{R}$. Since $k > 2$, corresponding circles describing functions $f_{1,b,c}$ do not have linear terms so that $m = 0$. More precisely, one first shows that $b = 0$, which follows by differentiating the identity $|x|^k + mx + t = f_{1,b,c}(x)$ twice and substituting $x = 0$. Then $f_{1,0,c}(x) = |x|^k + c$ when $c \leq 0$ and $f_{1,0,c}(x) = |x|^k + (1/c)$ when $c > 0$. In either case one obtains that $m = 0$. We then can without loss of generality assume that $t \geq 0$.

Finally, $\alpha(C_{1,1,0}) = \{(x, |x-1|^k + t) \mid x \in \mathbb{R}\} \cup \{(\infty, 1)\}$ is a circle $C_{1,b',c'}$ where $b', c' \in \mathbb{R}$. Furthermore, $c' > 0$ in the case $t > 0$. Hence, the identity

$$(|b'|^k + c')(|x-1|^k + t) = |b'x - 1|^k + c'|x|^k$$

for all $x \in \mathbb{R}$. Differentiation with respect to x gives us

$$k(|b'|^k + c')(x-1)|x-1|^{k-2} = k(b'(b'x-1)|b'x-1|^{k-2} + c'x|x|^{k-2})$$

for all $x \in \mathbb{R}$. When $x = 0$ one finds that $|b'|^k + c' = b'$. In particular $b' > 0$. Differentiating again with respect to x yields

$$k(k-1)(|b'|^k + c')|x-1|^{k-2} = k(k-1)((b')^2|b'x-1|^{k-2} + c'|x|^{k-2})$$

for all $x \in \mathbb{R}$. When $x = 0$ one obtains that $|b'|^k + c' = (b')^2$. Thus $(b')^2 = b'$ from above and so $b' = 1$. But then $c' = b' - |b'|^k = 0$—a contradiction to $c' > 0$.

This shows that $m = t = 0$. Therefore $C_{0,0,0}$ is fixed. □

Theorem 4.5. *The automorphism group of $\mathcal{L}(k)$, $k > 2$, is the group G as defined before Theorem 4.1. Furthermore, $\mathcal{L}(k)$ is of Kleinewillinghöfer type II.A.2.*

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Proof. By Theorem 4.1 we know that $\mathcal{L}(k)$ is a two-dimensional Laguerre plane. We first note that every automorphism α of $\mathcal{L}(k)$ fixes the point $(\infty, 0)$ and the circle $C_{0,0,0}$. Indeed, by Proposition 4.4 we know that $C_{0,0,0}$ is fixed by every automorphism. Corollary 4.3 shows that the derived affine plane $\mathcal{A}_{(\infty,0)}$ is Desarguesian whereas all other derived affine planes $\mathcal{A}_p$ where $p \in C_{0,0,0}$, $p \neq (\infty, 0)$ are non-Desarguesian. Thus $(\infty, 0)$ is fixed by α. As seen in the proof of Proposition 4.4 α induces a collineation α' of $\mathcal{A}_{(\infty,0)}$ and is of the form

$$\alpha' : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (rx + t, sy)$$

where $r, s, t \in \mathbb{R}$, $r, s \neq 0$. By applying $\gamma_{r,s}^{-1}$ we may without loss of generality assume that $r = s = 1$.

Now $\alpha(C_{1,0,1}) = \{(x, 1 + |x - t|^k) \mid x \in \mathbb{R}\} \cup \{(\infty, 1)\}$ has to be a circle $C_{1,b,c}$ of $\mathcal{L}(k)$. Furthermore this set is above $C_{0,0,0}$ and thus $c > 0$. Comparison with circle describing functions $f_{1,b,c}$ yields that $b = 0 = t$. Indeed, from

$$(|b|^k + c)(1 + |x - t|^k) = |bx - 1|^k + c|x|^k$$

for all $x \in \mathbb{R}$ we obtain by differentiation with respect to x and division by k that

$$(|b|^k + c)(x - t)|x - t|^{k-2} = b(bx - 1)|bx - 1|^{k-2} + cx|x|^{k-2} \tag{4.3}$$

for all $x \in \mathbb{R}$. When $x = 0$ this identity yields $(|b|^k + c)t|t|^{k-2} = b$. Differentiating (4.3) again and dividing by $k - 1$ gives us

$$(|b|^k + c)|x - t|^{k-2} = b^2|bx - 1|^{k-2} + c|x|^{k-2}$$

for all $x \in \mathbb{R}$. Evaluation at $x = 0$ shows that $(|b|^k + c)|t|^{k-2} = b^2$. Hence $tb^2 = b$ so that either $b = 0$ or $tb = 1$. In the former case it follows that $ct|t|^{k-2} = 0$ and thus $t = 0$. In the latter case we substitute $x = t$ into Equation (4.3) to obtain $0 = ct|t|^{k-2}$—a contradiction to $tb = 1$.

This shows that α' and thus α is the identity (on $\mathbb{R}^2$ and Z, respectively). In summary we have shown that $\mathrm{Aut}(\mathcal{L}(k))$ equals G as defined before Theorem 4.1.

Since G fixes each of the points $(\infty, 0)$ and $(0, 0)$ and the circle $C_{0,0,0}$, a Laguerre translation must be the identity. Thus $\mathcal{L}(q)$ has type A with respect to Laguerre translations.

Similarly, the centres of a Laguerre homothety must be $(\infty, 0)$ and $(0, 0)$, and the circle that forms the axis of a Laguerre homology must be $C_{0,0,0}$. On the other hand, as mentioned in the proof of Proposition 4.4 the group $\{\gamma_{1,s} \mid s \in \mathbb{R}, s \neq 0\}$ is a linearly transitive group of $C_{0,0,0}$-homologies. It is readily verified that $\{\gamma_{r,r} \mid r \in \mathbb{R}, r \neq 0\}$ is a linearly transitive group of $\{(\infty, 0), (0, 0)\}$-homotheties. Hence, it follows that $\mathcal{L}(q)$ has combined Kleinewillinghöfer type II.A.2 as claimed. □

In a similar fashion one shows that an isomorphism from $\mathcal{L}(k)$ to $\mathcal{L}(k')$ where $k, k' > 2$, takes $(\infty, 0)$ in $\mathcal{L}(k)$ to $(\infty, 0)$ in $\mathcal{L}(k')$. One then has an induced isomorphism between the derived affine planes at $(\infty, 0)$, which are both Desarguesian. An analysis as in the proof of Theorem 4.5 shows that $k = k'$. Thus one has the following result.

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THEOREM 4.6. *Two two-dimensional Laguerre planes $\mathcal{L}(k)$ and $\mathcal{L}(k')$ where $k, k' > 2$ are isomorphic if and only if $k = k'$.*

REMARK. The models of two-dimensional Laguerre planes of type II.A.2 can be generalized by starting with the planes of type II.E.4 from [20] instead of the two-dimensional Laguerre planes of translation type we used in this section. By [20, Proposition 8], given $k, l > 0$, $2 \neq kl > 1 \neq l$, the subsets of Z given by

$$D_{a,b,c} = \begin{cases} \{(x, bx + c) \mid x \in \mathbb{R}\} \cup \{(\infty, 0)\} & \text{if } a = 0, \\ \{(x, a(|x - b|^{kl} + c)) \mid x \in \mathbb{R}\} \cup \{(\infty, a)\} & \text{if } a \neq 0, c \leq 0, \\ \{(x, a(|x - b|^{k} + c)^{l}) \mid x \in \mathbb{R}\} \cup \{(\infty, a)\} & \text{if } a \neq 0, c > 0, \end{cases}$$

where $a, b, c \in \mathbb{R}$ form the set of circles of a two-dimensional Laguerre plane $\mathcal{L}_{k,l}$ of type II.E.4. (This is not quite the form given in [20] but equivalent to it in the case of type II.E.4.) In a fashion similar to the one used in the proof of Theorem 4.1 we can exchange circles that meet $D_{0,0,0}$ in two points with circles of an isomorphic copy of $\mathcal{L}_{k,l}$. In this way, one obtains two-dimensional Laguerre planes $\mathcal{L}(k, l)$ of type II.A.2 that depend on two parameters k and l.

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