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A GENERAL APPROACH TO LITTLEWOOD-PALEY THEOREMS FOR ORTHOGONAL FAMILIES

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ABSTRACT. A general lacunary Littlewood-Paley type theorem is proved, which applies in a variety of settings including Jacobi polynomials in $[0,1]$, SU(2), and the usual classical trigonometric series in $[0,2\pi)$. The theorem is used to derive new results for L^p multipliers on SU(2) and Jacobi L^p multipliers.

1. **Introduction.** Littlewood-Paley theorems have been investigated and applied in a wide variety of settings, with different technical methods which are particular to each setting. The purpose of this paper is to present a generic approach. While our results are not always new (although, in many cases they are), our method, which is based on ideas in [14] and [15], is elementary and unifies a range of examples.

The method applies to general orthogonal decompositions of L^2 which satisfy the following conditions: Assume $L^2(\mu) = \bigoplus_{k=0}^{\infty} H_k$ where the subspaces H_k are closed, closed under complex conjugation and pairwise orthogonal. We let $P_k: L^2 \longrightarrow H_k$ denote the orthogonal projection, and suppose that whenever $f, g \in L^2$, then

$$P_k(f)P_j(g) \in \bigoplus_{i=|k-j|}^{k+j} H_i. \tag{1}$$

Given such a decomposition of L^2 and a sequence $E = \{n_j\}_{j=1}^{\infty}$ of positive integers we define operators S_j and the square function S_E on L^2 by:

$$S_j(f) = \sum_{k\in[n_{j-1},n_j)} P_k(f), \quad j = 1, 2, \ldots \ (n_0 = 0)$$

and

$$S_E(f) = \Big(\sum_{j=1}^{\infty} |S_j(f)|^2\Big)^{1/2}.$$

Our main result, which is proved in Section 3, is

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THEOREM 1.1. *Suppose* $L^2(\mu) = \bigoplus_{k=0}^{\infty} H_k$ *where the subspaces* H_k *are closed, closed under conjugation, pairwise orthogonal and satisfy property (1). Suppose* $E = \{n_j\}_{j=1}^{\infty}$ *is a lacunary sequence of positive integers (i.e.* $\inf n_{j+1}/n_j > 1$*),* $s \in \mathbb{N}$ *and* $H_k \subseteq L^{2s}(\mu)$ *for all* k*. Then there is a constant* $c(s, E)$ *so that for all* $f \in L^{2s}(\mu)$

(2) $$\|f\|_{2s} \leq c(s, E)\|S_E f\|_{2s}.$$

Decompositions of L^2 of this type arise naturally in many different settings. For example, in $L^2[-1, 1]$ the subspaces $H_k = \mathrm{sp}\{e^{i\pi kx}, e^{-i\pi kx}\}$ or $H_k = \mathrm{sp}\{P_k^{(\alpha,\beta)}(x)\}$, where $P_k^{(\alpha,\beta)}$ is the Jacobi polynomial of degree k, have the required properties. These examples, as well as orthogonal decompositions of $L^2(\mathrm{SU}(2))$, are discussed in detail in Section 2 where we compare Theorem 1.1 to the Littlewood-Paley theorems which are already known in these settings.

In Sections 4 and 5 applications to the study of L^p multipliers on SU(2) and Jacobi L^p multipliers are examined.

2**. Examples.** In this section we will give a list of examples to which our theorem applies, and indicate how our theorem compares with what is currently known.

(1) *Classical Trigonometric Series on* $[0, 2\pi)$.

Let $L^2(\mu) = L^2(T$, Lebesgue measure$)$ and $H_k = \mathrm{sp}\{e^{ikx}, e^{-ikx}\}$ for $k = 0, 1, 2, \ldots$. The classical Littlewood-Paley theorem (a good reference is [10]) for this setting, states

THEOREM 2.1. *If* E *is a lacunary sequence, then for every* $1 < p < \infty$*, there are constants* $A(p, E)$ *and* $B(p, E) > 0$ *so that*

$$A(p, E)\|f\|_p \leq \|S_E f\|_p \leq B(p, E)\|f\|_p \quad \text{for all } f \in L^p(T).$$

In fact, the comparability of norms remains true when lacunary sequences are replaced by certain more general partitions of $\mathbb{Z}$ (*cf.* [11], [22] and [15]). In [15] it is also observed that if for a given set E

$$\|f\|_{2s} \leq c(s, E)\|S_E f\|_{2s} \quad \text{for all } f \in L^{2s}(T)$$

and for all $s \in \mathbb{N}$, then the usual two-sided Littlewood-Paley inequalities hold for all $1 < p < \infty$ (this is essentially a consequence of [21]), and thus we have a new proof of the classical theorem.

Before proceeding with the next two examples it is convenient to prove an elementary lemma.

LEMMA 2.2. *Let* H_k *be closed, closed under conjugation, orthogonal subspaces of* L^2 *and let* P_k *denote the orthogonal projection onto* H_k*. If for all* $k, j \in \mathbb{N}$ *and for all* $f, g \in L^2$

$$P_k(f)P_j(g) \in \bigoplus_{i=0}^{k+j} H_i,$$

then

$$P_k(f)P_j(g) \in \bigoplus_{i=|k-j|}^{k+j} H_i.$$

PROOF. Without loss of generality assume $k-j>0$ and $0\le l<k-j$. Let h be an arbitrary element of L^2. Since

$$P_j(g)P_l(h)\in\bigoplus_{i=0}^{j+l}H_i,$$

and $k>j+l$, it follows that $P_k(\overline{f})$ is orthogonal to $P_j(g)P_l(h)$. Thus

$$\int\overline{P_k(f)}P_j(g)P_l(h)=0$$

for all $h\in L^2$, and since the subspaces H_i are closed under conjugation this implies that $P_k(f)P_j(g)$ is orthogonal to H_l.

(2) *Classical orthogonal polynomials on* $[-1,1]$.

For $\alpha,\beta\ge-\frac{1}{2}$ let $P_n^{(\alpha,\beta)}(x)$ denote the Jacobi polynomial of degree n and order (α,β):

$$(1-x)^\alpha(1+x)^\beta P_{n(\alpha,\beta)}(x)\equiv\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}].$$

The Jacobi polynomials are well known [25] to be an orthogonal basis for $L^2(m_{\alpha,\beta})$ where

$$dm_{\alpha,\beta}=(1-x)^\alpha(1+x)^\beta\,dx.$$

Set $H_k=\mathrm{sp}\{P_k^{(\alpha,\beta)}\}$. It is easy to see that $\{P_0^{(\alpha,\beta)},\ldots,P_k^{(\alpha,\beta)}\}$ span the subspace of polynomials of degree k, consequently

$$P_k^{(\alpha,\beta)}(x)P_j^{(\alpha,\beta)}(x)\in\bigoplus_{i=0}^{k+j}H_i,$$

and $H_k\subseteq L^p$ for all $1\le p\le\infty$. An appeal to Lemma 2.2 shows that the conditions of Theorem 1.1 are satisfied.

Special cases of the Jacobi polynomials include Legendre polynomials ($\alpha=\beta=0$), the Gegenbauer or ultraspherical polynomials,

$$C_n^\lambda(x)=\frac{\Gamma(\lambda+\frac{1}{2})\Gamma(2\lambda+n)}{\Gamma(2\lambda)\Gamma(\lambda+n+\frac{1}{2})}P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x),$$

and the Chebyshev polynomials,

$$T_n(x)=\frac{n!\sqrt{\pi}}{\Gamma(n+\frac{1}{2})}P_n^{(-\frac{1}{2},-\frac{1}{2})}(x),\ U_n(x)=\frac{(n+1)!\sqrt{\pi}}{2\Gamma(n+\frac{3}{2})}P_n^{(\frac{1}{2},\frac{1}{2})}(x).$$

Littlewood-Paley theory has been studied extensively for the classical families of orthogonal polynomials (*cf.* [1], [7], [8], [9], [18], [19] and the references cited therein). There are theorems involving g-functions, maximal operators, Marcinkiewicz multiplier theorems and Littlewood-Paley diadic decomposition theorems. In particular Askey [1] (see also [9]) has shown that for the Jacobi polynomials of order (α,β) with $\alpha\ge\beta$ the full 2-sided Littlewood-Paley theorem (as in Theorem 2.1) holds for $E = \{2^j\}$, provided $4(\alpha+1)/(3+2\alpha) < p < 2(\alpha+1)/(\alpha+\frac{1}{2})$, and it is known that this range of p cannot be improved [2]. Our one-sided Littlewood-Paley theorem yields new inequalities for sufficiently large, even integers p.

(3) *Spherical Harmonics*.

Another example is to consider square integrable functions defined on the sphere in $\mathbb{R}^{n+1}$, and take H_k to be the space formed by the harmonic, homogeneous polynomials of degree k. Similar arguments to those used before show that the necessary conditions for our theorem are satisfied in this setting. Littlewood-Paley theory has been studied here as well (for *e.g.* [4] and [24]), however, our result appears to be new when $n \geq 2$.

(4) SU(2).

For each non-negative integer k let σ_k denote the irreducible unitary representation of SU(2) of degree $k+1$. For an orthogonal decomposition of $L^2(\mathrm{SU}(2))$ we take $H_k = \{\mathrm{Tr}\, A\sigma_k : A \text{ is a } (k+1)\times(k+1) \text{ matrix}\}$. It is well known that $\sigma_k \otimes \sigma_j \simeq \oplus_{i=|k-j|}^{k+j} \sigma_i$ [16; 29.26], and consequently

$$(\mathrm{Tr}\, A\sigma_k)(\mathrm{Tr}\, B\sigma_j) \in \bigoplus_{i=|k-j|}^{k+j} H_i.$$

Several authors have investigated Littlewood-Paley theorems for this decomposition including [5] and [26], however our one-sided, unweighted result appears to be new. Moreover, it is not in general true that $\|S_{\{2^j\}} f\|_{2s}$ is bounded over f in the unit ball of L^{2s}. This is due to Clerc [5] who has shown that the partial sums of the Fourier series of a function in L^{2s} can have unbounded L^{2s}-norms.

3. Proof of the Main Result.

PROOF OF THEOREM 1.1. The case $s = 1$ is trivial, so fix $s \in \{2, 3, 4, \dots\}$ and assume $E = \{n_j\}$ is a lacunary sequence of positive integers. Since $\inf n_{j+1}/n_j > 1$ we can choose an integer m so large that $n_{j-1} > (2s-1)n_{j-m}$ for all j.

Standard arguments show that it suffices to prove the inequality (2) for those $f \in L^2$ satisfying $P_k(f) = 0$ for all but finitely many k. For such $f \in L^2$ and each $i = 1, \dots, m$ set

$$F_i(f) = \sum_{k=0}^{\infty} S_{mk+i}(f).$$

Observe that $f = \sum_{i=1}^m F_i(f)$ and

$$\begin{aligned} \big(S_E(f)\big)^{2s} &= \Big(\sum_j |S_j(f)|^2\Big)^s = \Big(\sum_{i,j} \big|S_j\big(F_i(f)\big)\big|^2\Big)^s \\ &\geq \sum_{i=1}^m |S_E(F_i(f)|^{2s}, \end{aligned}$$

so without loss of generality we may assume $F_i(f) = f$. An important consequence of this assumption is that if $S_j(f) \neq 0$ then $S_k(f) = 0$ if $|j-k| < m$.

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Following the scheme of [15] we let $G_1 = 0$ and $G_j = \sum_{k=1}^{j-1} S_k(f)$ for $j = 2, 3, \ldots$, and we let $P_j = |G_j + S_j(f)|^{2s} - |G_j|^{2s}$. With this notation $\|f\|_{2s}^{2s} = \sum \int P_j$. Expanding gives

$$P_j = \sum_{\substack{a,b=0 \\ a+b\neq 0}}^{s} c(s,a,b) G_j^{s-a} \bar{G}_j^{s-b} (S_j f)^a (\overline{S_j f})^b$$

where $c(s,a,b) = \binom{s}{a}\binom{s}{b}$. There are two cases to consider.

CASE (1). $a+b=1$: Without loss of generality we may assume $a=1$, $b=0$. If $S_j(f) = 0$ then clearly

$$\int S_j(f)\bar{G}_j |G_j|^{2(s-1)} = 0,$$

so we assume otherwise. But then $S_k(f) = 0$ for $k = j-m+1, \ldots, j-1$, and thus $G_j = \sum_{k=1}^{j-m} S_k(f)$. This fact, together with property (1) of the orthogonal decomposition of L^2, ensures that

$$|G_j|^{2(s-1)} \in \bigoplus_{i=0}^{2(s-1)n_{j-m}} H_i,$$

while

$$S_j(f)\bar{G}_j \in \bigoplus_{i=n_{j-1}-n_{j-m}}^{n_j+n_{j-m}} H_i.$$

The choice of m ensures that these functions are orthogonal, *i.e.*, $\int S_j(f)\bar{G}_j|G_j|^{2(s-1)} = 0$.

CASE (2). $a+b \geq 2$: We will prove that in this case there is a constant c so that

$$\left| \int G_j^{s-a}\bar{G}_j^{s-b}\big(S_j(f)\big)^a\big(\overline{S_j(f)}\big)^b \right| \leq c \int \big(|S_j(f)|^{2s} + |S_j(f)|^2|f|^{2s-2}\big) \tag{3}$$

For this we obviously may assume $S_j(f) \neq 0$ and we set $B_j = \sum_{k=j+m}^{\infty} S_k(f)$. Since $S_j(f) \neq 0$, it follows that $G_j = \sum_{k=1}^{j-m} S_k(f)$. Because $a+b \geq 2$ we have the inequality

$$|G_j^{s-a}\bar{G}_j^{s-b}\big(S_j(f)\big)^a\big(\overline{S_j(f)}\big)^b| \leq |S_j(f)|^{2s} + |S_j(f)|^2|G_j|^{2s-2}. \tag{4}$$

Observe that the functions $G_j\bar{B}_j$, $S_j(f)\bar{B}_j$ and their conjugates belong to $\bigoplus_{i \geq n_{j+m-1}-n_j} H_i$, while the function $|G_j|^{2(s-2)}|S_j(f)|^2$ belongs to

$$\bigoplus_{i=0}^{2(s-2)n_{j-m}+2n_j} H_i.$$

The definition of m ensures that $n_{j+m-1} - n_j > (2s-1)n_j - n_j = (2s-2)n_j$. If $s=2$ then $2(s-2)n_{j-m} + 2n_j = 2n_j \leq (2s-2)n_j$; while if $s \geq 3$ then $2(s-2)n_{j-m} + 2n_j < n_{j-1} + 2n_j < 3n_j < (2s-2)n_j$. Consequently these are orthogonal subspaces of L^2 and so it follows that

$$\begin{aligned}\int |G_j|^{2(s-2)}|S_j(f)|^2|f|^2 &= \int |G_j|^{2(s-2)}|S_j(f)|^2|G_j + S_j(f) + B_j|^2 \\ &= \int |G_j|^{2(s-2)}|S_j(f)|^2\big(|G_j|^2 + |S_j(f)|^2 + |B_j|^2 + 2\,\mathrm{Re}\, G_j\overline{S_j(f)}\big).\end{aligned}$$

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This identity certainly suffices to show

$$\int |G_j|^{2s-2}|S_j(f)|^2 \leq \int |G_j|^{2(s-2)}|S_j(f)|^2|f|^2 + 2|G_j|^{2s-3}|S_j(f)|^3.$$

Now we use the elementary inequality $a^x b^{n-x} \leq \epsilon a^n + c(\epsilon, x)b^n$ for $a, b \geq 0, 0 \leq x < n$ and $0 < \epsilon < 1$, in the two forms:

$$|G_j|^{2(s-2)}|f|^2 \leq \epsilon|G_j|^{2s-2} + c(\epsilon, S)|f|^{2s-2}$$

and

$$|G_j|^{2s-3}|S_j(f)|^3 \leq \epsilon|G_j|^{2s-2}|S_j(f)|^2 + c_1(\epsilon, S)|S_j(f)|^{2s}.$$

Together with the previous estimate this yields the bound

$$\int |G_j|^{2s-2}|S_j(f)|^2 \leq 3\epsilon \int |G_j|^{2s-2}|S_j(f)|^2 \\ + c_2(\epsilon, s)\int \big(|S_j(f)|^2|f|^{2s-2} + |S_j(f)|^{2s}\big).$$

Taking $\epsilon = \frac{1}{4}$, using (4) and simplifying gives (3).

In order to complete the proof of the theorem we need to combine these two cases and sum over j to obtain

$$\begin{aligned} \|f\|_{2s}^{2s} &= \sum_j \int P_j \leq \sum_j \sum_{a+b\geq 2} c(s, a, b) \int |S_j(f)|^{2s} + |S_j(f)|^2|G_j|^{2s-2} \\ &\leq \sum_j c(s) \int |S_j(f)|^{2s} + |S_j(f)|^2|f|^{2s-2}. \end{aligned}$$

Again, use the elementary inequality

$$|S_j(f)|^2|f|^{2s-2} \leq \epsilon|f|^{2s} + c(\epsilon, s)|S_j(f)|^{2s}$$

for sufficiently small $\epsilon > 0$, and upon simplifying and observing that

$$\sum_j \int |S_j(f)|^{2s} \leq \|S_E(f)\|_{2s}^{2s},$$

the proof of the theorem is complete. ∎

An important corollary of the theorem is

COROLLARY 3.1. *Under the hypothesis of Theorem 1.1, for all $f \in L^{2s}(\mu)$ we have*

$$\|f\|_{2s} \leq c(s, E)\Big(\sum \|S_j(f)\|_{2s}^2\Big)^{\frac{1}{2}}.$$

PROOF. This simply follows from the theorem and Minkowski's inequality which implies that

$$\Big\|\Big(\sum |S_j(f)|^2\Big)^{\frac{1}{2}}\Big\|_{2s} \leq \Big(\sum \|S_j(f)\|_{2s}^2\Big)^{\frac{1}{2}}.$$ ∎

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4. **Multipliers on** SU(2)**.**

DEFINITION. Let G be a compact group. A bounded operator M mapping $L^p(G)$ to $L^q(G)$ which commutes with left translation is called an (L^p, L^q) *multiplier* (or simply an L^p *multiplier* if $p = q$).

This means that if $f \in \mathrm{Trig}(G)$ then $M(f)$ is the trigonometric polynomial

$$\sum_{\sigma\in\hat{G}} d_\sigma \operatorname{Tr} \hat{M}(\sigma)\hat{f}(\sigma)\sigma,$$

where $\hat{G}$ is the dual object of G, and for each $\sigma \in \hat{G}, \hat{M}(\sigma)$ is a matrix of size $d_\sigma \times d_\sigma$ ($d_\sigma = \deg\sigma$). We customarily identity M with the set $\{\hat{M}(\sigma)\}_{\sigma\in\hat{G}}$. If $\hat{M}(\sigma)$ is a scalar multiple of the identity for each σ then M is called *central*.

For a matrix A the notation $\|A\|_\infty$ will mean the maximum eigenvalue of the matrix $|A|$. We refer the reader to [16, Appendix D] for properties of this norm. Since M maps L^2 to L^p if and only if M^* maps $L^{p'}$ to L^2, an easy consequence of Parseval's theorem is that M is an (L^2, L^p) multiplier if the same is true for the central multiplier $\{\|\hat{M}(\sigma)\|_\infty I_{d_\sigma}\}_{\sigma\in\hat{G}}$.

Central L^p multipliers on compact Lie groups have been widely investigated. Weiss [26], for example, studies Hormander-type multiplier theorems, and Coifman and Weiss [6] consider the problem of transferring L^p multipliers from T^l to a compact Lie group of rank l.

In this section we will show how Theorem 1.1 allows us to construct (L^2, L^p) multipliers on SU(2) for $p > 2$ (and in particular L^p-multipliers) which do not satisfy their criteria. We also obtain a transference result for (L^2, L^p) multipliers.

The theorem from which the new examples and transference results follow is:

THEOREM 4.1. *Suppose E is a set of positive integers and that for some* $0 \le t \le 1$,

$$sup_j \frac{|E\cap[2^{j-1}, 2^j)|}{2^{jt}} \le C < \infty.$$

If A_n is an $(n+1)\times(n+1)$ *matrix with* $\|A_n\|_\infty \le 1$, *and if* $p > 2$, *then*

$$\hat{M}_p(n) = \frac{A_n}{n^{(1+t/2)(1-2/p)}}\chi_E(n).$$

defines an (L^2, L^p) *multiplier on* SU(2) *with operator norm (denoted* $\|M_p\|_{2,p}$*) at most* $B(p)C^{\frac{1}{2}-\frac{1}{p}}$ *(where $B(p)$ is a constant independent of C, E and t).*

To prove this it is convenient to first prove a lemma.

LEMMA 4.2. *Suppose E is as in the theorem. For each $s \in \mathbb{N}$ there is a constant $B(2s)$ so that for all* $f \in \mathrm{Trig}\,\mathrm{SU}(2)$,

$$\Big\|\sum_{n\in E}(n+1)Tr\hat{f}(\sigma)\sigma_n\Big\|_{2s} \le B(2s)C^{\frac{1}{2}-\frac{1}{2s}}\Big(\sum_{n\in E}(n+1)^{(2+t)(1-\frac{1}{s})+1}\operatorname{Tr}|\hat{f}(n)|^2\Big)^{\frac{1}{2}}.$$

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PROOF. Example 4 of Section 2 shows that Theorem 1.1 applies in this setting, so that if

$$g_j = \sum_{n\in[2^{j-1},2^j)\cap E} (n+1)\,\mathrm{Tr}\,\hat{f}(n)\sigma_n,$$

then Corollary 3.1 implies that $\|\sum g_j\|_{2s} \le c(s)(\sum \|g_j\|_{2s}^2)^{\frac{1}{2}}$.

An application of Holder's inequality gives $\|g_j\|_{2s} \le \|g_j\|_2^{1/s}\|g_j\|_\infty^{1-1/s}$. Since $\hat{f}(n)$ is an $(n+1)\times(n+1)$ matrix,

$$\mathrm{Tr}\,|\hat{f}(n)| \le \sqrt{n+1}\big(\mathrm{Tr}\,|\hat{f}(n)|^2\big)^{\frac{1}{2}}$$

and so the Cauchy Schwarz inequality, together with the assumption on the cardinality of $E\cap[2^{j-1},2^j)$, yields

$$\begin{aligned}\|g_j\|_\infty &\le \sum_{n\in[2^{j-1},2^j)\cap E} (n+1)\,\mathrm{Tr}\,|\hat{f}(n)|\\ &\le 2^{j(t/2+1)}\sqrt{C}\|g_j\|_2.\end{aligned}$$

Combining these estimates gives the result. ∎

PROOF OF THEOREM 4.1. Without loss of generality we assume $A_n = I_{n+1}$. The proof when $p=2s$, $s\in\mathbb{N}$, is a routine application of the lemma after observing that

$$\mathrm{Tr}\,|M_{2s}\hat{f}(n)|^2 \le \|\hat{M}_{2s}(n)\|_\infty^2\,\mathrm{Tr}\,|\hat{f}(n)|^2.$$

For arbitrary $2<p<\infty$, choose an integer s with $2<p<2s$, and let v denote the conjugate index to $2s$ (*i.e.* $\frac{1}{v}+\frac{1}{2s}=1$). By duality

$$\|M_{2s}\|_{v,2} = \|M_{2s}\|_{2,2s} \le B(2s)C^{\frac{1}{2}-\frac{1}{2s}}.$$

Given a complex number z with $\mathrm{Re}\,z\ge 0$, define an operator M^z by

$$\hat{M}^z(n) = \frac{I_{n+1}}{n^{(1+t/2)(1-1/s)z}}\chi_E(n).$$

If $\mathrm{Re}\,z=1$ then $\|M^zf\|_2 = \|M_{2s}f\|_2 \le B(2s)C^{\frac{1}{2}-\frac{1}{2s}}\|f\|_\nu$, while if $\mathrm{Re}\,z=0$, $\|M^zf\|_2 = \|f\|_2$. A consequence of Stein's interpolation theorem for operators [23] is that if z satisfies $1/p' = z/v+(1-z)/2$, then M^z maps $L^{p'}$ to L^2 with norm at most $\big(B(2s)C^{\frac{1}{2}-\frac{1}{2s}}\big)^z$. As $z(1-1/s)=1-2/p$, a duality argument completes the proof. ∎

Taking $t=0$ and $t=1$ respectively in Theorem 4.1 gives

COROLLARY 4.3. *Let $p\ge 2$. If either (i) $\hat{M}(n) = n^{2/p-1}I_{n+1}\chi_{\{2^j\}}(n)$ or (ii) $\|\hat{M}(n)\|_\infty \le O(n^{\frac{3}{p}-\frac{3}{2}})$ then M is an (L^2,L^p) multiplier.*

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REMARK. The second part can essentially be found in [17].

In [6] Coifman and Weiss describe a method for transferring L^p multipliers on T^l to central L^p multipliers on a compact Lie group of rank l. When $l = 1$ their theorem states that $\hat{M}(n) = \hat{m}(n)I_{n+1}$ is a central L^p multiplier of SU(2) provided

$$\hat{\mu}(\pm n) \equiv (n+1)\hat{m}(n) - (n-1)\hat{m}(n-2)$$

defines an L^p multiplier on the circle T. Our approach provides new examples of L^p multipliers even in the central case. For example, the multiplier defined in Corollary 4.3(i) is not one of Coifman and Weiss's type since the corresponding sequence $\{\hat{\mu}(\pm n)\}$ is not even bounded.

As far as we are aware there is no result analogous to [6] for transferring (L^q, L^p) multipliers with $q \neq p$. We consider here the case when $q = 2 < p$.

THEOREM 4.4. *Suppose $\{\hat{m}(n)\}$ defines an (L^2, L^p) multiplier on T for some $p > 2$. If $q > 2$ and A_n is an $(n+1) \times (n+1)$ matrix with $\|A_n\|_\infty \leq 1$, then*

$$\hat{M}(n) = \frac{\hat{m}(n)A_n}{n^{(1+1/p)(1-2/q)}}$$

defines an (L^2, L^q) multiplier on SU(2).

PROOF. As remarked earlier, it suffies to assume $A_n = I_{n+1}$. Also, without loss of generality we may assume $\sup_n |\hat{m}(n)| \leq 1$.

For each $\epsilon > 0$, let $E(\epsilon) = \{n : |\hat{m}(n)| > \epsilon\}$. Since m is an $\big(L^2(T), L^p(T)\big)$ multiplier it is known [13, 1.11] that there is a constant C so that for every $\epsilon > 0$ and for each j

$$|E(\epsilon) \cap [2^{j-1}, 2^j)| \leq C\epsilon^{-2} 2^{2j/p}.$$

For a given $2 < q < \infty$, define M_ϵ by

$$\hat{M}_\epsilon(n) = \frac{I_{n+1}\chi_{E(\epsilon)\cap\mathbb{Z}^+}(n)}{n^{(1+1/p)(1-2/q)}}$$

By Theorem 4.1 one can see that M_ϵ is an (L^2, L^q) multiplier of SU(2) with norm at most $C(q)\epsilon^{-(1-2/q)}$ (where $C(q)$ is independent of ϵ), and by duality it is an $(L^{q'}, L^2)$ multiplier with the same norm. Since

$$\operatorname{Tr} |\hat{M}(n)\hat{f}(n)|^2 \leq 2^{-2(j-1)} \operatorname{Tr} |\hat{M}_{2^{-j}}(n)\hat{f}(n)|^2 \quad \text{for } n \in E(2^{-j}) \setminus E(2^{-(j-1)}),$$

a consequence of Parseval's theorem and the bounds on the norms of the operators $M_{2^{-j}}$, is that

$$\begin{aligned}\|Mf\|_2^2 &\leq \sum_{j=1}^{\infty} 2^{-2(j-1)} \|M_{2^{-j}}\|_{q',2}^2 \|f\|_{q'}^2 \\ &\leq C(q)\|f\|_{q'}^2 \sum_{j=1}^{\infty} 2^{-2(j-1)} 2^{2j(1-2/q)}.\end{aligned}$$

Since the latter sum converges, M is an $(L^{q'}, L^2)$ multiplier, and an (L^2, L^q) multiplier by duality. ∎

REMARK. In contrast to the situation for the circle there are no central (L^2, L^p) multipliers M on SU(2) with $\limsup \|\hat{M}(n)\|_\infty > 0$. This is essentially because a central idempotent multiplier maps L^2 to L^p if and only if $\operatorname{supp} \hat{M}$ is a $\Lambda(p)$ set (see [12]), and SU(2) is known to admit no infinite $\Lambda(p)$ set [20].

5**. Jacobi multipliers.** In this section we derive similar results for multipliers on Jacobi expansions. We refer the reader to Example 2 of Section 2 for the notation. In addition we assume $\alpha \geq \beta \geq -1/2$.

First, we need estimates on the p-norms of the Jacobi polynomials.

THEOREM 5.1. *If $P_n^{(\alpha,\beta)}$ denotes the Jacobi polynomial of degree n and order (α, β) then, for $1 \leq p < \infty$,*

$$\|P_n^{(\alpha,\beta)}\|_p = \begin{cases} O(n^{-1/2}) & \text{if } p < 2(1+\alpha)/(\alpha+1/2) \\ O(n^{-1/2}(\log n)^{1/p}) & \text{if } p = 2(1+\alpha)/(\alpha+1/2) \\ O(n^{\alpha(1-2/p)-2/p}) & \text{if } p > 2(1+\alpha)/(\alpha+1/2) \end{cases}.$$

PROOF. These estimates are obtained by routine calculations based upon the fact [25, p. 169] that there is a constant $c > 0$ so that

$$P_n^{(\alpha,\beta)}(\cos\theta) = \begin{cases} \theta^{-(\alpha+1/2)} O(n^{-1/2}) & \text{if } \frac{c}{n} \leq \theta \leq \frac{\pi}{2} \\ O(n^\alpha) & \text{if } 0 \leq \theta \leq \frac{c}{n} \end{cases}.$$

We leave the details to the reader. ∎

Following the notation of [7] we let $R_n(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$ and $h_n^{-1} = \|R_n\|_2^2$. With this notation $f = \sum_{n=0}^{\infty} \hat{f}(n) h_n R_n$ where $\hat{f}(n) = \int_{-1}^{1} f(x) R_n(x)\, dm_{\alpha,\beta}$. A consequence of Theorem 5.1 is that $\|f\|_2^2 = \sum |\hat{f}(n)|^2 n^{2\alpha+1}$.

DEFINITION. A *Jacobi* (L^p, L^q) *multiplier* is a bounded map $M: L^p(m_{\alpha,\beta}) \longrightarrow L^q(m_{\alpha,\beta})$ defined by

$$Mf = \sum \hat{M}(n)\hat{f}(n) h_n R_n$$

for some sequence $\{\hat{M}(n)\}$.

Analogous to Theorem 4.1 we have

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THEOREM 5.2. *Suppose E is a set of positive integers and that for some* $0 \le t < 1$

$$\sup_j \frac{|E \cap [2^{j-1}, 2^j)|}{2^{jt}} \le D < \infty.$$

(i) If $2 < p < 2(1+\alpha)/(\alpha+1/2)$ and $\hat{M}(n) = n^{-t(1-2/p)(1+\alpha)}\chi_E(n)$ then M is a Jacobi (L^2, L^r) multiplier for all $2 < r < p$, with norm at most $O(\sqrt{D})$.

(ii) If $s \in \mathbb{N}$ and $2s \ge 2(1+\alpha)/(\alpha+1/2)$, then $\hat{M}(n) = n^{-(\frac{t+1}{2}-\frac{1}{s}+\alpha(1-\frac{1}{s}))}\chi_E(n)$ defines an (L^2, L^{2s}) multiplier of norm at most $O(\sqrt{D})$.

PROOF. The method of proof is similar in spirit to the proof of Theorem 4.1. First we note that

$$\begin{aligned}\Big\| \sum_{n\in E\cap[2^{j-1},2^j)} \hat{f}(n) h_n R_n \Big\|_p^2 &\le D2^{jt} \sum_{n\in E\cap[2^{j-1},2^j)} |\hat{f}(n)|^2 h_n^2 \|R_n\|_p^2 \\ &\le \begin{cases} 2^{jt} D C_p^2 \sum_{n\in[2^{j-1},2^j)} |\hat{f}(n)|^2 n^{2\alpha+2} n^{-1} & \text{if } p < 2(1+\alpha)/(\alpha+1/2) \\ 2^{jt} D C_p^2 \sum_{n\in[2^{j-1},2^j)} |\hat{f}(n)|^2 n^{2\alpha+2} n^{2\alpha(1-\frac{1}{s})-\frac{2}{s}} & \text{if } p \ge 2(1+\alpha)/(\alpha+1/2) \end{cases}.\end{aligned}$$

Part (ii) now follows easily from Corollary 3.1.

For part (i) we first use Askey's Littlewood-Paley theorem [1] and the estimate above to show that $\hat{M}(n) = n^{-t/2}\chi_E(n)$ is an (L^2, L^p) multiplier of norm $\sqrt{D}C_p$ when $p < 2(1+\alpha)/(\alpha+1/2)$. To complete the proof of the stronger result claimed in (i) we interpolate: For a complex number z set $\widehat{M^z}(n) = \frac{1}{n^{zt/2}}\chi_E$. Fix $2 < r < p < 2(1+\alpha)/(\alpha+1/2)$. Choose $r < q < 2(1+\alpha)/(\alpha+1/2)$ satisfying

$$\frac{1-\frac{2}{r}}{2(1-\frac{2}{q})} \le \Big(1-\frac{2}{p}\Big)(\alpha+1).$$

(This can be done since $1-2/q$ increases to $1/2(1+\alpha)$ as q increases to $2(1+\alpha)/(\alpha+1/2)$.)

If $\operatorname{Re} z = 0$ then M^z maps L^2 to L^2 with norm 1, while if $\operatorname{Re} z = 1$ one sees from the previous work that M^z maps $L^{q'}$ to L^2 with norm at most $O(\sqrt{D})$. If $0 < z < 1$ is chosen satisfying $1/r' = z/q' + (1-z)/2$, then Stein's complex interpolation theorem [23] again implies that M^z is an (L^2, L^r) multiplier of norm at most $O(\sqrt{D}^z)$. Since $z = (\frac{1}{2}-\frac{1}{r})/(\frac{1}{2}-\frac{1}{q})$ we obtain the desired result. ∎

REMARKS. 1. Clearly result (i) is optimal when $t = 0$ (in the sense that no larger power of n will work). That result (ii) is also optimal when $t = 0$ can be seen by considering the multiplier $\hat{M}(n) = \frac{1}{n^x}\chi_{\{2^j\}}$ where

$$x < \frac{1}{2} - \frac{1}{s} + \alpha\Big(1-\frac{1}{s}\Big).$$

Since $\|M(P_{2^j}^{(\alpha,\beta)})\|_{2s}/\|P_{2^j}^{(\alpha,\beta)}\|_2 \longrightarrow \infty$ as $j \longrightarrow \infty$, M is not an (L^2, L^{2s}) multiplier.

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2. A similar interpolation argument applied to (ii) in the case $t = 1$ gives a special case of Bavinck's Hardy and Littlewood type fractional integration theorem [3].

There are also similar transference results for (L^2, L^p) Jacobi multipliers.

THEOREM 5.3. *Suppose* $m: L^2(T) \longrightarrow L^q(T)$ *for some* $q > 2$. *If* $2s < 2(1+\alpha)/(\alpha+1/2)$, *and* $\hat{M}(n) = \hat{m}(n)n^{-2/q(1-1/s)(1+\alpha)}$, *or if* $s \in \mathbb{N}, 2s \geq 2(1+\alpha)/(\alpha+1/2)$ *and* $\hat{M}(n) = \hat{m}(n)n^{1/s-1/q-1/2-\alpha(1-1/s)}$ *then* M *maps* $L^2 \longrightarrow L^r$ *for all* $r < 2s$.

PROOF. The ideas here are very similar to those in Theorem 4.4. We leave the details to the reader. ∎

COROLLARY 5.4. *If* $m: L^2(T) \longrightarrow \bigcap_{q>2} L^q(T)$ *then, for any* $\epsilon > 0, \hat{M}_\epsilon(n) = \hat{m}(n)n^{-\epsilon}$ *maps* L^2 *to* L^p *for all* $p < 2(1+\alpha)/(\alpha+1/2)$.

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