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A note on induced central extensions

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A characterization of induced central extensions which gives an explicit relationship between induced central extensions and n-stem extensions is obtained. Using the characterization, necessary and sufficient conditions for a central extension of an abelian group by a nilpotent group of class n to be a Baer sum of an induced central extension and an extension of class n are obtained.

1. Introduction

Let G be a nilpotent group of class $n \geq 1$ and A an abelian group regarded as a trivial G-module. A central extension

$$E : 0 \to A \xrightarrow{i} M \xrightarrow{\alpha} G \to 1$$

of A by G, where i is the inclusion map, is called induced if there exists a central extension $0 \to L_{n+1} \xrightarrow{i} L \xrightarrow{\gamma} G \to 1$, L_k the kth term of the lower central series of L, and a homomorphism $\beta : L \to M$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_{n+1} & \xrightarrow{i} & L & \xrightarrow{\gamma} & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \bar{\beta}} & & \downarrow{\scriptstyle \beta} & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & M & \xrightarrow{\alpha} & G & \longrightarrow & 1 \end{array}$$

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commutative. Here $\overline{\beta}$ is the homomorphism β restricted to L_{n+1} . Induced central extensions were first studied by Passi [4] when $A = T$, the additive group of rationals mod 1 . He proved ([4], Theorem 3.3) that a central extension $0 \to T \to M \to G \to 1$ is induced if and only if the extension $0 \to T/M_{n+1} \to M/M_{n+1} \to G \to 1$ splits. This result is, however, true for any abelian group A .

Let ξ be the element of $H^2(G, A)$ which corresponds to E , $\delta_E : H_2(G, Z) \to A$ be the cotransgression associated with E and $\tau : H_2(G, Z) \to H_2(G/G_n, Z)$ be the coinflation homomorphism. Let B be the image of δ_E . In this note we prove

THEOREM 1.1. *E is induced if and only if*

(i) $H_2(G, Z) = \ker \delta_E + \ker \tau$ *and*

(ii) $\xi \in \ker\left(H^2(G, A) \to H^2(G, A/B)\right)$, *where* $H^2(G, A) \to H^2(G, A/B)$ *is the homomorphism induced by the natural projection* $A \to A/B$.

When $\operatorname{ext}\left(G/G_2, A\right) = 0$, we find that E is induced if and only if $H_2(G, Z) = \ker \delta_E + \ker \tau$. Recall ([6], pp. 129-130) that a central extension $0 \to A \to N \to K \to 1$ of A by a group K is called m-stem if $A \subseteq N_{m+1}$. We observe that a stem (= 1-stem) extension of A by G is induced if and only if it is n-stem. Using Theorem 1.1 we also obtain necessary and sufficient conditions for a central extension of A by G to be a Baer sum of an induced central extension and an extension of class n .

2. Preliminaries and notations

Let G be a group and A an abelian group regarded as a trivial G-module. It is well-known that if $1 \to R \to F \to G \to 1$ is a free presentation of G , then $H_2(G, Z) \cong \overline{R \cap F'}$ [6]. (For a subgroup S of F , we write $\widetilde{S}$ for $S[F, R]/[F, R]$. Also $K' = K_2$ denotes the derived group of K .) We identify the two groups and also write H_2G for

$H_2(G, Z)$. Let $\xi \in H^2(G, A)$ and $E : 0 \to A \to M \to G \to 1$ be a central extension to which ξ corresponds. F being a free group, we can define a homomorphism $g : \bar{F} \to M$ such that the diagram

(2.1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bar{R} & \longrightarrow & \bar{F} & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow f & & \downarrow g & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & M & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

is commutative. Here $f = g|_{\bar{R}}$. Recall [6] that the homomorphism $\sigma : H^2(G, A) \to \hom(H_2G, A)$ of the natural split exact sequence

(2.2)
$$0 \to \operatorname{ext}(G/G', A) \xrightarrow{\pi} H^2(G, A) \xrightarrow{\sigma} \hom(H_2G, A) \to 0$$

of the universal coefficient theorem is then given by

$$\sigma(\xi) = f|_{\overline{R \cap F'}} .$$

Also recall that corresponding to the central extension E there is a 5-term exact sequence ([6], p. 15),

$$H_2M \to H_2G \xrightarrow{\delta_E} A \to M/M' \to G/G' \to 0$$

and the homomorphism $\delta_E = \sigma(\xi)$ ([6], Proposition II.5.4).

Throughout this paper G will be a nilpotent group of class $n \geq 1$, $1 \to R \to F \to G \to 1$ a free presentation of G , A an abelian group regarded as a trivial G-module and $\tau : H_2G \to H_2G/G_n$ the coinflation map. Observe that with the identification of H_2G with $\overline{R \cap F'}$ that we have already made, $\ker \tau = \bar{F}_{n+1}$. We say that an extension $E : 0 \to A \to M \to G \to 1$ is of class n if M is nilpotent of class n .

PROPOSITION 2.3. *A central extension* $E : 0 \to A \to M \to G \to 1$ *is of class* n *if and only if* δ_E *vanishes on* $\ker \tau$.

Proof. Since F is a free group, we can obtain a commutative diagram (2.1). Then it follows that $M_k = g(\bar{F}_k)$ for all $k \geq 2$. Thus M is of class n if and only if g vanishes on $\bar{F}_{n+1} = \ker \tau$. But $g|_{\overline{R \cap F'}} = \delta_E$ and the result follows.

3. The main result

Let a central extension

$$E : 0 \to A \to M \to G \to 1$$

be given and let ξ be the element of $H^2(G, A)$ which corresponds to E.

LEMMA 3.1. *$H_2G = \ker \delta_E + \ker \tau$ if and only if $B\ (= \operatorname{im}(\delta_E)) = M_{n+1}$.*

Proof (see Diagram (2.1)). As observed earlier

$$M_{n+1} = g(F_{n+1}) = \delta_E(\ker \delta_E + \ker \tau)$$

and this equals the image of δ_E if $H_2G = \ker \delta_E + \ker \tau$.

Conversely suppose that $B = M_{n+1}$. The commutative diagram

$$\begin{tikzcd}
E : 0 \arrow[r] & A \arrow[r] \arrow[d, "i"] & M \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & M_nA \arrow[r] & M \arrow[r] & G/G_n \arrow[r] & 1 ,
\end{tikzcd}$$

where $G \to G/G_n$ is the natural projection, induces a commutative diagram

$$(3.2)\qquad \begin{tikzcd}
H_2M \arrow[r] \arrow[d, equal] & H_2G \arrow[r, "\delta_E"] \arrow[d, "\tau"] & A \arrow[r] \arrow[d, "\lambda"] & M/M' \arrow[r] \arrow[d, equal] & G/G' \arrow[r] \arrow[d, equal] & 0 \\
H_2M \arrow[r] & H_2G/G_n \arrow[r] & M_nA/M_{n+1} \arrow[r] & M/M' \arrow[r] & G/G' \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The hypothesis shows that $\lambda\delta_E$ is the zero map, and then an easy diagram chasing shows that $H_2G = \ker \delta_E + \ker \tau$.

PROPOSITION 3.3. *If E is induced, then $H_2G = \ker \delta_E + \ker \tau$.*

Proof. E is induced implies that $\lambda^*(\xi) = 0$, where $\lambda^* : H^2(G, A) \to H^2(G, A/M_{n+1})$ is the homomorphism induced by the natural projection $\lambda : A \to A/M_{n+1}$ ([4], Theorem 3.3). The commutative diagram

$$\text{(3.4)}\qquad \begin{array}{ccccccccc} 0 \longrightarrow & \operatorname{ext}(G/G',A) & \xrightarrow{\pi} & H^2(G,A) & \xrightarrow{\sigma} & \hom(H_2G,A) & \longrightarrow 0 \\ & \Big\downarrow{\lambda^*} & & \Big\downarrow{\lambda^*} & & \Big\downarrow{\lambda^*} & \\ 0 \longrightarrow & \operatorname{ext}(G/G',A/M_{n+1}) & \xrightarrow{\pi} & H^2(G,A/M_{n+1}) & \xrightarrow{\sigma} & \hom(H_2G,A/M_{n+1}) & \longrightarrow 0 \end{array}$$

where the vertical maps are all induced by λ , then shows that $\lambda\delta_E = 0$. Hence so is the composition $\lambda \circ \delta_E : H_2G \to A \to M_nA/M_{n+1}$ in the diagram (3.2). The result then follows from the commutativity of (3.2).

COROLLARY 3.5. *If E is a stem extension, then E is induced if and only if it is n-stem.*

Proof. If E is n-stem, then $A \subseteq M_{n+1}$. But $M_{n+1} \subseteq A$, G being nilpotent of class n . Thus $M_{n+1} = A$ and it follows from Theorem 3.3 of [4] that E is induced.

Now suppose that E is induced. That E is n-stem then follows from a result of Evens ([2], or also [6], Proposition V.8.1) and Proposition 3.3.

Observe that the converse of Prosposition 3.3 is not true in general. For example, let G be a non-abelian group of order p^3 , p a prime, and $A = Z_p$, the cyclic group of order p . Then $\operatorname{ext}(G/G', Z_p) \neq 0$. Let η be the image under $\pi : \operatorname{ext}(G/G', A) \to H^2(G, A)$ of a non-zero element of $\operatorname{ext}(G/G', A)$. Let E_1 be a central extension which corresponds to η . From the definition of π it follows that E_1 is of class n and hence is not induced. On the other hand $\delta_{E_1} = \sigma(\eta) = 0$, and so $H_2G = \ker \delta_{E_1} + \ker \tau$ is trivially satisfied. Thus although n-stem extensions of A by G are always induced, the converse is not true in general.

Theorem 1.1 follows from Lemma 3.1, Proposition 3.3, and Theorem 3.3 of [4].

THEOREM 3.6. *If $\operatorname{ext}(G/G', A) = 0$, then E is induced if and only if $H_2G = \ker \delta_E + \ker \tau$.*

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Proof. Suppose that $H_2G = \ker \delta_E + \ker \tau$. Then $B = M_{n+1}$ (Lemma 3.1) and in the commutative diagram (3.2), $\lambda\delta_E = 0$. This then shows that $\sigma\lambda^*(\xi) = \lambda^*\sigma(\xi) = 0$ (see Diagram (3.4)). Since $\operatorname{ext}(G/G', A) = 0$ and $\operatorname{ext}(G/G', A) \to \operatorname{ext}(G/G', A/B)$ is an epimorphism, $\sigma : H^2(G, A/B) \to \hom(H_2G, A/B)$ is an isomorphism. Therefore $\lambda^*(\xi) = 0$ in $H^2(G, A/M_{n+1})$ and E is induced.

Since $\operatorname{ext}(G/G', A) = 0$ if A is divisible or G/G' is free abelian, we have

COROLLARY 3.7. *If G/G' is free abelian or A is divisible, then E is induced if and only if* $H_2G = \ker \delta_E + \ker \tau$.

When $A = T$, the additive group of rationals mod 1, the result of Corollary 3.7 has also been independently proved by Beyl [*1*] and used to prove that the sum of two induced central extensions need not always be induced.

Passi ([*4*], Theorem 3.7) proved that every central extension of T by G is induced if and only if inflation : $H^2(G/G_n, T) \to H^2(G, T)$ is zero. This may suggest that perhaps every extension which is not in the image of inflation is induced. This is, however, not the case, as the following example shows.

EXAMPLE 3.8. Let $G = D_8 \oplus Z_4^{(3)}$, where D_8 is the dihedral group of order 8 and $Z_4^{(3)}$ is the direct sum of three copies of Z_4, the cyclic group of order 4. Since $H_2(D_8) \cong Z_2$, it follows from the Künneth Theorem ([*6*], pp. 29-30) that

$$H_2G \cong Z_2^{(7)} \oplus Z_4^{(3)}$$

and

$$H_2G/G' \cong Z_2^{(7)} \oplus Z_4^{(3)}$$

Also the kernel of the coinflation $H_2G \to H_2G/G'$ is cyclic of order 2 and is a direct summand of H_2G. Let us write $H_2G = \sum_{1\le i\le 7} A_i \oplus \sum_{1\le j\le 3} B_j$ where each A_i generated by a_i is of order 2 and each B_j generated by b_j is of order 4. Let $\ker \tau = A_1$. Define a homomorphism $f : H_2G \to T$ by

$$f(a_i) = \tfrac{1}{2} + Z\ ,\quad 1 \le i \le 7\ ,$$

and

$$f(b_j) = \tfrac{1}{4} + Z\ ,\quad 1 \le j \le 3\ .$$

It is easy to see that the order of $\ker f$ is 2^{11} and $a_1 \notin \ker f$. Thus the order of $\ker f + \ker \tau$ is 2^{12} which is strictly smaller than the order of H_2G and so $H_2G \ne \ker f + \ker \tau$.

Let $\eta \in H^2(G, T)$ such that $\sigma(\eta) = f$ and let E_1 be a central extension of T by G which corresponds to η. Then $f = \delta_{E_1}$ and, by what we have proved above it follows that E_1 is not induced. Also it follows from Proposition 2.3 that E_1 is not of class 2 and, hence, is not in the image of the inflation $H^2(G/G', T) \to H^2(G, T)$ ([5], Theorem 4.6).

4. Baer sums of extensions

Recall [3] that a central extension E of A by G is the Baer sum of extensions E' and E'' if the element of $H^2(G, A)$ corresponding to E is the sum of elements which correspond to E' and E''.

Again, *let* E *and* ξ *be as fixed in Section* 3.

THEOREM 4.1. *E is a Baer sum of an induced central extension and an extension of class n if and only if there exists a central extension $E' : 0 \to A \to X \to G \to 1$ of class n and a commutative diagram*

$$(4.2)\qquad \begin{array}{ccccccccc} E' : 0 & \longrightarrow & A & \longrightarrow & X & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow\lambda & & \downarrow & & \| & & \\ E_1 : 0 & \longrightarrow & A/M_{n+1} & \longrightarrow & M/M_{n+1} & \longrightarrow & G & \longrightarrow & 1 . \end{array}$$

Proof. Suppose that there exists a central extension E' of class n and a commutative diagram (4.2). Let η be the element of $H^2(G, A)$ which corresponds to E'. In view of the commutative diagrams (4.2) and

$$\begin{array}{ccccccccc} E : 0 & \longrightarrow & A & \longrightarrow & M & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow\lambda & & \downarrow\mu & & \| & & \\ E_1 : 0 & \longrightarrow & A/M_{n+1} & \longrightarrow & M/M_{n+1} & \longrightarrow & G & \longrightarrow & 1 , \end{array}$$

where λ, μ are natural projections, it follows that $\delta_{E_1} = \lambda\delta_{E'}$ and $\delta_{E_1} = \lambda\delta_E$. Let E'' be a central extension to which $\xi - \eta = \chi$ (say) corresponds. Then $\delta_E = \delta_{E'} + \delta_{E''}$ and $\lambda\delta_{E''} = 0$. Then, for any $a \in H_2G$, $\delta_{E''}(a) \in M_{n+1} = \delta_E(\ker \tau) = \delta_{E''}(\ker \tau)$, as $\delta_{E'}$ vanishes on $\ker \tau$ (Proposition 2.3). Hence $M_{n+1} = \operatorname{im}(\delta_{E''})$ and $H_2G = \ker \delta_{E''} + \ker \tau$. Then (see Diagram (3.4)) there exists $u \in \operatorname{ext}(G/G', A)$ such that $\chi - \pi(u)$ gets mapped onto 0 under the homomorphism $\lambda^* : H^2(G, A) \to H^2(G, A/(\operatorname{im} \delta_{E''}))$.

Let E''' be a central extension to which $\chi - \pi(u)$ corresponds. Then $\delta_{E''} = \delta_{E'''}$. Therefore, $H_2G = \ker \delta_{E'''} + \ker \tau$ and $\chi - \pi(u)$ gets mapped onto zero under the homomorphism $H^2(G, A) \to H^2(G, A/(\operatorname{im}\delta_{E'''}))$. Hence E''' is induced (Theorem 1.1). Now $\xi = \chi - \pi(u) + \eta + \pi(u)$ and both η, $\pi(u)$ correspond to extensions of class n. But such elements form a subgroup of $H^2(G, A)$. Therefore E is the Baer sum of E''' which is induced and an extension of class n.

Conversely, suppose that E is the Baer sum of an induced central extension E' and an extension E'' of class n. Then $\delta_E = \delta_{E'} + \delta_{E''}$. Also $H_2G = \ker \delta_{E'} + \ker \tau$ (Proposition 3.3). Therefore $\lambda\delta_E = \lambda\delta_{E''}$ and if η is the element of $H^2(G, A)$ which corresponds to E'', then there exists $u \in \operatorname{ext}(G/G', A)$ such that $\lambda^*(\xi) = \lambda^*(\eta+\pi(u))$ (see Diagram (3.4)). Since $\eta + \pi(u)$ corresponds to an extension of class n, the existence of such an extension and a commutative diagram (4.2) follows from the definition of the homomorphism $\lambda^* : H^2(G, A) \to H^2(G, A/M_{n+1})$.

THEOREM 4.3. *If* M_{n+1} *is a direct summand of* A, *then* E *is a Baer sum of an induced central extension and an extension of class* n.

Proof. Let C be a subgroup of A such that $A = C \oplus M_{n+1}$. Let $i : C \to A$ be the natural injection and $j : A \to C$ the natural projection. Let $i^* : H^2(G, C) \to H^2(G, A)$ and $j^* : H^2(G, A) \to H^2(G, C)$ be the homomorphisms induced by i and j respectively. Let $\eta = i^*j^*(\xi)$. It is clear that $\lambda^*(\xi) = \lambda^*(\eta)$. Also $\sigma(\eta) = \sigma(i^*j^*(\xi)) = ij\delta_E$, and so $\sigma(\eta)$ vanishes on $\ker \tau$. Thus η corresponds to a central extension of class n. The result then follows from $\lambda^*(\xi) = \lambda^*(\eta)$, the definition of the homomorphism λ^*, and Theorem 4.1.

COROLLARY 4.4. *Every central extension of an elementary abelian p-group by a nilpotent group of class* n *is a Baer sum of an induced central extension and an extension of class* n.

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