# SOME $p$-ALGEBRAS AND DOUBLE $p$-ALGEBRAS HAVING ONLY PRINCIPAL CONGRUENCES

*by* R. BEAZER

(Received 5 November, 1990)

**1. Introduction.** In [**6**], Blyth and Varlet characterize those algebras having only principal congruences in some well known classes of algebras having distributive lattice reducts. In particular, they characterize those Stone algebras having only principal congruences. In this paper we characterize those quasi-modular $p$-algebras having only principal congruences and show on specializing that distributive $p$-algebras having only principal congruences can be described in exactly the same way as Blyth and Varlet described Stone algebras having the same property. The same problem is addressed for some distributive double $p$-algebras.

**2. Preliminaries.** A (*distributive*) *p-algebra* is an algebra $\langle L; \vee, \wedge, {}^*, 0, 1\rangle$ in which the deletion of the unary operation* yields a bounded (distributive) lattice and * is the operation of pseudocomplementation; that is, $x \leq a^*$ iff $a \wedge x = 0$. A $p$-algebra is said to be *quasi-modular* if it satisfies the identity

$$[(x \wedge y) \vee z^{**}] \wedge x = (x \wedge y) \vee (z^{**} \wedge x).$$

This important class of $p$-algebras was introduced by T. Katriňák in [**11**]; it properly contains the class of modular $p$-algebras and is properly contained in the class of $p$-algebras satisfying the identity $x = x^{**} \wedge (x \vee x^*)$. See [**12**] for an overview. We assume that the reader is familiar with the standard rules of computation in $p$-algebras. If, for any $p$-algebra $L$, we write $B(L) = \{x \in L : x = x^{**}\}$ and $D^*(L) = \{x \in L : x^{**} = 1\}$ then $\langle B(L); \cup, \wedge, {}^*, 0, 1\rangle$ is a Boolean algebra when $a \cup b$ is defined to be $(a^* \wedge b^*)^*$, for any $a, b \in B(L)$, and $D^*(L)$ is a filter of $L$. The map $\beta : L \to B(L)$ defined by $\beta(x) = x^{**}$ is a homomorphism from $L$ onto $B(L)$ and its kernel $\varphi$ is called *the Glivenko congruence* of $L$ so that $a \equiv b(\varphi)$ iff $a^* = b^*$.

A (*distributive*) *double p-algebra* is an algebra $\langle L; \vee, \wedge, {}^*, {}^+, 0, 1\rangle$ in which the deletion of the unary operation $^+$ yields a (distributive) $p$-algebra and, for every $a \in L$, $a^+$ is the dual pseudocomplement of $a$; that is $x \geq a^+$ iff $a \vee x = 1$. For the standard rules of computation in double $p$-algebras we refer the reader to [**3**]. For any $a$ in a double $p$-algebra $L$, we define elements $a^{n(+*)} \in L$ by $a^{0(+*)} = a$ and $a^{(k+1)(+*)} = a^{k(+*)+*}$, for any integer $k \geq 0$. Elements $a^{n(*+)}$ are defined analogously. If $L$ is distributive then $a^{(n+1)(+*)} \leq a^{n(+*)}$, for any $n < \omega$, and $L$ is said to have *finite range* if, for any $a \in L$, there exists $n < \omega$ such that $a^{(n+1)(+*)} = a^{n(+*)}$; equivalently, if $a^{n(+*)}$ belongs to $\mathrm{Cen}(L)$, the centre of $L$, for some $n < \omega$.

If, for any double $p$-algebra $L$, we write $D^+(L) = \{x \in L : x^{++} = 0\}$ then $D^+(L)$ is an ideal of $L$ and the set $C(L) = D^*(L) \cap D^+(L)$ is called *the core of* $L$. A filter $F$ in a double $p$-algebra is said to be *normal* if $f^{+*} \in F$ whenever $f \in F$. If $F$ is a filter in a $p$-algebra or double $p$-algebra $L$ then $\Theta(F)$ will denote the smallest $p$-algebra or double $p$-algebra congruence of $L$ collapsing $F$ and $\theta_{lat\,L}(F)$ will denote the corresponding (lattice) congruence of the lattice reduct of $L$. If $L$ is a $p$-algebra or double $p$-algebra then

AMS (1980) Mathematics Subject Classification: Primary 06D15.

$\theta(a, b)$ will denote the principal $p$-algebra or double $p$-algebra congruence collapsing a pair $a, b \in L$ and if $S$ is any sublattice of the lattice reduct of $L$ then $\theta_{\text{lat } S}(a, b)$ will denote the principal (lattice) congruence of $S$ collapsing a pair $a, b \in S$.

The relation $\Phi$ defined on a double $p$-algebra $L$ by $a \equiv b(\Phi)$ iff $a^* = b^*$ and $a^+ = b^+$ is a congruence, called the *determination congruence of* $L$, and the members of $L/\Phi$ will be referred to as *determination classes of* $L$. For example, if the core of $L$ is non-empty then it is a determination class of $L$. For any algebra $A$, be it a lattice, $p$-algebra or double $p$-algebra, Con($A$) will denote the congruence lattice of $A$. Given such an algebra $A$, subalgebra $S$ of $A$ and congruence $\theta$, of $A$, $\theta \restriction S$ will denote the restriction of $\theta$ to $S$, and $[a]\theta$ will denote the class of $\theta$ containing $a \in L$. $A$ is said to have the *principal join property*, abbreviated P.J.P., if the join of any pair of principal congruences of $A$ is again principal.

For all other unexplained notation and terminology we refer the reader to [**2**], [**7**] or [**9**].

**3. Some $p$-algebras having only principal congruences.** Our objective in this section is to describe those quasi-modular $p$-algebras whose congruences are all principal. In order to effect this we will first list some key results about principal congruences on $p$-algebras the proofs of which may be found in [**5**].

LEMMA 3.1. *Let $L$ be a $p$-algebra, $x, y \in L$, $x \leq y$ and $x \equiv y(\varphi)$. Then*
(i) $\theta(x, y) = \theta_{\text{lat } L}(x, y)$
*and if, in addition, $L$ is quasi-modular then*
(ii) (a) $\theta(x, y) = \theta(x \vee x^*, y \vee x^*)$,
(b) $\theta_{\text{lat } L}(d, e) \restriction D^*(L) = \theta_{\text{lat } D^*(L)}(d, e)$, for any $d, e \in D^*(L)$.

LEMMA 3.2. *A congruence relation of a $p$-algebra $L$ is principal iff it is of the form $\theta(0, a) \vee \theta(d, e)$, for some $a \in B(L)$, $d, e \in L$ with $d \leq e$ and $d \equiv e(\varphi)$. If $L$ is quasi-modular then $d$ and $e$ may be chosen from $D^*(L)$.*

We are now sufficiently well-armed to tackle our immediate objective.

LEMMA 3.3. *If a $p$-algebra $L$ has only principal congruences then $B(L)$ is finite.*

*Proof.* If $L$ has only principal congruences then so does $B(L)$. Indeed, $B(L)$ is a homomorphic image of $L$ and it is straightforward to show that any homomorphic image of an algebra having only principal congruences enjoys the same property. However, it is well known that principal congruences of Boolean lattices are complemented and that the congruence lattice of a Boolean lattice $B$ is Boolean iff $B$ is finite. Therefore $B(L)$ is finite.

LEMMA 3.4. *If a quasi-modular $p$-algebra $L$ has only principal congruences then so does $D^*(L)$.*

*Proof.* Let $\psi$ be a congruence of $D^*(L)$ and let us define a congruence $\bar{\psi}$ of $L$ by $\bar{\psi} = \vee \{\theta(d, e) : d \equiv e(\psi)\}$. By hypothesis, $\bar{\psi}$ is principal and it is below $\varphi$. Therefore $\bar{\psi} = \theta_{\text{lat } L}(p, q)$, for some $p, q \in D^*(L)$, by Lemma 3.1(i) and (ii)(a). Thus $\bar{\psi} \restriction D^*(L) = \theta_{\text{lat } D^*(L)}(p, q)$, by Lemma 3.1(ii)(b).

However, on using Lemma 3.1 and the fact that $D^*(L)$ is a convex sublattice of the lattice reduct of $L$, we see that

$$\begin{aligned}\theta_{\text{lat } D^*(L)}(p, q) &= (\vee \{\theta_{\text{lat } L}(d, e) : d \equiv e(\psi)\}) \restriction D^*(L) \\ &= \vee \{\theta_{\text{lat } L}(d, e) \restriction D^*(L) : d \equiv e(\psi)\} \\ &= \vee \{\theta_{\text{lat } D^*(L)}(d, e) : d \equiv e(\psi)\} = \psi.\end{aligned}$$

Therefore $D^*(L)$ has only principal congruences.

LEMMA 3.5. *Let $L$ be a quasi-modular $p$-algebra. If $B(L)$ is finite and $D^*(L)$ has only principal congruences then so does $L$.*

*Proof.* Let $\theta$ be a congruence of $L$. Then $\theta$ is a join of principal congruences of $L$ and so, by Lemma 3.2, there exist $a_\alpha \in B(L)$ and $d_\alpha, e_\alpha \in D^*(L)$ with $d_\alpha \leqslant e_\alpha$, $\alpha \in I$, such that

$$\theta = \vee \{\theta(0, a_\alpha) : \alpha \in I\} \vee \vee \{\theta(d_\alpha, e_\alpha) : \alpha \in I\}.$$

Since $B(L)$ is finite, there is a finite subset $J$ of $I$ such that

$$\vee \{\theta(0, a_\alpha) : \alpha \in I\} = \vee \{\theta(0, a_\alpha) : \alpha \in J\} = \theta(0, a),$$

where $a = (\vee \{a_\alpha : \alpha \in J\})^{**}$. Also, $\vee \{\theta_{\text{lat } D^*(L)}(d_\alpha, e_\alpha) : \alpha \in I\} = \theta_{\text{lat } D^*(L)}(d, e)$, for some $d, e \in D^*(L)$ with $d \leqslant e$, since $D^*(L)$ has only principal congruences. It is straightforward to deduce that $\vee \{\theta_{\text{lat } L}(d_\alpha, e_\alpha) : \alpha \in I\} = \theta_{\text{lat } L}(d, e)$ and so $\theta = \theta(0, a) \vee \theta(d, e)$ which is principal by Lemma 3.2.

Summarizing the contents of Lemmas 3.3, 3.4 and 3.5 we have:

THEOREM 3.6. *A quasi-modular $p$-algebra $L$ has only principal congruences iff $B(L)$ is finite and $D^*(L)$ has only principal congruences.*

There exist infinite modular $p$-algebras having only principal congruences; consider, for example, any infinite bounded simple modular lattice with a new zero adjoined and construed as a $p$-algebra. As we will soon see, this contrasts sharply with the situation for distributive $p$-algebras. Henceforth, we will denote by $J(L)$ the poset of non-zero join-irreducibles of a finite distributive lattice $L$ and write $l(J(L))$ for its length.

The next lemma was proved by Blyth and Varlet in **[6]** and its successor was proved in **[5]**.

LEMMA 3.7. *A distributive lattice $L$ has only principal congruences iff it is finite and $l(J(L)) \leqslant 1$.*

LEMMA 3.8. *A distributive $p$-algebra $L$ has an $(n+1)$-element chain in its poset of prime ideals iff $D^*(L)$ has an $n$-element chain in its poset of prime ideals.*

As an immediate consequence of Theorem 3.6, Lemma 3.7 and Lemma 3.8 we have the following extension of Blyth and Varlet's characterization of Stone algebras having only principal congruences:

COROLLARY 3.9. *Let $L$ be a distributive $p$-algebra. Then the following are equivalent:*

(i) *$L$ has only principal congruences,*
(ii) *$L$ is finite and $l(J(D^*(L))) \leqslant 1$,*
(iii) *$L$ is finite and $l(J(L)) \leqslant 2$.*

https://doi.org/10.1017/S0017089500008673 Published online by Cambridge University Press

**4. Some double p-algebras having only principal congruences.** Unlike the situation for distributive $p$-algebras, there are infinite distributive double $p$-algebras having only principal congruences. Of course, if an algebra has only principal congruences then it has the P.J.P. and if a finite algebra has the P.J.P. then it has only principal congruences. Distributive double $p$-algebras having the P.J.P. have been described variously in **[1]** and we record from there the following:

THEOREM 4.1. *Let L be a distributive double p-algebra. Then the following are equivalent*:

(i) *L has the P.J.P.*,
(ii) *every determination class of L has the P.J.P.*,
(iii) *there is no 5-element chain in the poset of prime ideals of L.*

Next, we list some properties of congruences which will be required in our study of distributive double $p$-algebras having only principal congruences; their proofs may be found **[4]**, **[3]**, **[5]**, **[1]**, **[5]** and **[1]**, respectively.

LEMMA 4.2. *Let L be a distributive double p-algebra, F be a normal filter of L and* $\theta$ *be a congruence of L. Then*

(i) $\Theta(F)=\Theta_{\mathrm{lat}\,L}(F)$,
(ii) $\theta=\Theta(\mathrm{cok}\,\theta)\vee(\theta\wedge\Phi)$, *where* $\mathrm{cok}\,\theta=[1]\theta$,
(iii) $\mathrm{cok}(\theta\vee\psi)=\mathrm{cok}\,\theta$, *for any congruence* $\psi\leq\Phi$,
(iv) $\theta$ *is principal iff it is of the form* $\theta(a,1)\vee\theta(e,f)$, *for some* $a\in L$ *and* $e,f\in L$ *with* $e\leq f$ *and* $e\equiv f(\Phi)$,
(v) $\theta(a,1)=\Theta(N(a))$, *where* $N(a)$ *is the principal normal filter generated by* $a\in L$,
(vi) *if* $e,f\in L$, $e\leq f$ *and* $e\equiv f(\Phi)$ *then*

$$\begin{aligned}\theta(e,f)&=\theta_{\mathrm{lat}\,L}(e,f)\\&=\theta(e\wedge f^{+},f\wedge f^{+})\\&=\theta((e^{*}\wedge x)\vee e,(e^{*}\wedge x)\vee f)),\qquad \textit{for any } x\in L.\end{aligned}$$

THEOREM 4.3. *Let L be a distributive double p-algebra for which* $D^{+}(L)$ *is a principal ideal. Then the following are equivalent*:

(1) *L has only principal congruences*,
(2)(i) *every normal filter of L is principal and*
(ii) *every determination class of L has only principal congruences.*
(3)(i) *every normal filter of L is principal*,
(ii) *every determination class of L is finite and*
(iii) *there is no 5-element chain in the poset of prime ideals of L.*

*Proof.* Suppose that (1) holds. Let $F$ be a normal filter of $L$. Then $\Theta(F)$ is principal and so there exist $a,e,f\in L$ with $e\leq f$ and $e\equiv f(\Phi)$ such that $\Theta(F)=\theta(a,1)\vee\Theta(e,f)$, by Lemma 4.2(iv). By Lemma 4.2(i), (iii) and (iv), $F=\mathrm{cok}\,\Theta(F)=\mathrm{cok}(\theta(a,1)\vee\theta(e,f))=\mathrm{cok}(\theta(a,1))=\mathrm{cok}(\Theta(N(a))=N(a)$ and so (2)(i) holds.

Next, suppose that $C$ is a determination class of $L$. Let $\psi$ be a congruence of $C$ and let us define a congruence $\bar{\psi}$ of $L$ by $\bar{\psi}=\vee\{\theta(c,d):c\equiv d(\psi)\}$. By hypothesis, $\bar{\psi}$ is principal and so it is compact. Therefore $\bar{\psi}=\vee\{\theta(c_i,d_i):1\leq i\leq n\}$, for some $c_i,d_i\in L$ with $c_i\equiv d_i(\psi)$ and $1\leq i\leq n$. Now $c_i\equiv d_i(\Phi)$ and so $\theta(c_i,d_i)=\theta_{\mathrm{lat}\,L}(c_i,d_i)$, by Lemma

4.2(vi). As a consequence of this and the fact that $C$ is a convex sublattice of $L$, we have

$$\bar{\psi} \restriction C = (\vee \{\theta_{\text{lat } L}(c_i, d_i) : 1 \leq i \leq n\}) \restriction C = \vee \{\theta_{\text{lat } L}(c_i, d_i) \restriction C : 1 \leq i \leq n\}$$
$$= \vee \{\theta_{\text{lat } C}(c_i, d_i) : 1 \leq i \leq n\}.$$

But $L$ has the P.J.P. and therefore so does $C$, by Theorem 4.1. Thus, $\bar{\psi} \restriction C = \theta_{\text{lat } C}(p, q)$, for some $p, q \in C$. However, arguing as above, $\bar{\psi} \restriction C = \vee \{\theta_{\text{lat } C}(c, d) : c \equiv d(\psi)\} = \psi$. Therefore $\psi = \theta_{\text{lat } C}(p, q)$. It follows that $C$ has only principal congruences. Thus, (1) implies (2).

Suppose now that (2) holds. Then 3(ii) holds by virtue of Lemma 3.10, and 3(iii) holds by virtue of Theorem 4.1. Thus (2) implies (3).

Finally, suppose that (3) holds. Let $\theta$ be an arbitrary congruence of $L$. By Lemma 4.2(ii), $\theta = \Theta(\text{cok } \theta) \vee (\theta \wedge \Phi)$. By 3(i), $\text{cok } \theta = N(a)$, for some $a \in L$, and so $\Theta(\text{cok } \theta) = \theta(a, 1)$, by Lemma 4.2(v). Furthermore, $\theta \wedge \Phi$ is below $\Phi$ and so there is a family $\{(e_\alpha, f_\alpha) : \alpha \in I\} \subseteq \Phi$ with $e_\alpha \leq f_\alpha$, for all $\alpha \in I$, such that $\theta \wedge \Phi = \vee \{\theta(e_\alpha, f_\alpha) : \alpha \in I\}$. We can assume that $e_\alpha, f_\alpha \in D^+(L)$, for all $\alpha \in I$, by Lemma 4.2(vi). By hypothesis, $D^+(L) = (p]$, for some $p \in L$, and, for each $\alpha \in I$, $\theta(e_\alpha, f_\alpha) = \theta(p_\alpha, q_\alpha)$, where $p_\alpha = (e_\alpha^* \wedge p) \vee e_\alpha$ and $q_\alpha = (e_\alpha^* \wedge p) \vee f_\alpha$, by Lemma 4.2(vi). Now observe that, for any $\alpha, \beta \in I$, $p_\alpha \in D^+(L)$ and $p_\alpha^{**} = [(e_\alpha^* \vee e_\alpha) \wedge p]^{**} = (e_\alpha^* \vee e_\alpha)^{**} \wedge p^{**} = p^{**}$, since $e_\alpha^* \vee e_\alpha \in D(L)$, so that $p_\alpha^{**} = p_\beta^{**}$. Therefore $\{p_\alpha, q_\alpha : \alpha \in I\}$ is contained in some determination class of $L$. By 3(ii), there is a finite subset $J$ of $I$ such that $\vee \{\theta(e_\alpha, f_\alpha) : \alpha \in I\} = \vee \{\theta(p_\alpha, q_\alpha) : \alpha \in J\}$. However, $L$ has the P.J.P., by 3(iii) and Theorem 4.1, and so there exist $p, q \in L$ with $p \leq q$ and $p \equiv q(\Phi)$ such that $\vee \{\theta(p_\alpha, q_\alpha) : \alpha \in J\} = \theta(p, q)$. Summarizing, we have $\theta = \theta(a, 1) \vee \theta(p, q)$, which is principal by Lemma 4.2(iv). Thus, $L$ has only principal congruences.

COROLLARY 4.4. *Let $L$ be a distributive double $p$-algebra having finite range and $D^+(L)$ principal. Then $L$ has only principal congruences iff the following conditions hold*:

(i) Cen($L$) *and every determination class of $L$ is finite*,

(ii) *there is no 5-element chain in the poset of prime ideals of $L$.*

*Proof.* Suppose that $L$ has only principal congruences and let $\psi$ be a congruence of Cen($L$). Define a congruence $\bar{\psi}$ of $L$ by $\bar{\psi} = \vee \{\theta(a, b) : a \equiv b(\psi)\}$. We claim that $\bar{\psi} \restriction \text{Cen}(L) = \psi$. Clearly, $\psi \leq \bar{\psi} \restriction \text{Cen}(L)$. If $z \equiv w(\bar{\psi} \restriction \text{Cen}(L))$ then there is a sequence $z = x_0, x_1, \ldots, x_n = w$ in $L$ such that $x_{i-1} \equiv x_i(\theta(a_i, b_i))$, for some $a_i, b_i \in \text{Cen}(L)$ with $a_i \equiv b_i(\psi)$, $1 \leq i \leq n$. Now, for each $i$ with $1 \leq \text{i} \leq \text{n}$, there exists an integer $m_i$ such that $x_i^{m_i(+*)} \in \text{Cen}(L)$, since $L$ has finite range. Define $m = \max\{m_i : 1 \leq i \leq n\}$. Then $z = z^{m(+*)}$, $w = w^{m(+*)}$ and, for all $i$ with $1 \leq i \leq n$, $x_{i-1}^{m(+*)} \equiv x_i^{m(+*)}(\theta(a_i, b_i))$. Thus, $z \equiv w(\vee \{\theta(a, b) \restriction \text{Cen}(L) : a \equiv b(\psi)\})$. But Cen($L$) is a subalgebra of $L$ and so $\theta(a, b) \restriction \text{Cen}(L) = \theta_{\text{Cen}(L)}(a, b)$, by the congruence extension property which is known to hold for the variety of distributive double $p$-algebras (see [**10**]). Therefore $z \equiv w(\vee \{\theta_{\text{Cen}(L)}(a, b) : a \equiv b(\psi)\})$; in other words, $z \equiv w(\psi)$. Thus, $\bar{\psi} \restriction \text{Cen}(L) = \psi$. However, $\bar{\psi}$ is principal and so it is compact. Therefore $\bar{\psi} = \vee \{\theta(a, b) : (a, b) \in \theta\}$, for some finite subset $\theta \subseteq \psi$. Arguing as above, we have $\bar{\psi} \restriction \text{Cen}(L) = \vee \{\theta_{\text{Cen}(L)}(a, b) : (a, b) \in \theta\}$ and so $\psi$ is a principal congruence of Cen($L$), since Boolean lattices have the P.J.P.. Therefore Cen($L$) has only principal congruences and so it is finite. Thus, if $L$ has only principal congruences then (i) and (ii) hold.

https://doi.org/10.1017/S0017089500008673 Published online by Cambridge University Press

Suppose now that (i) and (ii) hold. By Theorem 4.3, we need only show that every normal filter of $L$ is principal. Let $F$ be a normal filter of $L$ and observe that $F = \vee \{N(a) : a \in F\}$. For each $a \in F$ there is an $n < \omega$ such that $a^{n(+*)} \in \mathrm{Cen}(L)$, since $L$ has finite range, and so $N(a) = [z)$, where $z = a^{n(+*)}$. Thus, $F = \vee \{[z) : z \in F \cap \mathrm{Cen}(L)\} = N(w)$, where $w = \wedge (F \cap \mathrm{Cen}(L))$ which exists, since $\mathrm{Cen}(L)$ is finite.

COROLLARY 4.5. *Let $L$ be a distributive double $p$-algebra whose $p$-algebra reduct is Stone. Then $L$ has only principal congruences iff there exists $t \in D^+(L)$ such that $D^+(L) \subseteq (t^{**}]$ and conditions* (i) *and* (ii) *in the statement of Corollary* 4.4 *hold.*

*Proof.* First, observe that $L$ has finite range. Indeed, for any $a \in L$, $a^{+*} \in \mathrm{Cen}(L)$ and so $a^{+*} = a^{2(+*)}$, since the $p$-algebra reduct of $L$ is Stone. Now suppose that $L$ has only principal congruences. It is clear, on examination of the proof of Corollary 4.4, that (i) and (ii) hold. Furthermore, if $\psi = \vee \{\theta(0, s) : s \in D^+(L)\}$ then $\psi$ is principal and so compact. Therefore there is a finite subset $S$ of $D^+(L)$ such that $\psi = \vee \{\theta(0, s) : s \in S\} = \theta(0, t)$, where $t = \vee S \in D^+(L)$. However, $\theta(0, t) = \theta(t^*, 1) = \theta_{\mathrm{lat}\, L}(t^*, 1)$, since $t^* \in \mathrm{Cen}(L)$ and it is easily verified that $\theta_{\mathrm{lat}\, L}(z, 1)$ preserves $^*$ and $^+$ whenever $z \in \mathrm{Cen}(L)$. Consequently, for any $s \in D^+(L)$, we have $s \wedge t^* = 0$, by the well known description of principal congruences of distributive lattices. Therefore $D^+(L) \subseteq (t^{**}]$.

To prove the converse, we need only examine the proof of Corollary 4.4 to realize that it suffices to show that, given any family $\{(e_\alpha, f_\alpha) : \alpha \in I\} \subseteq \Phi$ with $e_\alpha \leq f_\alpha \in D^+(L)$, we can find $\{(p_\alpha, q_\alpha) : \alpha \in I\}$ with $p_\alpha \leq q_\alpha$ such that, for all $\alpha \in I$, $\theta(e_\alpha, f_\alpha) = \theta(p_\alpha, q_\alpha)$ and $\{(p_\alpha, p_\beta) : \alpha, \beta \in I\} \subseteq \Phi$. For each $\alpha \in I$, let $p_\alpha = (e_\alpha^* \wedge t) \vee e_\alpha$ and $q_\alpha = (e_\alpha^* \wedge t) \vee f_\alpha$. Then, for all $\alpha \in I$, $p_\alpha \leq q_\alpha$ and $\theta(e_\alpha, f_\alpha) = \theta(p_\alpha, q_\alpha)$, by Lemma 4.2(vi). Furthermore, for all $\alpha \in I$, $p_\alpha \in D^+(L)$ and $p_\alpha^{**} = [(e_\alpha^* \vee e_\alpha) \wedge (t \vee e_\alpha)]^{**} = (t \vee e_\alpha)^{**} = t^{**} \vee e_\alpha^{**} = t^{**}$, since $e_\alpha \leq t^{**}$, so that $\{(p_\alpha, p_\beta) : \alpha, \beta \in I\} \subseteq \Phi$.

We point out that there are infinite distributive double $p$-algebras whose $p$-algebra reduct is Stone and which have only principal congruences; for example, any infinite Boolean algebra with a new zero adjoined and construed as a double $p$-algebra. In sharp contrast we have:

COROLLARY 4.6. *A double Stone algebra $L$ has only principal congruences iff $L$ is finite and $l(J(L)) \leq 3$.*

*Proof.* If $L$ is a double Stone algebra having only principal congruences then $L/\Phi$, being a homomorphic image of $L$, is a regular double Stone algebra (alias, a 3-valued Łukasiewicz algebra) having only principal congruences. Now, $\mathrm{Cen}(L/\Phi)$ is finite, by Corollary 4.5, and so $L/\Phi$ is finite, by Corollary 3 in [**3**]. However, every member of $L/\Phi$ is finite, by Corollary 4.5, and therefore $L$ is finite.

Distributive double $p$-algebras whose core is non-empty play an important role in the general theory. For such algebras significant improvements of Theorem 4.3 and Corollary 4.4 are attainable. Henceforth $L$ will denote a distributive double $p$-algebra having non-empty core $C(L)$.

The next two lemmas are proved in [**1**] and are the vital ingredients in our simplifications.

Lemma 4.7. *A congruence of $L$ below $\Phi$ is principal iff it is of the form $\theta(k,l)$, for some $k,l \in C(L)$ with $k \leq l$.*

Lemma 4.8. *For $L$, the following are equivalent:*
(i) *$L$ has the P.J.P.,*
(ii) *$C(L)$ has the P.J.P.,*
(iii) *there is no 3-element chain in the poset of prime ideals of $C(L)$.*

Theorem 4.9. *$L$ has only principal congruences iff*
(i) *every normal filter of $L$ is principal and*
(ii) *$C(L)$ has only principal congruences.*

*Proof.* If $L$ has only principal congruences then (i) and (ii) can be proved as in Theorem 4.3.

Suppose that (i) and (ii) hold. Let $\theta$ be a congruence of $L$. Using (i) we can prove, exactly as in the proof of Theorem 4.3, that

$$\theta = \theta(a,1) \vee \vee \{\theta(e_\alpha, f_\alpha) : \alpha \in I\},$$

for some $a \in L$ and $\{(e_\alpha, f_\alpha) : \alpha \in I\} \subseteq \Phi$ with $e_\alpha \leq f_\alpha$ for all $\alpha \in I$. By Lemma 4.7, we can assume that $e_\alpha, f_\alpha \in C(L)$ for all $\alpha \in I$. However, by Lemma 3.7 and (ii), $C(L)$ is finite and has the P.J.P.. Therefore $L$ has the P.J.P., by Lemma 4.8, and so $\vee \{\theta(e_\alpha, f_\alpha) : \alpha \in I\} = \theta(k,l)$, for some $k,l \in C(L)$ with $k \leq l$. Thus $\theta$ is principal by Lemma 4.2(iv).

On examining the proof of Corollary 4.4 and bearing Lemma 3.7 in mind, we obtain

Corollary 4.10. *If $L$ has finite range then it has only principal congruences iff $\mathrm{Cen}(L)$ and $C(L)$ are finite and $l(J(C(L))) \leq 1$.*

Concluding remarks

(i) It may be instructive to consider the distributive double $p$-algebra $L_2$ associated with the double $p$-space depicted in Example 2 of Section 3 in **[8]**. The congruence lattice of $L_2$ is a 3-element chain and so $L_2$ has only principal congruences. Furthermore, $L_2$ does not have finite range, $\Phi = \omega$, $D^+(L_2)$ is non-principal and $C(L_2)$ is empty.

(ii) We have been concerned with distributive lattices having no $n$-element chain, where $n = 3$, 4 or 5, in their poset of prime ideals. For intrinsic characterizations of such lattices we refer the reader to **[1]**.

## REFERENCES

**1.** M. E. Adams and R. Beazer, Congruence properties of distributive double $p$-algebras *Czechoslovak Math. J.* **41** (1991), 216–231.

**2.** R. Balbes and P. Dwinger, *Distributive Lattices*, (University of Missouri Press, 1974).

**3.** R. Beazer, The determination congruence on double $p$-algebras, *Algebra Universalis* **6** (1976), 121–129.

**4.** R. Beazer, Congruence uniform algebras with pseudocomplementation, *Studia Sci. Math. Hungar* **20** (1985), 43–48.

**5.** R. Beazer, Principal congruence properties of some algebras with pseudocomplementation *Portugaliae Math.* (to appear).

**6.** T. S. Blyth and J. C. Varlet, Principal congruences on some lattice-ordered algebras, *Discrete Math.* **81** (1990), 323–329.

https://doi.org/10.1017/S0017089500008673 Published online by Cambridge University Press

**7.** S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Graduate Texts in Mathematics, (Springer-Verlag, 1981).

**8.** B. A. Davey, Subdirectly irreducible distributive double $p$-algebras, *Algebra Universalis* **8** (1978), 73–88.

**9.** G. Grätzer, *General lattice theory*, (Birkhäuser Verlag, 1978).

**10.** T. Katriňák, Congruence extension property for distributive double $p$-algebras, *Algebra Universalis* **4** (1974), 273–276.

**11.** T. Katriňák, Subdirectly irreducible $p$-algebras, *Algebra Universalis* **9** (1979), 116–126.

**12.** T. Katriňák, $p$-algebras, *Colloq. Math. Soc. J. Bolyai*, *Szeged* (1980), 549–573.

UNIVERSITY OF GLASGOW,
UNIVERSITY GARDENS,
G12 8QW,
SCOTLAND