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EXISTENCE OF AN ORDER-PRESERVING FUNCTION ON NORMALLY PREORDERED SPACES

GHANSHYAM MEHTA

The object of this paper is to generalize the classic theorems of Eilenberg and Debreu on the existence of continuous order-preserving transformations on ordered topological spaces and to prove them in a different way. The proof of the theorems is based on Nachbin's generalization to ordered topological spaces of Urysohn's separation theorem in normal topological spaces.

Introduction.

Eilenberg [2] has proved the existence of a continuous order-preserving transformation on a connected and topologically separable ordered topological space. Debreu [1] has proved the existence of a continuous order-preserving transformation on a second countable ordered topological space.

The object of this paper is to prove a theorem which generalizes the theorems of Eilenberg and Debreu and to deduce these classic theorems as corollaries. The proof of this theorem is based on Nachbin's generalization to ordered topological spaces of Urysohn's separation

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theorem in normal topological spaces and is different from the proofs given by Eilenberg and Debreu. It will be proved that a partially preordered space which is normally preordered and order-separable, has a continuous order-preserving representation. Such a space need not be connected or second countable.

In a related paper (see Mehta [4]) it has been proved that this theorem can also be used to extend Fleischer's theorem [3] and to prove it in a different way.

Preliminaries.

A preorder $\lesssim$ on a topological space X is a reflexive transitive binary relation on X. We say $x < y$ if and only if $x \lesssim y$ and not $y \lesssim x$. The preorder $\lesssim$ is said to be decisive or complete if for two elements x, y belonging to X, either $x \lesssim y$ or $y \lesssim x$. It is said to be continuous if the sets $\{x \in X: x \lesssim y\}$ and $\{x \in X: y \lesssim x\}$ are closed for every y in X. A topology in which sets of the form $\{x \in X: x \lesssim y\}$ and $\{x \in X: y \lesssim x\}$ are closed for y in X, is said to be a natural topology for X.

A subset E of X is said to be decreasing if $b \in E$, $a \lesssim b$ imply that $a \in E$. Each subset E of X determines uniquely a smallest decreasing subset $d(E)$ containing E. Similarly, one defines the concept of an increasing set and the smallest increasing subset $i(E)$ containing E.

A topological space equipped with a preorder is said to be normally preordered, if, for every two disjoint closed subsets F_0 and F_1 of X, F_0 being decreasing and F_1 increasing, there exist two disjoint open subsets A_0 and A_1 such that A_0 contains F_0 and is decreasing and A_1 contains F_1 and is increasing.

A preordered topological space $(X, \lesssim)$ is said to be order-separable if there exists a countable subset Z such that if x, y belong to X and $x < y$ then there exists a z in Z such that $x < z < y$.

Let x, y be two elements such that $x < y$. This pair is said to be a jump if $(x, y) = \{a \in X: x < a < y\}$ is empty.

Let E_1 and E_2 be two preordered sets. A function f on E_1 to E_2 is said to be increasing if x, y in E and $x \lesssim y$ imply $f(x) \leq f(y)$. A function f on E_1 to E_2 is said to be order-preserving if it is increasing and $x < y$ implies $f(x) < f(y)$.

Order-preserving functions.

Before proving the main theorem, we consider the following proposition.

PROPOSITION 1. *Let $(X, \lesssim)$ be a preordered topological space. Assume that the preorder is decisive and continuous. Then $(X, \lesssim)$ is normally preordered.*

Proof. Let F_0 and F_1 be two disjoint closed subsets of X, with F_0 decreasing and F_1 increasing. If F_0 and F_1 exhaust X then F_0 and F_1 are open and X is normally preordered. If they do not exhaust X there is a point d not in F_0 or F_1. Since the preorder is decisive, $a < d < b$ for every a in F_0 and b in F_1. Hence, $\{x \in X: x < d\}$ and $\{x \in X: d < x\}$ are the required decreasing and increasing open sets containing F_0 and F_1 respectively. Thus $(X, \lesssim)$ is normally preordered. □

We now prove the main result of this paper.

THEOREM 1. *Let $(X, \lesssim)$ be a normally preordered topological space and suppose that the preorder $\lesssim$ is continuous. Then if $(X, \lesssim)$ is order-separable there exists a continuous order-preserving real function on $(X, \lesssim)$.*

Proof. Since the preorder $\lesssim$ is continuous, it follows that $d(x) = \{y: y \lesssim x\}$ and $i(x) = \{y: x \lesssim y\}$ are closed in X for every x in X. Hence, for every x, $d(x)$ is a closed decreasing set, and $i(x)$ is a closed increasing set. Now if $x < y$, $d(x)$ is a closed decreasing set containing x and $i(y)$ is a closed increasing set containing y. Clearly, $d(x)$ and $i(y)$ are disjoint.

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Consequently, by Nachbin's separation theorem [5, p. 30] there is an increasing continuous real valued function f on X to the unit interval $[0, 1]$ such that f is zero on $d(x)$ and one on $i(y)$.

Let Z be a countable order-dense subset of X. Such a set exists since X is assumed to be order-separable. Then apply Nachbin's theorem to each pair (z_i, z_j), z_i, z_j belonging to Z with $z_i < z_j$ to obtain an increasing continuous function f_{ij} on X to the unit interval $[0, 1]$ such that $f_{ij}(d(z_i)) = 0$ and $f_{ij}(i(z_j)) = 1$. Let S be the subset of all pairs of points (z_i, z_j) belonging to Z such that $z_i < z_j$. Then S is countable, and we may suppose that S is equal to the set of positive integers N.

Define $f(x) = \sum_{k \in S} f_k(x) \frac{1}{2^k}$. Then f is an increasing continuous function on X. If $x < y$, there exist z_i and z_j belonging to Z such that $x < z_i < z_j < y$. It follows that $x < y$ implies $f(x) < f(y)$. Hence, f is the required order-preserving function. □

Remark 1. In the above theorem it has not been assumed that the preorder is decisive. If we assume that the preorder is decisive then the above theorem gives the result that $x \lesssim y$ if and only if $f(x) \leq f(y)$.

Remark 2. In the above theorem, it has not been assumed that X is second countable or that it is connected. All that has been assumed is that X is normally preordered and order-separable. As an example, consider the real line with the discrete topology and the natural ordering. Then it is easy to check that this space is not connected and does not satisfy the second axiom of countability (it is not a Lindelöf space). However, it is order-separable and normally preordered. Hence, it has a continuous order-preserving representation by Theorem 1, a fact that cannot be deduced from the Eilenberg-Debreu theorems.

The theorems of Eilenberg and Debreu will now be shown to be consequences of Theorem 1.

COROLLARY 1. *(Eilenberg) Let $(X, \leq)$ be a decisively preordered connected and separable topological space. If the preorder is continuous, then there exists a continuous order-reserving real-valued function on* X.

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Proof. Since the preorder is decisive and continuous, Proposition 1 implies that X is normally preordered.

Since X is topologically separable, it has a countable dense subset Z. Let $x < y$. The sets $d(x)$ and $i(y)$ are closed because the preorder is continuous. They are clearly non-empty and disjoint. The sets $d(x)$ and $i(y)$ do not exhaust X because X is connected. Hence the set $K = \{k \in X : x < k < y\}$ is non-empty. It is also open since the preorder is continuous. Consequently, there exists z in Z such that $x < z < y$. This proves that X is order-separable.

Eilenberg's theorem now follows from Theorem 1. □

COROLLARY 2. *(Debreu) Let $(X, \lesssim)$ be a decisively preordered topological space satisfying the second axiom of countability. If the preorder is continuous, then there exists a continuous order-preserving real-valued function on* X.

Proof. The proof of Corollary 2 is accomplished by constructing a certain quotient space Y from X, embedding Y in a larger space Y' and then by applying theorem 1 to Y'.

We start by defining a relation $\sim$ on X as follows: $x \sim y$ if and only if $x \lesssim y$ and $y \lesssim x$. It is easily verified that $\sim$ is an equivalence relation on X. Denote an equivalence class of an element $a \in X$ by $[a]$. Let $Y = X/\sim$ be the set of equivalence classes. Endow Y with the quotient topology relative to the canonical projection P of X onto Y.

The relation $<$ on X induces a relation $<'$ on Y as follows: $[a] <' [b]$ if and only if $x \in [a]$, $y \in [b]$ and $x < y$. We can now define a new relation $\lesssim'$ on Y by saying that $[a] \lesssim' [b]$ if and only if either $[a] = [b]$ or if $[a] <' [b]$. Clearly $\lesssim'$ is a decisive preorder on Y. Furthermore, since Y has the quotient topology relative to X and the canonical projection P, it follows that $\lesssim'$ is a continuous preorder on Y.

We prove next that $(Y, \lesssim')$ has at most denumerably many jumps. To this end, let $S = \{S_1, S_2, \ldots\}$ be a countable base for the topology of X. Suppose that $([x], [y])$ and $([p], [q])$ are two distinct jumps in Y. Since the preorder on Y is decisive only two cases can

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arise. Either $[x] <' [y] <' [p] <' [q]$ or $[p] <' [q] <' [x] <' [y]$. In the first case, there exists $x \in [x]$ and $y \in [y]$ such that $x \in \{a \in X : a < y\}$, an open set because the preorder $\lesssim$ is decisive and continuous. Since S is a base, there exists an i such that $x \in S_i \subseteq \{a \in X : a < y\}$. Hence, $x \in S_i$ and $y \notin S_i$. A similar argument shows that there is a j such that $p \in S_j \subseteq \{a \in X : a < q\}$ and $q \notin S_j$, where $p \in [p]$ and $q \in [q]$. Now $y \notin S_i$ implies that $p \notin S_i$ by transitivity of the preorder. Since $p \in S_j$ it follows that $S_i \neq S_j$. The argument in the second case is analogous. Hence, there is an injection from the set of jumps of Y to the countable set S and we may conclude that Y has at most denumerably many jumps.

The main idea of the rest of the proof is to extend the space Y so that it satisfies the conditions of theorem 1. Suppose, first, that there are only finitely many jumps $([a_i], [b_i])$. $i = 1, \ldots, n$, in Y. Interpose between the endpoints of each jump a copy of the open real interval $(i, i+1)$, $i = 1, \ldots, n$ with its usual preorder. If there are countably infinite jumps, interpose between the endpoints of each jump $([a_n], [b_n])$, $n = 1, 2, \ldots,$ a copy of the open real interval $(n, n+1)$, $n = 1, 2, \ldots,$ with its usual preorder. Call this enlarged space Y'. The preorder on Y' is obtained in a natural manner from the preorder on Y and the usual preorder of the real numbers. The topology on Y' is the natural topology on the disjoint union of two topological spaces.

The preorder on Y' is clearly decisive and continuous. So again by Proposition 1 we may conclude that Y' is normally preordered.

It remains to be proved that Y' is order-separable. Since X is second countable, it has a countable topologically dense subset B. Let $Z = \{[z] \in Y : b \in [z]$ for some $b \in B\}$. Then, clearly, Z is countable. If Y has finitely many jumps $([a_i], [b_i])$, $i = 1, \ldots, n$, let Z' be the set of all non-negative non-integral rational numbers smaller than $n+1$. If Y has countably infinite jumps, let Z' be the set of all non-negative, non-integral rational numbers. Then it is easily verified that $Z \cup Z'$ is order-dense in Y', so that Y' is order-separable.

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Hence, all the conditions of Theorem 1 are satisfied for the space Y' and we may conclude that there exists a continuous order-preserving function T on Y'.

Finally, we define a real-valued function f on X by $f(x) = T([P(x)])$, where $P : X \to Y$ is the natural projection. The function f is continuous as a composition of continuous functions. If $x \sim y$, $[P(x)] = [P(y)]$ and $f(x) = f(y)$. If $x < y$, $[P(x)] <' [P(y)]$ and $f(x) < f(y)$ since T is order-preserving. Thus f is the required continuous order-preserving function. □

Remark 1. Observe that Theorem 1 is a common generalization of the theorems of Eilenberg and Debreu.

Remark 2. The proofs of the Eilenberg and Debreu theorems given above are based on Nachbin's separation theorem and are different from the original proofs.

Remark 3. Fleischer [3] has proved that a linearly ordered set that is separable in its order topology and has countably many jumps is order-isomorphic to a subset of the real numbers. It is shown in Mehta [4] that Theorem 1 can be used to extend Fleischer's theorem to partially preordered spaces and to prove it in a different way.

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Departments of Economics and Mathematics
University of Queensland
St. Lucia. Qld. 4067.

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