A topological version of a theorem of Mather on twist maps

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Abstract. In this report we show that a twist map of an annulus with a periodic point of rotation number p/q must have a Birkhoff periodic point of rotation number p/q. We use topological techniques so no assumption of area-preservation or circle intersection property is needed. If the map is area-preserving then this theorem and the fixed point theorem of Birkhoff imply a recent theorem of Aubry and Mather. We also show that periodic orbits of (significantly) smallest period for a twist map must be Birkhoff.

1. *Introduction*

We say a diffeomorphism f of an annulus onto itself satisfies a twist condition if the angular component of the image of a point under f increases as the radial component of the point increases (see § 2 for precise definitions). Such maps were first studied in connection with the three-body problem by Poincaré, and in this context the map has a natural invariant measure. Birkhoff [4], [5] showed that such area-preserving twist maps have many periodic orbits, however his theorem gave no insight into the nature of these orbits. Recently, Aubry [2] and Mather [15] (see also [14]) have shown that area-preserving twist maps possess periodic orbits such that f preserves the angular ordering of points on the orbit, such orbits are called Birkhoff periodic orbits (see [14] and § 2). Moreover, they also showed the existence of 'quasi-periodic' orbits for area-preserving twist maps. The variational techniques used by Mather have proved useful in the study of other aspects of area-preserving twist maps (see, for example [16]).

Twist maps also occur frequently as non-area-preserving maps. For example, near the rest point of a map of the plane which has undergone Hopf bifurcation a twist condition will be satisfied. For such dissipative maps there will be no invariant measure.

In this report we show that a twist map of the annulus having a periodic orbit of some rotation number will have a Birkhoff periodic orbit of the same rotation number. One can think of this as replacing the area-preserving hypothesis with an assumption on the existence of periodic orbits, so this theorem can be applied to dissipative maps (see [9], [10]). Given an area-preserving twist map, Birkhoff's theorem mentioned above gives the existence of many periodic orbits, and hence the Aubry–Mather theorem may be derived as a corollary.

The idea of the proof is to make use of the twist condition to show that periodic orbits which are not Birkhoff periodic orbits are topologically complicated. For example, if we suspend the twist map and look at the orbit of a non-Birkhoff periodic orbit it forms a non-trivial braid. The way in which these orbits link, given by the twist condition, can be used to imply the existence of new periodic points either by a geometric argument or a theorem on braids and periodic orbits of Matsuoka [17]. In particular, if a twist map has Birkhoff and non-Birkhoff periodic orbits of some given rotation number then it must in fact have two distinct Birkhoff periodic orbits with that rotation number.

In §2 we give the notation used throughout. In §3 we state the main result, proved in §5 using lemmas of §4. §6 is used to give some related theorems for twist maps which say basically that a periodic orbit with period (much) smaller than the period of every other periodic orbit of a given twist map must be a Birkhoff periodic orbit.

2. Definitions and notation

We let $A = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le 1\}$, π_1 (respectively π_2): $A \to \mathbb{R} : (x, y) \to x$ (respectively y), be the usual projections, and for each $\alpha \in \mathbb{R}$

$$l_\alpha = \{(\alpha, y) \in A : 0 \le y \le 1\}.$$

Definition. A map $f : A \to A$ is called a *twist map* if
    (1) f is a C^2 diffeomorphism, preserving boundary components and orientation;
    (2) for all $(x, y) \in A$, $f((x, y)) + (1, 0) = f((x+1, y))$;
    (3) there exists $\delta > 0$, such that for all $z \in A$, $\partial(\pi_1 \circ f)(z)/\partial y > \delta$.

Remarks. Condition (2) states that f is the lift of a map on the annulus of which A is the universal cover. Condition (3) is the 'twist condition', it implies that for any $\alpha, \beta \in \mathbb{R}$, $f(l_\alpha) \cap l_\beta$ is at most one point (see figure 1). This is sometimes called a 'monotone' twist condition.
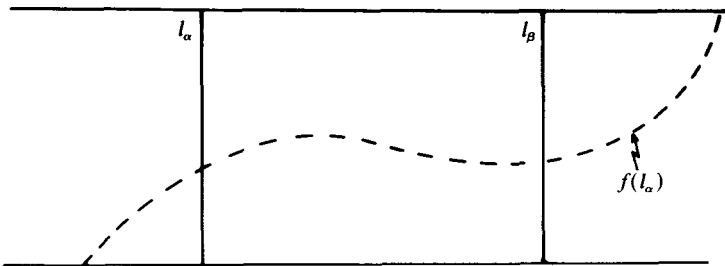
FIGURE 1

Definition. The *orbit* of a point $z \in A$ under a twist map $f : A \to A$ is defined to be the set $\mathcal{O}(f, z) = \{f^k(z) + (l, 0) : k, l \in \mathbb{Z}\}$.

Definition. A point $z \in A$ is called a *p/q-periodic point* for a twist map $f : A \to A$ if

$$f^q(z) - (p, 0) = z.$$

Similarly, a point $z \in A$ is called a p/q-*Birkhoff periodic point* for f if z is a p/q-periodic point and for any $z_1, z_2 \in \mathcal{O}(f, z)$

$$\pi_1(z_1) < \pi_1(z_2) \Rightarrow \pi_1(f(z_1)) < \pi_1(f(z_2)).$$

Remarks. Recall that f is the lift of an annulus map and hence the 'orbit' defined above and the definition of p/q-periodic point correspond to the lifts from the annulus of the usual orbit and periodic points. The p/q-Birkhoff periodic points are those p/q-periodic points z for which f restricted to $\mathcal{O}(f, z)$ is 'order preserving' in the x-coordinate.

Definition. If $f: A \to A$ is a twist map and $z \in A$ then the *rotation number* of f at z is

$$\rho(f, z) = \lim_{n \to \infty} \frac{1}{n} (\pi_1(f^n(z)))$$

if it exists.

Notation. The maps $f|_{\{(x,i): x \in \mathbb{R}\}}$, $i = 0, 1$ are lifts of circle diffeomorphisms, hence we let

$$\rho_0(f) = \rho(f(x, 0)) \quad \text{and} \quad \rho_1(f) = \rho(f(x, 1))$$

where these definitions are independent of $x \in \mathbb{R}$ and the limits exist (see [12]).

3. *Statement of the main theorem*
In this paper we prove the following theorem:

THEOREM 1. *If $f: A \to A$ is a twist map and f has a p/q-periodic point then f has a p/q-Birkhoff periodic point.*

This theorem applies whether or not f is area-preserving. If f is area-preserving then we can combine it with the following two results to obtain a theorem of Aubry and Mather.

THEOREM 2 (Birkhoff [4], [5], see also Chenciner [9]). *If $f: A \to A$ is an area-preserving twist map and $p/q \in [\rho_0(f), \rho_1(f)]$ then f has a p/q-periodic point.*

Remark. Birkhoff's theorem (known as 'Poincaré's Last Geometric Theorem') is actually true under a much weaker twist condition.

LEMMA 0 (Katok [14]). *If $f: A \to A$ is a twist map, $\{p_n/q_n\}_{n \geq 0}$ is a sequence of rationals with $p_n/q_n \to \alpha \notin \mathbb{Q}$ as $n \to \infty$ and for each n, f has a (p_n/q_n)-Birkhoff periodic point z_n, then any limit point z of $\{z_n\}_{n \geq 0}$ satisfies $\rho(f, z) = \alpha$.*

Remark. In fact, much more is true of the orbit of such a limit point z of $\{z_n\}_{n \geq 0}$. The map f restricted to it is order preserving in the x-coordinate and the orbit lies on the graph of a periodic Lipschitz function (see [15], [14], [13]). These additional facts follow merely from the geometry of twist maps.

THEOREM 3 (Aubry [2] and Mather [15]). *If $f: A \to A$ is an area-preserving twist map and $\alpha \in [\rho_0(f), \rho_1(f)]$ then there exists $z_\alpha \in A$ with $\rho(f, z_\alpha) = \alpha$.*

Proof of theorem 3. By Birkhoff's theorem (theorem 2), for each rational $p/q \in [\rho_0(f), \rho_1(f)]$ there is a $z_{p/q} \in A$ which is a p/q-periodic point of f. By theorem 1 we see that there must then exist a p/q-Birkhoff periodic point. Applying lemma 0 to sequences of Birkhoff periodic points we can obtain points with irrational rotation number for any irrational in $[\rho_0(f), \rho_1(f)]$, which completes the proof of the theorem.   □

(See the remark above and the papers of Mather [15], Katok [14], and Herman [13] for the other properties of the orbits of these 'Birkhoff points'.)

The proof of theorem 1 proceeds as follows: Assume we have a twist map $f: A \to A$ with a p/q-periodic point. Then we show that f is homotopic to a twist map $g: A \to A$ which has a p/q-Birkhoff periodic point such that each map in the homotopy has p/q-periodic points. The set of maps which have p/q-Birkhoff periodic points will then be seen to comprise an open-closed set in the parameter of this homotopy. Closure follows easily from the definition of Birkhoff periodic point (see lemma 1 below) while openness follows from the fixed point lemma of the next section, lemma 4. Basically this lemma says that a map with a p/q-Birkhoff periodic orbit and another p/q-periodic orbit must in fact have two p/q-Birkhoff periodic orbits, moreover this second orbit must persist under small perturbations.

The next section contains some lemmas needed for the proof of theorem 1 which is in § 5. In § 6 we show, using similar, but technically easier techniques that if $f: A \to A$ is a twist map and for some relatively prime integers p, q the map f satisfies the following condition:

(†) Every r/s-periodic point of f has $s = q$ or $s > (\frac{3}{2})q$,

then every p/q-periodic point of f is a p/q-Birkhoff periodic point. Conditions on p/q, $\rho_0(f)$ and $\rho_1(f)$ can be given which imply the condition, (†) above saying essentially that the map isn't twisting very much. For area-preserving twist maps we can improve this theorem, replacing (†) with the following:

(‡) Every r/s-periodic point of f has $s \ge q$,

obtaining the same conclusion, that every p/q-periodic orbit is a p/q-Birkhoff periodic orbit.

Remark. A. Katok informed the author that D. Bernstein has recently given a proof of a version of theorem 3 for which the area-preservation property is replaced by the 'circle-intersection' property, see [3]. There is also a theorem of P. Carter [8] showing a result analogous to theorem 2 with area-preservation replaced with the circle-intersection property.

4. Some lemmas

In this section we prove lemmas useful in the proof of theorem 1.

LEMMA 1. *Suppose $f_n: A \to A$ is a sequence of twist maps, $n = 1, 2, \ldots$, and for some fixed p, q relatively prime integers, each f_n has a p/q-Birkhoff periodic point $z_n \in A$. If f_n converges, in the sup norm topology, to a twist map $f_0: A \to A$ (i.e. $\sup_{z \in A} \|f_n(z) - f_0(z)\| \to 0$ as $n \to \infty$ where $\| \cdot \|$ is the usual $\mathbb{R}^2$ norm) and z_n converges to $z_0 \in A$ as n tends to infinity then z_0 is a p/q-Birkhoff periodic point of f_0.*

Proof of lemma 1. Since $f_n^q(z_n) - (p, 0) = z_n$ for all $n = 1, 2, \ldots$, it follows that $f_0^q(z_0) - (p, 0) = z_0$. (Moreover, $f_0^s(z_0) - (r, 0) \neq z_0$ for any r, s with $s < q$ since p and q are relatively prime.) Hence, z_0 is a p/q-periodic point of f_0.

To show that z_0 is a p/q-Birkhoff periodic point of f_0 we fix $k_1, k_2, l_1, l_2 \in \mathbb{Z}$. Then for $n = 1, 2, \ldots$

$$\pi_1(f_n^{k_1}(z_n)) + l_1 < \pi_1(f_n^{k_2}(z_n)) + l_2 \Rightarrow \pi_1(f_n^{k_1+1}(z_n)) + l_1 < \pi_1(f_n^{k_2+1}(z_n)) + l_2.$$

But then the same statement holds by continuity when we replace f_n and z_n by f_0 and z_0 respectively and put '$\leq$' into the second inequality. But suppose

$$\pi_1(f_0^{k_1}(z_0)) + l_1 < \pi_1(f_0^{k_2}(z_0)) + l_2, \qquad (*)$$

and

$$\pi_1(f_0^{k_1+1}(z_0)) + l_1 = \pi_1(f_0^{k_2+1}(z_0)) + l_2.$$

Then, by the twist condition we have $\pi_2(f_0^{k_1+1}(z_0)) > \pi_2(f_0^{k_2+1}(z_0))$ (see figure 2). So, again by the twist condition $\pi_1(f_0^{k_1+2}(z_0)) + l_1 > \pi_1(f_0^{k_2+2}(z_0)) + l_2$. But this implies

$$\pi_1(f_n^{k_1+2}(z_n)) + l_1 > \pi_1(f_n^{k_2+2}(z_n)) + l_2$$

for n sufficiently large while $(*)$ above implies

$$\pi_1(f_n^{k_1}(z_n)) + l_1 < \pi_1(f_0^{k_2}(z_n)) + l_2$$

for n sufficiently large and this contradicts the fact that z_n is a p/q-Birkhoff periodic point of f_n. So z_0 must be a p/q-Birkhoff periodic point of f_0 and the proof is complete. ☐
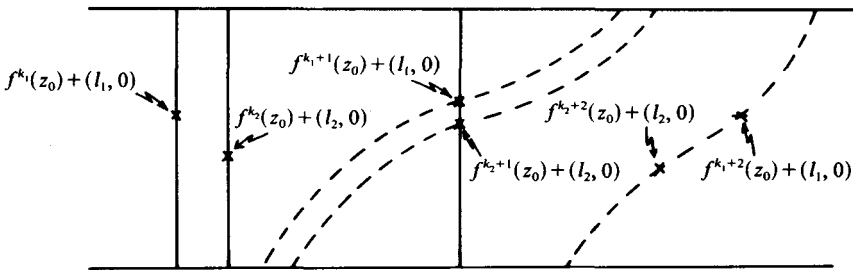

FIGURE 2

The next lemma says we can suspend a given twist map so that the intervening maps are also twist maps.

LEMMA 2. *Given a twist map $f: A \to A$ there exists a C^1 map $\phi: A \times \mathbb{R} \to A$ satisfying*
  (1) *$\phi(\cdot, 0) = identity$ on A, $\phi(\cdot, 1) = f(\cdot)$;*
  (2) *for all $t \in [0, 1]$ and all $n \in \mathbb{Z}$, $\phi(\cdot, t+n) = \phi(f^n(\cdot), t)$;*
  (3) *for all $t \in [0, 1]$, $\phi(\cdot, t)$ is a twist map;*
  (4) *for all $z \in A$ and all $t \in \mathbb{R}$, $\phi(z + (1, 0), t) = \phi(z, t) + (1, 0)$.*

Proof of lemma 2. It suffices to define ϕ on $A \times [0, 1]$ so that (1), (3) and (4) are satisfied since (2) can then be used to extend the definition to all of $\mathbb{R}$.

Let $f_0: \mathbb{R} \to \mathbb{R}$ be given by $f_0(x) = \pi_1(f(x, 0))$ and $G: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$G(z) = Df(f^{-1}(z))(0, 1)$$

for all $z \in A$, where $Df(w)$ is the derivative (matrix) of f at $w \in A$. Then f is determined by the map f_0 and the vector field G as follows: Let ψ be the local flow with domain in $A \times \mathbb{R}$ determined by the initial value problem

$$\begin{cases} \dfrac{d\psi}{dt}(z, t) = G(\psi(z, t)) \\ \psi(z, 0) = z, \qquad \forall z \in A. \end{cases}$$

Then f is given, for all $z \in A$, by

$$f(z) = \psi((f_0(\pi_1(z)), 0), \pi_2(z)). \tag{$**$}$$

The required homotopy of f can then be easily constructed by:

(a) deforming f_0 to the identity on $\mathbb{R}$ through diffeomorphisms which are lifts of circle diffeomorphisms;

(b) forming a smooth, one-parameter family $G_s : A \to \mathbb{R}^2$ of vector fields with parameter $s \in [0, 1]$ so that G_0 is the constant $(0, 1)$ and G_1 is equal to G. To assure condition (3) we construct G_s so that the angle between $G_s(z)$ and the x-axis is always between $-\pi/2$ and $\pi/2$ and increasing with s. For condition (4) we require $G_s(z + (1, 0)) = G_s(z)$ for all $z \in A$, $s \in [0, 1]$.

The maps $\phi(\cdot, s)$ are now given by using the solution of the vector field G_s and the initial conditions specified by the diffeomorphism in (a) above in an equation of the form $(**)$. Of course, to extend ϕ to a globally smooth map on $A \times \mathbb{R}$ we must 'match up' the families of circle maps and vector fields given in (a) and (b) near $s = 0$ and $s = 1$. These details are left to the reader.   □

It is the topological nature of the orbits of p/q-Birkhoff periodic points under these flows which is the key to the proof of theorem 1, i.e. the fact that they are not 'linked', as is made precise in the next lemma.

Suppose $f : A \to A$ is a twist map and p, q are relatively prime integers. Let $g : A \to A$ be defined for all $z \in A$ by

$$g(z) = f^q(z) - (p, 0).$$

Then the fixed points of g correspond precisely with the p/q-periodic points of f. Suppose f has a p/q-Birkhoff periodic point $z_0 \in A$, then we have

LEMMA 3. *There exists a C^1 map $\phi : A \times \mathbb{R} \to A$ satisfying*:

(i) *for all $z \in A$ and all $t \in \mathbb{R}$, $\phi(z + (1, 0), t) = \phi(z, t) + (1, 0)$*;

(ii) *$\phi(\cdot, 0) = identity$, $\phi(\cdot, 1) = g(\cdot)$*;

(iii) *for all $t \in [0, 1]$ and all $n \in \mathbb{Z}$, $\phi(\cdot, t + n) = \phi(g^n(\cdot), t)$*;

(iv) *for all $t \in [0, 1/q]$ and $i = 0, \ldots, q - 1$, $\phi(\phi(\cdot, i/q)^{-1}, i/q + t)$ is a twist map*;

(v) *for $i = 1, \ldots, q$, and any $z, w \in A$, $\pi_1(\phi(z, i/q)) < \pi_1(\phi(w, i/q))$ if and only if $\pi_1(f^i(z)) < \pi_1(f^i(w))$*;

(vi) *for all $\zeta \in \mathcal{O}(f, z_0)$ and all $t \in \mathbb{R}$, $\phi(\zeta, t) = \zeta$*.

Proof of lemma 3. Let $\phi_1 : A \times \mathbb{R} \to A$ be the one parameter family associated with f by lemma 2. We may assume, by composing the maps $\phi_1(\cdot, t)$ with a map which preserves the $x = $ constant foliation (i.e. adjusting the norms of the vectors given by

the vector field G_t), that if $\zeta_1, \zeta_2 \in \mathcal{O}(f, z_0)$ with $\pi_1(\zeta_1) < \pi_1(\zeta_2)$ then

$$\pi_1(\phi_1(\zeta_1, t)) < \pi_1(\phi_1(\zeta_2, t)) \qquad \text{for all } t \in [0, 1].$$

But then letting $\phi_2 : A \times [0, 1] \to A$ be defined for all $z \in A$ and all $t \in [0, 1]$ by

$$\phi_2(z, t) = \phi_1(z, qt) - t(p, 0)$$

we see ϕ_2 satisfies conditions (i), (ii), (iv) and (v) and that if $\zeta_1, \zeta_2 \in \mathcal{O}(f, z_0)$, $\pi_1(\zeta_1) < \pi_1(\zeta_2)$ then $\pi_1(\phi_2(\zeta_1, t)) < \pi_1(\phi_2(\zeta_2, t))$ for all $t \in [0, 1]$. Hence, we may deform ϕ_2 to a family $\phi : A \times [0, 1] \to A$ so that (i), (ii), (iv), (v) and (vi) are satisfied and extend ϕ to a map on $\mathbb{R}$ by condition (iii). This is the required map ϕ and the proof is complete. $\qquad\square$

Remarks. (1) Informally we can say that the braid given by $\phi_2(\zeta, t)$ for $\zeta \in \mathcal{O}(f, z_0)$ is trivial hence it may be 'straightened out', (see figure 3).
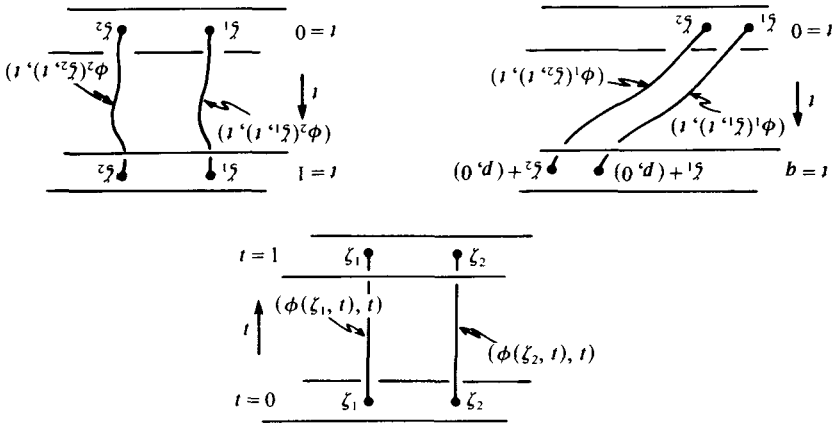

FIGURE 3

(2) The hypothesis that z_0 be a p/q-Birkhoff periodic point is necessary in the above lemma. In fact, if z_0 is a p/q-periodic point, but not a p/q-Birkhoff periodic point then the above lemma can *not* hold. To see this, note that if z_0 is not a p/q-Birkhoff periodic point then there exist $\zeta_1, \zeta_2 \in \mathcal{O}(f, z_0)$ with $\pi_1(\zeta_1) < \pi_1(\zeta_2)$ and $\pi_1(f(\zeta_1)) > \pi_1(f(\zeta_2))$. But as we will see in the proof of the next lemma, this implies that the orbits of ζ_1 and ζ_2 under ϕ_1 'link' non-trivially (see figure 4) and hence cannot be straightened out simultaneously.

(3) Finally we note that the proof of lemma 3 relies only on the fact that points of a Birkhoff periodic orbit stay in order, i.e. the same proof serves to show

LEMMA 3'. *With $f : A \to A$, $g : A \to A$ as above and ζ_1, ζ_2 p/q-periodic points of f satisfying $\pi_1(f^i(\zeta_1)) < \pi_1(f^i(\zeta_2))$ for all $i = 0, \ldots, q$ there exists a map $\phi : A \times \mathbb{R} \to A$ satisfying* (ii)–(v) *of lemma 3 and*
  (vi') *for all $t \in \mathbb{R}$, $\phi(\zeta_i, t) = \zeta_i$, $i = 1, 2$.*

Remark. It is important to note that we cannot require condition (i), periodicity in x, for this lemma since ζ_1 may not link with ζ_2 while linking with $\zeta_2 - (1, 0)$. Hence
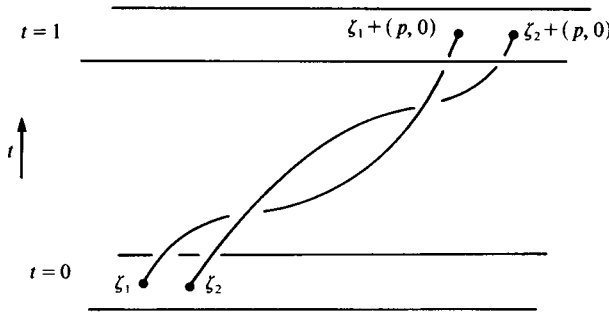
FIGURE 4

the 'straightening out' in this case takes place only in the covering space A, not in the annulus.

Proof of lemma 3'. Same as the proof of lemma 3.   □

The final lemma of this section is the 'fixed point' lemma required for theorem 1.

LEMMA 4. *Suppose $f: A \to A$ is a twist map and p, q are relatively prime integers. Suppose f has a p/q-Birkhoff periodic point $z_0 \in A$ and a p/q-periodic point $w_0 \in A$ which is not a Birkhoff periodic point. Then f has another p/q-Birkhoff periodic point $z_1 \in A$ with $z_1 \notin \mathcal{O}(f, z_0)$. Moreover, if $\tilde{f}: A \to A$ is a twist map and $\sup_{z \in A} \|f(z) - \tilde{f}(z)\|$ is sufficiently small, where $\|\cdot\|$ is the usual norm in $\mathbb{R}^2$, then $\tilde{f}$ has a p/q-Birkhoff periodic point.*

Remark. This lemma is reminiscent of the theorem of Birkhoff for area-preserving twist maps (of which theorem 2 in §3 is a corollary) which states that if $0 \in [\rho_0(f), \rho_1(f)]$ for a twist map f then f must have two fixed points. That f must have two fixed points in the generic case follows easily from index arguments, however f must have two fixed points even when they are degenerate (see [4], [5] and [7]). It should be emphasized that we are not assuming area-preservation or any circle intersection properties in lemma 4.

Proof of lemma 4. Let $g(z) = f^q(z) - (p, 0)$ for all $z \in A$ as above and let $\phi: A \times \mathbb{R} \to A$ be the one parameter family associated with g and the orbit of $z_0 \in A$ by lemma 3. For convenience we introduce the following technical notation:

We say two points $\zeta, \eta \in A$ *get out of order* if $\pi_1(\zeta) < \pi_1(\eta)$ and $\pi_1(f^i(\zeta)) > \pi_1(f^i(\eta))$ for some $i, 0 < i < q$, or if $\pi_1(\zeta) > \pi_1(\eta)$ and $\pi_1(f^i(\zeta)) < \pi_1(f^i(\eta))$ for some $i, 0 < i < q$.

An immediate consequence is that if ζ, η are p/q-periodic points of f then ζ, η get out of order if and only if $f^k(\zeta), f^k(\eta)$ get out of order for all $k \in \mathbb{Z}$.

Let $\mu_0, \mu_1, \ldots, \mu_{q-1} \in \mathcal{O}(f, z_0)$ be chosen and ordered so that

$$0 \le \pi_1(\mu_0) < \pi_1(\mu_1) < \cdots < \pi_1(\mu_{q-1}) < 1$$

and let $\nu_0, \ldots, \nu_{q-1} \in \mathcal{O}(f, w_0)$ be chosen and ordered so that

$$\pi_1(\mu_0) \le \pi_1(\nu_0) < \pi_1(\nu_1) < \cdots < \pi_1(\nu_{q-1}) < \pi_1(\mu_0 + (1, 0)).$$

(We may assume $\pi_1(\nu_i) \ne \pi(\nu_{i+1})$ by changing coordinates slightly if necessary.)

Now we specify two cases by the Pigeon Hole Principle:

Case 1. There exists i_0, $0 \leq i_0 < q$, such that for all $j = 0, 1, \ldots, q-1$, $\pi_1(\nu_j) \notin [\pi_1(\mu_{i_0}), \pi_1(\mu_{i_0+1}))$ (or $\pi_1(\nu_i) \notin [\pi_1(\mu_{q-1}), \pi_1(\mu_0)+1)$ if $i_0 = q-1$).

Case 2. For each i, $0 \leq i < q$, $\pi_1(\nu_i) \in [\pi_1(\mu_i), \pi_1(\mu_{i+1}))$ (and $\pi_1(\nu_{q-1}) \in [\pi_1(\mu_{q-1}), \pi_1(\mu_0)+1)$.)

If case 1 holds take $\zeta_0 = \mu_{i_0}$ and $\zeta_1 = \mu_{i_0+1}$; in case 2 take $\zeta_0 = \mu_0$ and $\zeta_1 = \mu_1$.

Next we note that since w_0 is not a Birkhoff periodic point, there must exist $z \in \mathcal{O}(f, z_0)$ such that z, w_0 get out of order. Let $\bar{z} \in \mathcal{O}(f, z_0)$ be such that $\bar{z}, w_0$ get out of order and if $z \in \mathcal{O}(f, z_0)$ and z, w_0 get out order then $z = \bar{z}$ or $\pi_1(z) < \pi_1(\bar{z})$. Similarly let $\underline{z} \in \mathcal{O}(f, z_0)$ be such that $\underline{z}, w_0$ get out of order and if $z \in \mathcal{O}(f, w_0)$ and z, w_0 get out of order then $z = \underline{z}$ or $\pi_1(\underline{z}) < \pi_1(z)$, (it is possible that $\bar{z} = \underline{z}$). Fix $r_1, r_2, s_1, s_2 \in \mathbb{Z}$ so that $f^{r_1}(\bar{z}) = \zeta_0 + (s_1, 0)$ and $f^{r_2}(\underline{z}) = \zeta_1 + (s_2, 0)$. Then it follows that $\zeta_0 + (s_1, 0), f^{r_1}(w_0)$ get out of order and $\zeta_1 + (s_2, 0), f^{r_2}(w_0)$ get out of order. Letting

$$\eta_0 = f^{r_1}(w_0) - (s_1, 0), \qquad \eta_1 = f^{r_2}(w_0) - (s_2, 0)$$

we have that ζ_0, η_0 get out of order and ζ_1, η_1 get out of order. Now, ζ_0, η_1 do not get out of order, since if they did then $\zeta_0 + (s_2, 0), f^{r_2}(w_0)$ would get out of order, so $f^{-r_2}(\zeta_0 + (s_2, 0)), w_0$ would get out of order. But z_0 is a Birkhoff periodic orbit, so $\pi_1(\zeta_0) < \pi_1(\zeta_1)$ implies

$$\pi_1(f^{-r_2}(\zeta_0 + (s_2, 0))) < \pi_1(f^{-r_2}(\zeta_1 + (s_2, 0))) = \pi_1(\underline{z})$$

and we have a contradiction of the choice of $\underline{z}$. Similarly, ζ_1, η_0 do not get out of order, i.e. for $i = 0, \ldots, q-1$, $\pi_1(f^i(\zeta_0)) < \pi_1(f^i(\eta_1))$ and $\pi_1(f^i(\zeta_1)) > \pi_1(f^i(\eta_0))$. If we are in case 2 then either

  (i) $\pi_1(\eta_0) < \pi_1(\zeta_0)$ and $\pi_1(\zeta_1) \leq \pi_1(\eta_1)$;
  (ii) $\pi_1(\eta_0) < \pi_1(\zeta_0)$ and $\pi_1(\zeta_0) \leq \pi_1(\eta_1) < \pi_1(\zeta_1)$; or
  (iii) $\pi_1(\zeta_0) \leq \pi_1(\eta_0) < \pi_1(\zeta_1)$ and $\pi_1(\zeta_1) < \pi_1(\eta_1)$;

while in case 1 condition (i) must hold.

Suppose $\pi_1(\eta_0) < \pi_1(\zeta_0)$ and for some i, $0 < i < q$, $\pi_1(f^i(\eta_0)) > \pi_1(f^i(\zeta_0))$. Fix the smallest $i > 0$ such that $\pi_1(\phi(\eta_0, (i-1)/q)) < \pi_1(\zeta_0)$ and $\pi_1(\phi(\eta_0, i/q)) > \pi_1(\zeta_0)$. By the twist condition we see that if $t \in [0, 1/q]$ and $\pi_1(\phi(\eta_0, i/q + t)) = \pi_1(\zeta_0)$ then

$$\pi_2(\phi(\eta_0, (i-1)/q + t)) > \pi_2(\zeta_0).$$

Fixing the smallest $j > i$ such that $\pi_1(\phi(\eta_0, (j-1))/q)) > \pi_1(\zeta_0)$ and $\pi_1(\phi(\eta_0, j/q)) \leq \pi_1(\zeta_0)$ we see similarly that if $t \in [0, 1/q]$ and $\pi_1(\phi(\eta_0, (j-1)/q + t) = \pi_1(\zeta_0)$ then

$$\pi_2(\phi(\eta_0, (j-1)/q + t)) < \pi_2(\zeta_0).$$

Repeating this argument until $i > q$ we see that $\phi(\eta_0, [0, 1])$ is not contractible in $A \sim \{\zeta_0\}$. Similarly, this statement holds if $\pi_1(\eta_0) \geq \pi_1(\zeta_0)$. Also, the same argument implies that $\phi(\eta_1, [0, 1])$ is not contractible in $A \sim \{\zeta_1\}$. Schematically the situation is as pictured in figure 5.

Claim. There exists a point $z_1 \in A$, $\pi_1(\zeta_0) < \pi_1(z_1) < \pi_1(\zeta_1)$, such that z_1 is a fixed point of g and the loop $\phi(z_1, [0, 1])$ is contractible in $A \sim \{\zeta_0, \zeta_1\}$.
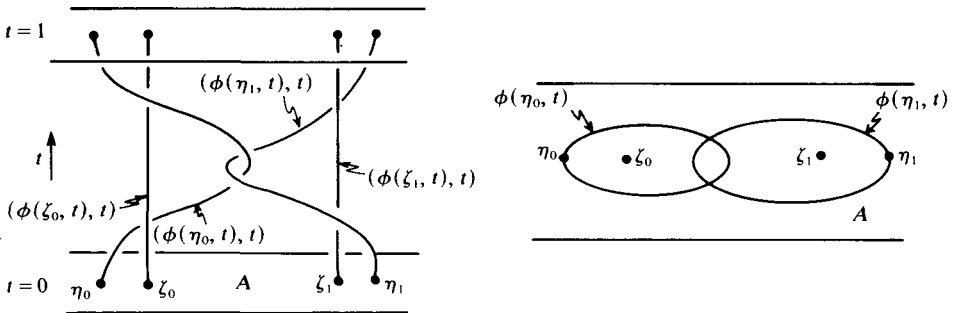
FIGURE 5

Remarks. (1) There are several alternatives for the proof of this claim. As indicated in figure 5, the orbits of ζ_0, ζ_1, η_0 and η_1 form a braid of a fairly simple form and hence it is not surprising that a proof of the claim can be obtained via a computation and an application of a theorem of Matsuoka [17] on braids and periodic orbits for time periodic o.d.e.'s on surfaces. The elementary proof given below gives a geometrical view of some simple cases of Matsuoka's theorem. (See also recent work of Asimov and Franks [1] relating 'removable' periodic orbits and Nielson theory.)

(2) The author would like to thank M. Handel for suggestions regarding this proof, particularly that the 'stability' statement of lemma 4 could be shown via the geometrical approach. This simplifies several steps in the next section.

Proof of the claim. First we show the proof for case 1; case 2 follows from similar, easier, arguments.

Case 1: $(\pi_1(\eta_0) < \pi_1(\zeta_0)$ and $\pi_1(\eta_1) \geq \pi_1(\zeta_1))$. The following notation will be useful. Let

$$B = \{z \in A: \pi_1(\zeta_0) \leq \pi_1(z) \leq \pi_1(\zeta_1)\}$$
$$b_i^+ = \{(\pi_1(\zeta_i), y) \in A: y > \pi_2(\zeta_i)\}, \qquad i = 0, 1$$
$$b_i^- = \{(\pi_1(\zeta_i), y) \in A: y < \pi_2(\zeta_i)\}, \qquad i = 0, 1 \quad \text{(see figure 6)}.$$
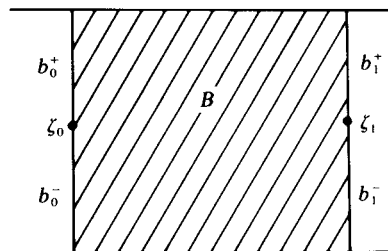

FIGURE 6

The following definition will also be useful:

For $t \in [0, 1]$ we say a point $\zeta \in B$ with $\phi(\zeta, t) \in \phi(B, t) \cap B$ is *null in B* at time t if the loop formed by $\phi(\zeta, [0, t])$ and the segment joining $\phi(\zeta, t)$ and ζ is contractible

in $A \sim \{\zeta_0, \zeta_1\}$. We let

$$S_t = \{\phi(\zeta, t) \in B : \zeta \in B \text{ and is null in } B \text{ at time } t\}.$$

Remark. Several other characterizations of S_t are available. For example, if we let $\hat{A}$ be the universal covering space of $A \sim \{\zeta_0, \zeta_1\}$, $\hat{\phi} : \hat{A} \times \mathbb{R} \to \hat{A}$ the lift of ϕ with $\hat{\phi}(\cdot, 0)$ the identity on $\hat{A}$ and $\hat{B} \subseteq \hat{A}$ a particular fixed lift of B then the points $\zeta \in S_t$ are precisely the projections of the points in $\hat{\phi}(\hat{B}, t) \cap \hat{B}$.

We note that if ζ is null in B then it does not necessarily follow that $\phi(\zeta, [0, t]) \subseteq B$. However, it does follow that $\phi(\zeta, i/q) \in B$ whenever $0 \le i/q \le t$ and that if $\pi_1(\phi(\zeta, t')) = \pi_1(\zeta_i)$, $i = 0$ or 1 for some $t' \in [0, t]$ then $\phi(\zeta, t') \in b_0^- \cup b_1^+$. This follows from the twist condition.

Similarly, the twist condition implies that for each $t \in [0, 1]$, we have

$$S_t \cap (b_0^+ \cup b_1^-) = \varnothing$$

and if $\sigma : [0, 1] \to B$ has $\sigma(0) \in b_0^-$, $\sigma(1) \in b_1^+$ then for each $t \in [0, 1]$, $\phi(\sigma([0, 1]), t) \cap S_t$ contains an arc with one end point in b_0^- and the other in b_1^+. Finally, we note that $\partial S_t \cap (\text{interior } B)$ is made up of arcs of the form $\phi(J, t)$ where J is an interval in b_0^+ or b_1^- and

$$\partial S_t \cap \partial B \subseteq b_0^- \cup b_1^+ \cup \{\zeta_0, \zeta_1\} \cup \{(x, j) \in B : j = 0, 1\}.$$

Next we use the existence of η_0 and η_1 and the properties of their orbits under ϕ to show that when $t = 1$, S_t must contain a connected component T_t such that

(α) $\partial T_t \cap \{\zeta_0, \zeta_1\} = \varnothing$;

(β) ∂T_t contains arcs of the form $\phi(J_0, t)$, $\phi(J_1, t)$ where $J_0 \subseteq b_0^+$, $J_1 \subseteq b_1^-$ and both $\phi(J_0, t)$ and $\phi(J_1, t)$ contain points of both $b_0^- \cup \{(x, 0) \in B\}$ and $b_1^+ \cup \{(x, 1) \in B\}$, (see figure 7).
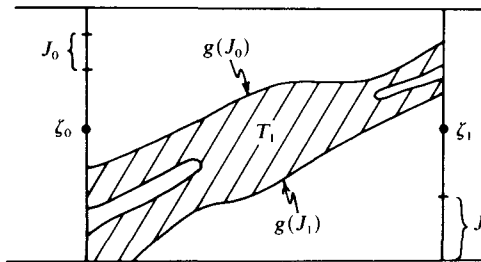

FIGURE 7

An easy index argument applied to the set $g^{-1}(T_1)$ will then give the required fixed point $z_1 \in A$.

Fix the smallest i_0, i_1 such that $\phi(\eta_0, i_0/q), \phi(\eta_1, i_1/q) \in B$ and the smallest $j_0 > i_0$, $j_1 > i_1$ such that $\phi(\eta_0, j_0/q), \phi(\eta_1, j_1/q) \notin B$. We assume first that $j_0 \le j_1$ and consider the following cases:

Case a. For all $i = i_0, i_0 + 1, \ldots, j_0$, $\pi_1(\phi(\eta_0, i/q)) < \pi_1(\phi(\eta_1, i/q))$.

Case b. For some $i, i_0 \le i < j_0$, $\pi_1(\phi(\eta_0, i/q)) > \pi_1(\phi(\eta_1, i/q))$, (i.e. either the orbits of η_0, η_1 do not link, or they do, respectively).

Case a. The loop formed by $\phi(\eta_0, [0, j_0/q])$ followed by the segment connecting $\phi(\eta_0, j_0/q)$ and η_0 is not contractible in $A \sim \{\zeta_0\}$. Hence, $S_{j_0/q}$ must contain a component $T_{j_0/q}$ satisfying (β) above and $\zeta_0 \notin T_{j_0/q}$ (see figure 8b).

Similarly, $\phi(\phi^{-1}(T_{j_0/q}, j_0/q), j_1/q)$ must contain a component $T_{j_1/q}$ satisfying (α) and (β) and $g(\phi^{-1}(T_{j_1/q}, j_1/q))$ will therefore contain the desired component of S_1 (see figure 8c).
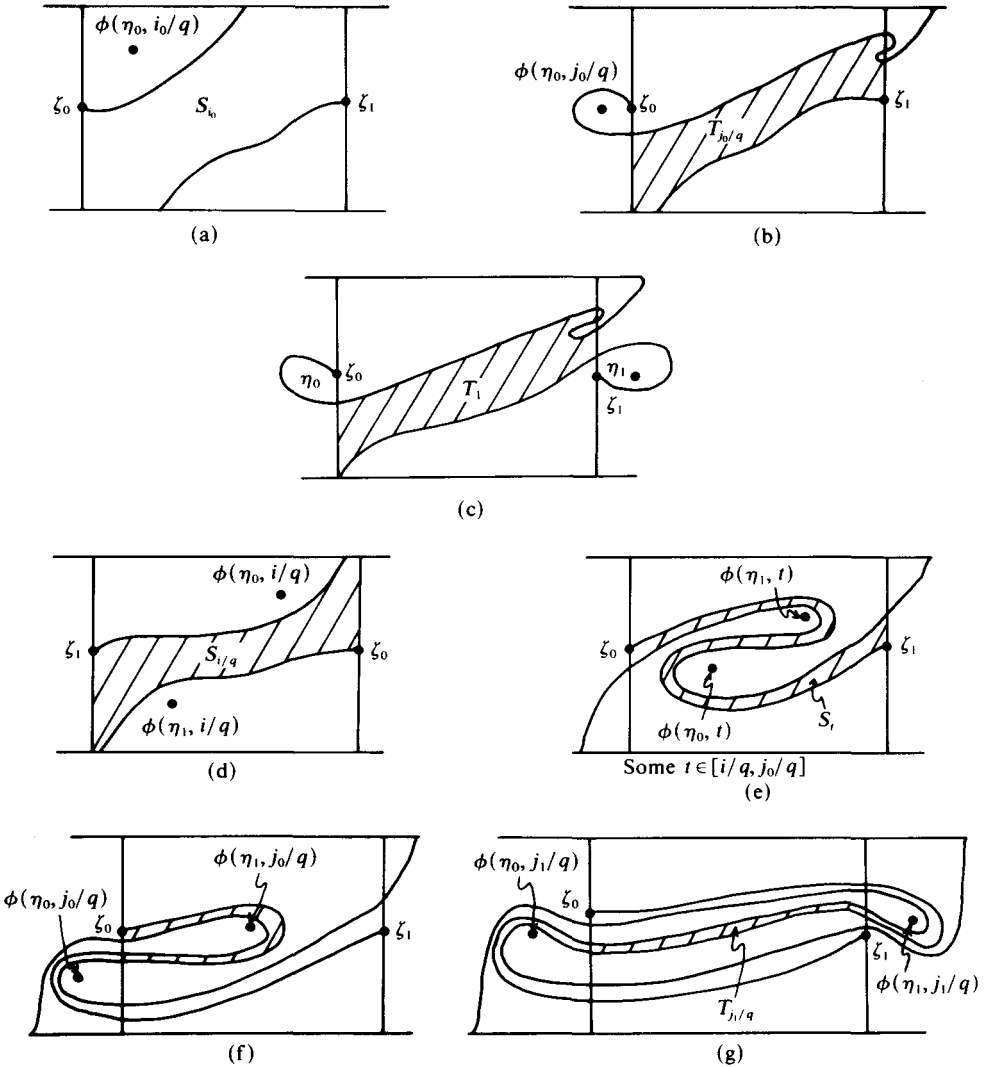

FIGURE 8

In fact, in this case we also have that

$$\pi_1(\phi(\eta_0, i/q)) < \pi_1(\phi(z, i/q)) < \pi_1(\phi(\eta_1, i/q))$$

for $z \in g^{-1}(T)$, $i = 0, \ldots, q$, so no point in $g^{-1}(T)$ gets out of order with η_0 or η_1. (This will be used in case 2 below.)

Case b. Since the loop $\phi(\eta_0, [0, j_0/q])$ followed by the segment connecting η_0 to $\phi(\eta_0, j_0/q)$ is not contractible in $A \sim \{\zeta_0\}$, and $\pi_1(\phi(\eta_0, i/q)) > \pi_1(\phi(\eta_1, i/q))$ for some $i, 0 < i < j_0$, we see that $S_{j_0/q}$ either contains a component $T_{j_0/q}$ satisfying (α) and (β) above (in which case a component of $g(\phi^{-1}(T_{j_0/q}, j_0/q))$ is the required set) or $S_{j_0/q}$ contains a component $T_{j_0/q}$ disjoint from $\{\zeta_1\}$ with $\partial T_{j_0/q}$ containing arcs $\phi(J_0, j_0/q)$ and $\phi(J_1, j_0/q)$ where J_0, J_1 are intervals in b_0^+, b_1^- respectively, and $\phi(J_0, j_0/q)$ has both end points in b_0^- while $\phi(J_1, j_0/q)$ either has both end points in b_0^- or one end point in $b_0^- \sim \{\zeta_0\}$ and one in $b_1^+ \sim \{\zeta_1\}$ (see figure 8f). When both end points of these arcs are in b_0^- the arcs are not homotopic to b_0^- in $B \sim \phi(\eta_1, j_0/q)$ with end points restricted to b_0^-. But then $\phi(\phi^{-1}(T_{j_0/q}, j_0/q), j_1/q)$ must contain a connected component $T_{j_1/q}$ satisfying (α) and (β) and hence $g(\phi^{-1}(T_{j_1/q}, j_1/q))$ has a component which is the required set (see figure 8g). The proof when $j_1 < j_0$ is symmetric to the above proof.

Let $B_1 = g^{-1}(T_1)$. Then by properties $(\alpha), (\beta)$ of T_1 we see that if we let $z \in A$ move around ∂B_1 and compute the total change in the angle between the vector $z - g(z)$ and the x-axis, the result will be $\pm 2\pi$ (depending on orientation) (see figure 9). Hence g must have a fixed point $z_1 \in B_1 \cap T_1$. Moreover, since $T_1 \subseteq S_1$ we see that $\phi(z_1, [0, 1])$ must be contractible in $A \sim \{\zeta_0, \zeta_1\}$ and since $\zeta_0, \zeta_1 \notin T_1, z_1 \notin \{\zeta_0, \zeta_1\}$.
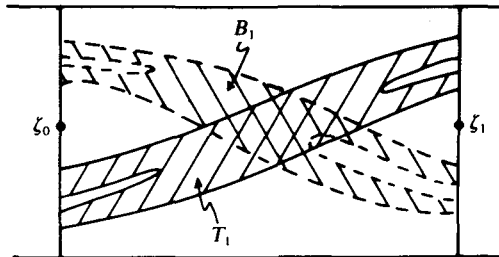

FIGURE 9

Since z_1 is a fixed point of g, it must be a p/q-periodic point of f. Since $\mathcal{O}(f, z_0)$ contains no points in $B \sim \{\zeta_0, \zeta_1\}$ we see that $z_1 \notin \mathcal{O}(f, z_0)$. Finally, since $\phi(z_1, [0, 1])$ is contractible in $A \sim \{\zeta_0, \zeta_1\}$ it follows that

$$\pi_1(f^i(\zeta_0)) < \pi_1(f^i(z_1)) < \pi_1(f^i(\zeta_1))$$

for $i = 0, \ldots, q$ and hence z_1 must be a p/q-Birkhoff periodic point of f. In case a, as noted above, we have that $\pi_1(f^i(\eta_0)) < \pi_1(f^i(z_1)) < \pi_1(f^i(\eta_1))$ as well. (This will be used in case 2 below.)

To obtain the stability portion of the lemma, we note that if $\tilde{f}$ is sufficiently close to f and $\tilde{g}(\cdot) \equiv \tilde{f}(\cdot) - (p, 0)$ then the index of $\tilde{g}: B_1 \to A$ will still be non-zero and hence $\tilde{g}$ will have a fixed point $\tilde{z}_1 \in \tilde{g}(B_1) \cap B_1$. Since $T_1 \cap B_1$ is contained in the interior of B (relative to A) this will also hold for $\tilde{g}(B_1) \cap B_1$ and hence

$$\pi_1(\tilde{f}^i(\zeta_0)) < \pi_1(\tilde{f}^i(\tilde{z}_1)) < \pi_1(\tilde{f}^i(\zeta_0))$$

for $i = 0, \ldots, q$ when $\tilde{f}$ is sufficiently close to f. It then follows as above (even though

ζ_0, ζ_1 are not necessarily near periodic points of $\tilde{f}$) that $\tilde{z}_1$ is a p/q-Birkhoff periodic point of $\tilde{f}$ and the proof of case 1 is complete.

Case 2. In this case we know $\pi_1(f^i(\eta_0)) < \pi_1(f^i(\eta_1))$ for $i = 0, \ldots, q$. Hence when $\pi_1(\zeta_0) \leq \pi_1(\eta_0) < \pi_1(\zeta_1)$ we may apply lemma 3' to obtain $\tilde{\phi} : A \times \mathbb{R} \to A$ associated with g which has $\tilde{\phi}(\eta_0, t) = \eta_0$, $\tilde{\phi}(\zeta_1, t) = \zeta_1$ for all t. Then $\tilde{\phi}(\zeta_0, [0, 1])$ is not contractible in $A \sim \{\eta_0\}$, but $\tilde{\phi}(\zeta_0, [0, 1])$ and $\tilde{\phi}(\eta_1, [0, 1])$ do not link. Hence we may apply case 1a above taking $B = \{z \in A : \pi_1(\eta_0) \leq \pi_1(z) \leq \pi_1(\zeta_1)\}$ and obtaining a fixed point z_1 of g. From case 1a we know that $\tilde{\phi}(z_1, [0, 1])$ links with none of the $\tilde{\phi}$ orbits of ζ_0, ζ_1, η_0 or η_1. The rest of the proof proceeds as before.

   The case when $\pi_1(\zeta_0) < \pi_1(\eta_1) \leq \pi_1(\zeta_1)$ is, of course symmetric to the above and the proofs of the claim and the lemma are complete.   □

5. *Proof of theorem* 1

Let $f : A \to A$ be a twist map, p, q relatively prime integers and $w_0 \in A$ a p/q-periodic point of f. We may assume w_0 is not a Birkhoff periodic point since if it were we would be done.

   Next we note that since f has a p/q-periodic point we have that $p/q \in [\rho_0(f), \rho_1(f)]$. We may assume $p/q \in (\rho_0(f), \rho_1(f))$ since if $\rho_i(f) = p/q$, $i = 0$ or 1 then $f|_{\{(x,i):x\in\mathbb{R}\}}$ is a lift of a circle diffeomorphism with rotation number p/q and the existence of the p/q-Birkhoff periodic point follows from the usual arguments for circle maps (see [12]). Moreover, we may assume that for each $\alpha \in \mathbb{R}$ there exists $y_\alpha \in (0, 1)$ such that $\pi_1(f(\alpha, y_\alpha)) = \alpha + p/q$. If this is not the case for the given map f then we may extend f to a map $f_1 : A_1 \to A_1$ where $A_1 = \{(x, y) \in \mathbb{R}^2 : -\frac{1}{2} \leq y \leq \frac{3}{2}\}$, f_1 is a twist map satisfying the above condition and f and f_1 agree on A. Since $\rho_0(f) < p/q < \rho_1(f)$, if we can find a p/q-Birkhoff periodic point for f_1 on A_1 then in fact it must have orbit in A (see figure 10).
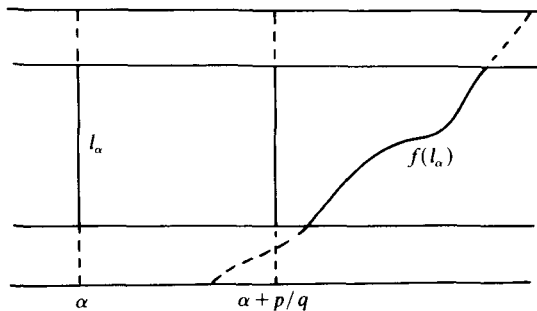

FIGURE 10

Claim 1. There exists a C^2 homotopy $H : A \times [0, 1] \to A$ satisfying
   (a) $H(\cdot, 0) = f(\cdot)$;
   (b) for all $s \in [0, 1]$, $H(\cdot, s)$ is a twist map;
   (c) for all $s \in [0, 1]$, w_0 is a p/q-periodic point of $H(\cdot, s)$;
   (d) $H(\cdot, 1)$ has a p/q-Birkhoff periodic point.

Proof of claim 1. Fix $z_0 \in A$ so that z_0 is in the interior of A,

$$\{\pi_1(z_0) + jp/q, j = 0, \ldots, q-1\} \cap (\{\pi_1(w): w \in \mathcal{O}(f, w_0)\} \cup \mathbb{Z}) = \varnothing$$

and $\pi_1(f(z_0)) = \pi_1(z_0) + p/q$. Fix $y_i \in [0, 1]$ so that $\pi_1(f(\pi_1(z_0) + ip/q, y_i)) = \pi_1(z_0) + (i+1)p/q$ for $i = 0, \ldots, q-1$ (so $y_0 = \pi_2(z_0)$). For any $\varepsilon > 0$ we may define $h_{i,s}: A \to A$, one parameter families of diffeomorphisms, $i = 1, \ldots, q$, $s \in [0, 1]$ so that

(1) for all $s \in [0, 1]$, $i = 1, \ldots, q$, $h_{i,s}(x+1, y) = h_{i,s}(x, y) + (1, 0)$ for all $(x, y) \in A$;

(2) for all $s \in [0, 1]$, $i = 1, \ldots, q$, support$(h_{i,s} - \text{identity}) \cap \{(x, y) \in A: 0 \le x < 1\}$ is contained in $\{(x, y) \in A: |x - (\pi_1(z_0) + ip/q - [\pi_1(z_0) + ip/q])| < \varepsilon\}$ (where $[\cdot]$ denotes the greatest integer function) and $h_{i,\cdot}(\cdot)$ is C^∞ on $A \times [0, 1]$,

(3) for all $s \in [0, 1]$, $i = 1, \ldots, q$, and all $(x, y) \in A$, $\pi_1(h_{i,s}(x, y)) = x$,

(4) $i = 1, \ldots, q$, $h_{i,0} = $ identity and

$$\pi_2(h_{i,1}(\pi_1(z_0) + ip/q - [\pi_1(z_0) + ip/q], \pi_2(f(\pi_1(z_0) + (i-1)p/q, y_{i-1}))))$$
$$= \begin{cases} y_i & \text{if } i = 1, \ldots, q-1, \\ y_0 & \text{if } i = q. \end{cases}$$

Choose ε so small that

$$\left(\bigcup_{i=1}^{q} \bigcup_{s \in [0,1]} \text{support}(h_{i,s} - \text{identity}) \right) \cap (\mathcal{O}(f, w_0) \cup \{(j, y) \in A: j \in \mathbb{Z}\}) = \varnothing$$

and define

$$H: A \times [0, 1] \to A$$
$$H(z, s) = h_{q,s} \circ f \circ h_{q-1,s} \circ h_{q-2,s} \circ \cdots \circ h_{1,s}(z).$$

Then H has properties (a)–(d) above. In particular, for all $s \in [0, 1]$, w_0 is a p/q periodic point of $H(\cdot, s)$ since f and $H(\cdot, s)$ agree on a neighbourhood of $\mathcal{O}(f, w_0)$ and $H(\cdot, s)$ is a twist map since the composition of a twist map with a map preserving the $x = $ constant foliation is a twist map. The map $H(\cdot, 1)$ has z_0 as a p/q-periodic point by condition (4) above and z_0 is a p/q-Birkhoff periodic point since any $\zeta \in \mathcal{O}(H(\cdot, 1), z_0)$ satisfies $\pi_1(H(\zeta, 1)) = \pi_1(\zeta) + p/q$ again by condition (4). Hence, H is the required homotopy and the proof of the claim is complete. $\qquad \square$

Let $\Xi = \{s \in [0, 1]: \text{for all } s_1 \ge s, H(\cdot, s_1) \text{ has a } p/q\text{-Birkhoff periodic point}\}$. Since $1 \in \Xi \ne \varnothing$, if we can show that the point $s_1 = \inf\{s \in [0, 1]: s \in \Xi\}$ is in the interior of Ξ i.e. that Ξ is open and closed, then we must have $\Xi = [0, 1]$. Hence $H(\cdot, 0) = f(\cdot)$ would have a p/q-Birkhoff periodic point and the proof would be complete.

Claim 2. Ξ is closed.

Proof of claim 2. This follows immediately from lemma 1. $\qquad \square$

Claim 3. $\Xi = [0, 1]$.

Proof of claim 3. Suppose $\Xi \ne [0, 1]$. Then $s_1 = \inf\{s \in [0, 1]: s \in \Xi\} > 0$. By claim 2 we have that $s_1 \in \Xi$ and hence that $H(\cdot, s_1)$ has a p/q-Birkhoff periodic point $\zeta_0 \in A$. By construction, w_0 is a p/q-periodic point of $H(\cdot, s_1)$ which is not a Birkhoff periodic point. Hence, we may apply lemma 4 to show that $H(\cdot, s_1)$ has another p/q-Birkhoff periodic point $z_1 \in A$ with $z_1 \notin \mathcal{O}(H(\cdot, s_1), z_1)$ and, more importantly, for s sufficiently close to s_1, the stability statement in lemma 4 implies $H(\cdot, s)$ also

has a p/q-Birkhoff periodic point. Hence s_1 is in the interior of Ξ contradicting the definition of s_1. This contradiction implies that we must have $s_1 = 0$, i.e. $\Xi = [0, 1]$, and the proof of the claim is complete.   ☐

As noted above, $0 \in \Xi$ implies that $H(\cdot, 0) = f(\cdot)$ has a p/q-Birkhoff periodic point and the proof of theorem 1 is complete.   ☐

6. Some related theorems
In this section we show

THEOREM 4. *Suppose $f: A \to A$ is a twist map and p, q are relatively prime integers. If every r/s-periodic point of f satisfies $s = q$ or $s > 3q/2$ then every p/q-periodic point of f is a p/q-Birkhoff periodic point.*

Remark. The theorem states that a periodic orbit of a twist map, with period significantly smaller than all larger periods, will be a Birkhoff periodic orbit. Since the possible periods are always contained between the rotation numbers of the map restricted to the boundaries, we may give conditions under which the hypotheses hold as follows:

Let $\mathscr{F}_n$ denote the Farey series of order n, i.e. $\mathscr{F}_n$ is a series of irreducible fractions between zero and one in ascending order with denominator less than or equal to n. So

$$\mathscr{F}_1 = \{\tfrac{0}{1}, \tfrac{1}{1}\}, \quad \mathscr{F}_2 = \{\tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{1}\}, \quad \mathscr{F}_3 = \{\tfrac{0}{1}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{1}{1}\}, \cdots.$$

(see [11]).

COROLLARY. *Suppose $f: A \to A$ is a twist map, p, q are relatively prime integers and $\rho_0(f)$, p/q, $\rho_1(f)$ are consecutive elements in $\mathscr{F}_n$ for some n. If $\rho_0(f) = r_0/s_0$, $\rho_1(f) = r_1/s_1$ in lowest form and $s_0 > 3q/2$, $s_1 > 3q/2$ then all p/q-periodic points of f (if any) are p/q-Birkhoff periodic points.*

Proof of the corollary. Suppose $r/s \in \mathscr{F}_{n+i} \sim \mathscr{F}_{n+i-1}$ for $i > 0$ and $r_0/s_0 < r/s < p/q$. Then, by theorems 29–31 of [11], $r/s = (a_1 + a_2)/(b_1 + b_2)$ where a_1/b_1, a_2/b_2 are consecutive elements of $\mathscr{F}_{n+i-1}$ and $a_1 + a_2$, $b_1 + b_2$ are relatively prime. But then a_1/b_1, $a_2/b_2 \in [r_0/s_0, p/q]$ and hence $a_1/b_1 \in \mathscr{F}_n$ iff $a_1/b_1 = r_0/s_0$ and $a_2/b_2 \in \mathscr{F}_n$ iff $a_2/b_2 = p/q$. Hence $b_1 \geq q$ and $b_2 \geq q$ so $s \geq 2q$. Similarly, if $p/q < r/s < r_1/s_1$ then $s \geq 2q$ so f, p, q satisfy the hypotheses of theorem 4 and the proof of the corollary is complete.   ☐

Remark. Essentially the theorem and corollary above say that if $f: A \to A$ is a twist map and f is not 'twisting' very much, then the periodic orbits with smallest period must be Birkhoff periodic orbits. We know of no examples implying that the conditions of theorem 4 are the best possible, however if f is area-preserving then the theorem can be improved significantly.

THEOREM 5. *If $f: A \to A$ is an area-preserving twist map, p, q are relatively prime integers and if every r/s-periodic point of f has $s \geq q$ then every p/q-periodic point of f is a p/q-Birkhoff periodic point.*

The proof of theorem 4 follows from ideas similar, but much easier technically, to those in theorem 1. The proof of theorem 5 is vaguely related to the 'graph-theoretical' approach to Šarkovskiĭ's theorem in [6].

Proof of theorem 4. Fix $f: A \to A$ and p, q as in the theorem. Suppose $z_0 \in A$ is a p/q-periodic point, but z_0 is not a p/q-Birkhoff periodic point of f. Then we fix $\zeta_1, \zeta_2 \in \mathcal{O}(f, z_0)$ so that $\pi_1(\zeta_1) < \pi_1(\zeta_2)$, $\pi_1(f(\zeta_1)) > \pi_1(f(\zeta_2))$ and $\pi_1(f^j(\zeta_1)) < \pi_1(f^j(\zeta_2))$ for some j, $1 < j \le q/2 + 1$. Then there exists an integer r so that $f^k(\zeta_2) = \zeta_1 + (r, 0)$ for some k, $j \le k \le 3q/2$, $k \ne q$.

Now fix $N > 0$ so large that $\pi_1(f^i(\pi_1(\zeta_2), 1)) < \pi_1(f^i(\pi_1(\zeta_1) + N, 0))$ for $i = 0, \ldots, 2q$ and let $B = \{(x, y): \pi_1(\zeta_1) \le x \le \pi_1(\zeta_1) + N\}$. Let $g: A \to A$ be defined for all $z \in A$ by, $g(z) = f^k(z) - (r, 0)$. Then $g(\zeta_2) = \zeta_1$. Let

$$J_1 = \{(\pi_1(\zeta_1), y): \text{for all } y_1 \ge y, g(\pi_1(\zeta_1), y_1) \in B\},$$

and

$$J_2 = \{(\pi_1(\zeta_1) + N, y): \text{for all } y_1 \le y, g(\pi_1(\zeta_1) + N, y_1) \in B\}.$$

Then the component T of $g(B) \cap B$ with boundary containing $g(J_1) \cup g(J_2)$ must have a fixed point. This is easily seen by computing the change of the argument of the vector $z - g(z)$ as z moves around the boundary of $g^{-1}(T)$, this change is $\pm 2\pi$ depending on orientation (see figure 11).
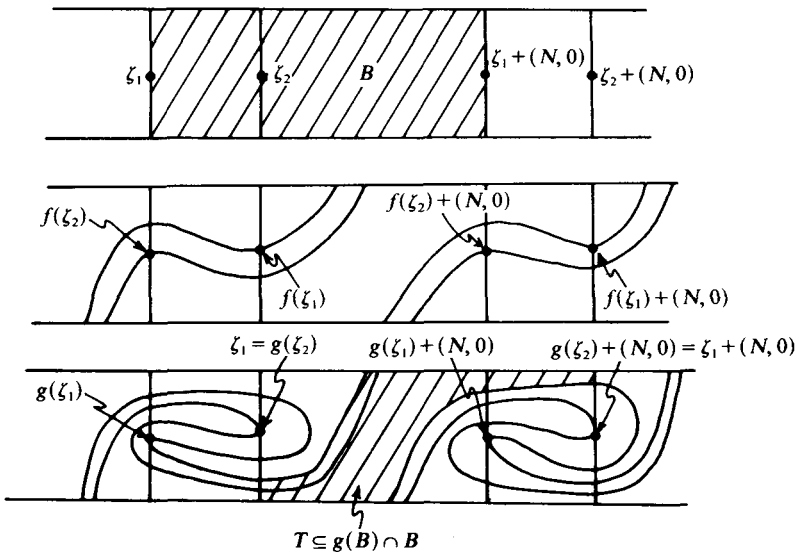

FIGURE 11

But this fixed point of g corresponds to an r/k-periodic point of f. Hence, if f has no r/s_1-periodic points with $s_1 \ne q$, $s_1 \le 3q/2$ then every p/q-periodic point of f must be a p/q-Birkhoff periodic point and the proof is complete. $\square$

Proof of theorem 5. Fix $f: A \to A$ an area preserving twist map, p, q relatively prime integers and z_0 a p/q-periodic point of f. Let

$$\{\zeta_0, \ldots, \zeta_{q-1}\} = \mathcal{O}(f, z_0) \cap \{z \in A: \pi_1(z) \in [0, 1)\}$$

and number so that $\pi_1(\zeta_0) \leq \pi_1(\zeta_1) \leq \cdots \leq \pi_1(\zeta_{q-1})$. We may assume, by changing variables slightly if necessary, that $\pi_1(\zeta_i) \neq \pi_1(\zeta_{i+1})$ for $i = 0, \ldots, q-1$. We define a directed graph with $\zeta_0, \ldots, \zeta_{q-1}$ as nodes as follows:

There is an edge from ζ_i to ζ_j if and only if

(a) $\pi_1(f(\pi_1(\zeta_i), 0) \in (\pi_1(\zeta_{j-1}) + s, \pi_1(\zeta_j) + s]$ for some $s \in \mathbb{Z}$; *or*

(b) $\pi_1(f(\pi_1(\zeta_i), 1) \in [\pi_1(\zeta_j) + s, \pi_1(\zeta_{j+1}) + s)$ for some $s \in \mathbb{Z}$.

Hence each ζ_i has at least one edge exiting it, moreover if z_0 is not a p/q-Birkhoff periodic point then some ζ_i has two edges exiting from it since for some $\zeta_i, \zeta_j, \pi_1(\zeta_i) < \pi_1(\zeta_j)$ but $\pi_1(f(\zeta_i)) > \pi_1(f(\zeta_j))$ (see figure 12), so

$$\pi_1(f(\pi_1(\zeta_i), 0)) < \pi_1(f(\zeta_j)) < \pi_1(f(\zeta_i)) < \pi_1(f(\pi_1(\zeta_i), 1)).$$
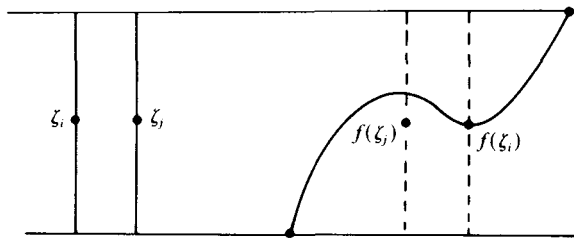

FIGURE 12

In this case the graph formed has loops of length less than q. But this implies that there exist rationals $r/s \in [\rho_0(f), \rho_1(f)]$ with $s < q$. By Birkhoff's theorem (theorem 2) (as noted by Birkhoff, $f: A \to A$ need only preserve a measure with non-zero density for theorem 2 to hold, see [4], so the change of variables above is no problem) we see that f must have an r/s-periodic point. Hence, if $f: A \to A$ has no r/s-periodic points with $s < q$ then every p/q-periodic point of f is a p/q-Birkhoff periodic point. □

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