6
Gauge fields on a lattice

The modern formulation of non-Abelian lattice gauge theories is due to Wilson [Wil74]. Independently, gauge theories were discussed on a lattice by Wegner [Weg71] as a gauge-invariant extension of the Ising model and in an unpublished work by A. Polyakov in 1974 which deals mostly with Abelian theories.

Placing gauge fields on a lattice provides, first, a nonperturbative regularization of ultraviolet divergences. Secondly, the lattice formulation of QCD possesses some nonperturbative terms in addition to perturbation theory. A result of this is that one has a nontrivial definition of QCD beyond perturbation theory which guarantees confinement of quarks.

The lattice formulation of gauge theories deals with phase-factor-like quantities, which are elements of the gauge group, and are natural variables for quantum gauge theories.

The gauge group on the lattice is therefore compact, offering the possibility of nonperturbative quantization of gauge theories without fixing the gauge. The lattice quantization of gauge theories is performed in such a way as to preserve the compactness of the gauge group.

The continuum limit of lattice gauge theories is reproduced when the lattice spacing is many times smaller than the characteristic scale. This is achieved when the non-Abelian coupling constant tends to zero as it follows from the renormalization-group equation.

In this chapter we consider the Euclidean formulation of lattice gauge theories. First, we introduce the lattice terminology and discuss the action of lattice gauge theory at the classical level. Then, we quantize gauge fields on the lattice using the path-integral method, where the integration is over the invariant group measure. We explain Wilson's criterion of confinement and demonstrate it using calculations in the strong-coupling limit. Finally, we discuss how to pass to the continuum limit of lattice gauge theories.

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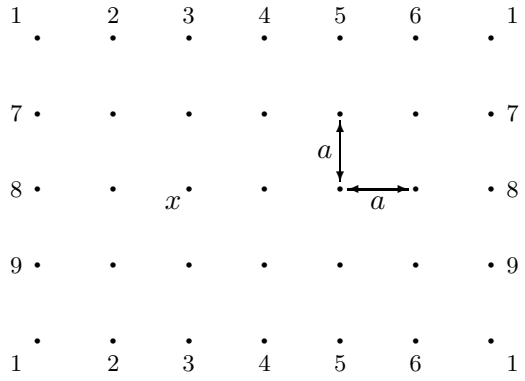

Fig. 6.1. Two-dimensional lattice with periodic boundary conditions. The sites labeled by the same numbers are identified. The lattice spacing equals a, while the spatial size of the lattice corresponds to $L_1 = 6$ and $L_2 = 4$.

6.1 Sites, links, plaquettes and all that

The first step in constructing a lattice gauge theory is to approximate the continuous space by a discrete set of points, i.e. a *lattice*. In the Euclidean formulation, the lattice is introduced along all four coordinates, while the time is left as continuous in the Hamiltonian approach.* We shall discuss only the Euclidean formulation of lattice gauge theories.

The lattice is defined as a set of points of the d-dimensional Euclidean space with the coordinates

$$x_\mu \;=\; n_\mu a\,, \tag{6.1}$$

where the components of the vector

$$n_\mu \;=\; (n_1, n_2, \ldots, n_d) \tag{6.2}$$

are integer numbers. The points (6.1) are called the lattice *sites*.

The dimensional constant a, which is equal to the distance between the neighboring sites, is called the *lattice spacing*. Dimensional quantities are usually measured in units of a, therefore setting $a = 1$.

A two-dimensional lattice is depicted in Fig. 6.1. A four-dimensional lattice for which the distances between sites are the same in all directions (as for the lattice in Fig. 6.1) is called a *hypercubic* lattice.

The next concept is the *link* of a lattice. A link is a line which connects two neighboring sites. A link is usually denoted by the letter l and is

* A Hamiltonian formulation of lattice gauge theories was developed by Kogut and Susskind [KS75].

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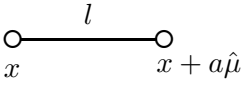

Fig. 6.2. A link of a lattice. The link connects the sites x and $x + a\hat{\mu}$.

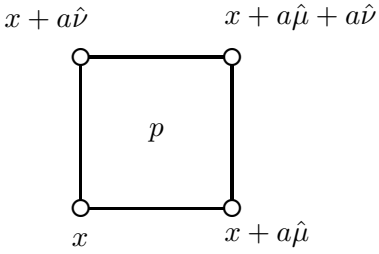

Fig. 6.3. A plaquette of a lattice. The plaquette boundary is made of four links.

characterized by the coordinate x of its starting point and its direction $\mu = 1, \ldots, d$:

$$l \;=\; \{x; \mu\}\,. \tag{6.3}$$

The link l connects sites with coordinates x and $x + a\hat{\mu}$, where $\hat{\mu}$ is a unit vector along the μ-direction, as shown in Fig. 6.2. The lengths of all links are equal to a for a hypercubic lattice.

The elementary square enclosed by four links is called the *plaquette*. A plaquette p is specified by the coordinate x of a site and by the two directions μ and ν along which it is constructed:

$$p \;=\; \{x; \mu, \nu\}\,. \tag{6.4}$$

A plaquette is depicted in Fig. 6.3. The set of four links which bound the plaquette p is denoted as ∂p.

If the spatial size of the lattice is infinite, then the number of dynamical degrees of freedom is also infinite (but enumerable). In order to limit the number of degrees of freedom, one deals with a lattice which has a finite size $L_1 \times L_2 \times \cdots \times L_d$ in all directions (see Fig. 6.1).

Usually, one imposes *periodic boundary conditions* to reduce finite-size effects that are due to the finite extent of the lattice. In other words, one identifies pairs of sites which lie on parallel bounding hyperplanes. Usually the sites with the coordinates $(0, n_2, \ldots, n_d)$ and $(L_1, n_2, \ldots, n_d)$ are identified and similarly along other axes.

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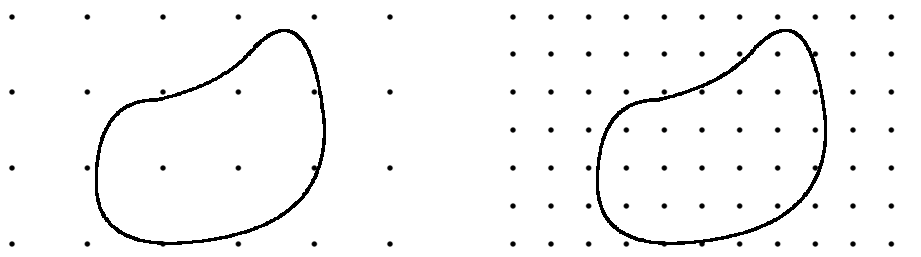

Fig. 6.4. Description of continuum field configurations using (a) "coarse" and (b) "fine" lattices. Lattice (a) can represent the given continuum field configuration very roughly, while lattice (b) has a spacing which is small enough.

Problem 6.1 Calculate the numbers of sites, links and plaquettes for a symmetric hypercubic lattice with periodic boundary conditions.

Solution Let us denote $L_1 = L_2 = \cdots = L_d = L$. Then

$$N_s \;=\; L^d, \qquad N_l \;=\; dL^d, \qquad N_p \;=\; \frac{d(d-1)}{2} L^d. \tag{6.5}$$

Problem 6.2 Label the lattice links by a natural number $l \in [1, N_l]$.

Solution One of the choices is as follows:

$$l \;=\; \mu + n_1 d + n_2 dL + \cdots + n_d dL^{d-1}, \tag{6.6}$$

where $n_\nu = x_\nu / a$ and μ is the direction of the link $\{x; \mu\}$.

6.2 Lattice formulation

The next step is to describe how matter fields and gauge fields are defined on a lattice.

A matter field, say a quark field, is attributed to the lattice sites. One can just think that a continuous field $\varphi(x)$ is approximated by its values at the lattice sites:

$$\varphi(x) \quad \Longrightarrow \quad \varphi_x \,. \tag{6.7}$$

It is clear that, in order for the lattice field φ_x to be a good approximation of a continuous field configuration $\varphi\,(x)$, the lattice spacing should be much smaller than the characteristic size of a given configuration. This is explained in Fig. 6.4.

The gauge field is attributed to the links of the lattice:

$$\mathcal{A}_\mu(x) \quad \Longrightarrow \quad U_\mu(x)\,. \tag{6.8}$$

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It looks natural since a link is characterized by a coordinate and a direction (see Eq. (6.3)) – the same as $\mathcal{A}_\mu(x)$. Sometimes the notation $U_{x,\mu}$ is used as an alternative for $U_\mu(x)$ to emphasize that it is attributed to links.

The link variable $U_\mu(x)$ can be viewed as

$$U_\mu(x) \;=\; \boldsymbol{P}\, \mathrm{e}^{\mathrm{i}\int_x^{x+a\hat{\mu}} \mathrm{d}z^\mu \mathcal{A}_\mu(z)}, \tag{6.9}$$

where the integral is along the link $\{x;\mu\}$. As $a \to 0$, this yields

$$U_\mu(x) \;\to\; \mathrm{e}^{\mathrm{i}a\mathcal{A}_\mu(x)} \tag{6.10}$$

so that $U_\mu(x)$ is expressed via the exponential of the μth component of the vector potential, say, at the center of the link to agree with Eq. (5.25).

Since the path-ordered integral in Eq. (6.9) depends on the orientation, the concept of the orientation of a given link arises. The same link, which connects the points x and $x + a\hat{\mu}$, can be written either as $\{x;\mu\}$ or as $\{x + a\hat{\mu}; -\mu\}$. The orientation is positive for $\mu > 0$ in the former case (i.e. the same as the direction of the coordinate axis) and is negative in the latter case.

We have assigned the link variable $U_\mu(x)$ to links with positive orientations. The U-matrices which are assigned to links with negative orientations are given by

$$U_{-\mu}(x + a\hat{\mu}) \;=\; U_\mu^\dagger(x)\,. \tag{6.11}$$

This is a one-link analog of Eq. (5.46).

It is clear from the relation (6.9) between the lattice and continuum gauge variables how one can construct lattice analogs of the continuum phase factors – one should construct the contours from the links of the lattice.

An important role in the lattice formulation is played by the phase factor for the simplest closed contour on the lattice: the (oriented) boundary of a plaquette, as is shown in Fig. 6.5. The plaquette variable is composed from the link variables (6.9) as

$$U(\partial p) \;=\; U_\nu^\dagger(x)\, U_\mu^\dagger(x + a\hat{\nu})\, U_\nu(x + a\hat{\mu})\, U_\mu(x)\,. \tag{6.12}$$

The link variable transforms under the gauge transformation, according to Eq. (5.52), as

$$U_\mu(x) \;\xrightarrow{\text{g.t.}}\; \Omega(x + a\hat{\mu})\, U_\mu(x)\, \Omega^\dagger(x)\,, \tag{6.13}$$

where the matrix $\Omega(x)$ is attributed to the lattice sites. This defines the lattice gauge transformation.

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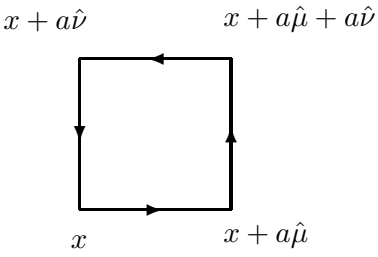

Fig. 6.5. A contour in the form of an oriented boundary of a plaquette.

The plaquette variable transforms under the lattice gauge transformation as

$$U(\partial p) \quad \xrightarrow{\text{g.t.}} \quad \Omega(x)\, U(\partial p)\, \Omega^{\dagger}(x)\,. \tag{6.14}$$

Therefore, its trace over the color indices is gauge invariant:

$$\operatorname{tr} U(\partial p) \quad \xrightarrow{\text{g.t.}} \quad \operatorname{tr} U(\partial p)\,. \tag{6.15}$$

The invariance of the trace under the lattice gauge transformation is used in constructing an action of a lattice gauge theory. The simplest (Wilson) action is

$$S_{\text{lat}}[U] \quad = \quad \sum_{p} \left[1 - \frac{1}{N} \operatorname{Re} \operatorname{tr} U(\partial p) \right]. \tag{6.16}$$

The summation is over all the elementary plaquettes of the lattice (i.e. over all x, μ, and ν), regardless of their orientations.

Since a reversal of the orientation of the plaquette boundary results, according to Eq. (5.46), in complex conjugation:

$$\operatorname{tr} U(\partial p) \quad \xrightarrow{\text{reor.}} \quad \operatorname{tr} U^{\dagger}(\partial p) \quad = \quad [\operatorname{tr} U(\partial p)]^{*}, \tag{6.17}$$

one can rewrite the action (6.16) in the equivalent form

$$S_{\text{lat}}[U] \quad = \quad \frac{1}{2} \sum_{\text{orient } p} \left[1 - \frac{1}{N} \operatorname{tr} U(\partial p) \right], \tag{6.18}$$

where the sum is also over the two possible orientations of the boundary of a given plaquette.

In the limit $a \to 0$, the lattice action (6.16) becomes (in $d = 4$) the action of a continuum gauge theory. In order to show this, let us first note that

$$U(\partial p) \quad \to \quad \exp\left[\mathrm{i} a^2 \mathcal{F}_{\mu\nu}(x) + \mathcal{O}\big(a^3\big) \right], \tag{6.19}$$

where $\mathcal{F}_{\mu\nu}(x)$ is defined using Eq. (5.14).

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In the Abelian theory, the expansion (6.19) is easily found from the Stokes theorem. The commutator of $\mathcal{A}_\mu(x)$ and $\mathcal{A}_\nu(x)$, which arises in the non-Abelian case, complements the field strength to the non-Abelian one, as is ensured by the gauge invariance. Equation (6.19) was, in fact, already derived in Problem 5.8 on p. 94.

The transition to the continuum limit is performed by virtue of

$$a^4 \sum_p \quad \stackrel{a\to 0}{\longrightarrow} \quad \frac{1}{2} \int \mathrm{d}^4 x \sum_{\mu,\nu} . \tag{6.20}$$

Expanding the exponential on the RHS of Eq. (6.19) in a, we obtain

$$S_{\mathrm{lat}} \quad \stackrel{a\to 0}{\longrightarrow} \quad \frac{1}{4N} \int \mathrm{d}^4 x \sum_{\mu,\nu} \mathrm{tr}\, \mathcal{F}^2_{\mu\nu}(x) \,, \tag{6.21}$$

which coincides modulo a factor with the action of the continuum gauge theory.

Problem 6.3 Derive the lattice version of the non-Abelian Maxwell equation (5.17).

Solution Let us perform the change of the link variable

$$U_\mu(x) \to U_\mu(x)\,[1 - \mathrm{i}\epsilon_\mu(x)] \,, \qquad U^\dagger_\mu(x) \to [1 + \mathrm{i}\epsilon_\mu(x)]\, U^\dagger_\mu(x) \,, \tag{6.22}$$

where $\epsilon_\mu(x)$ is an infinitesimal traceless Hermitian matrix.

A given link $\{x;\mu\}$ enters $4(d-1)$ plaquettes $p = \{x;\mu,\nu\}$ in the action (6.18). One-half of them have a boundary with a positive orientation and the other half with a negative one. The variation of the action (6.18) under the shift (6.22) is

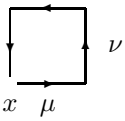
$$\delta S[U] \;=\; \frac{\mathrm{i}}{2N} \sum_{\nu\neq\pm\mu} \left[\mathrm{tr}\, U(\partial p)\epsilon_\mu(x) - \mathrm{tr}\,\epsilon_\mu(x) U^\dagger(\partial p)\right] . \tag{6.23}$$

Since $\epsilon_\mu(x)$ is arbitrary, we obtain

$$\sum_{\nu\neq\pm\mu} \left[U(\partial\{x;\mu,\nu\}) - U^\dagger(\partial\{x;\mu,\nu\})\right] \;=\; 0 \,, \tag{6.24}$$

or, graphically,

$$\sum_{\nu\neq\pm\mu} \left(\quad \right. \tag{6.25}$$



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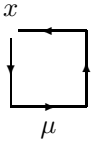

$$\left. \right) \;=\; 0 \,. \tag{6.25}$$

In the latter equation we have depicted only plaquettes with positive orientation, while those with negative orientation are recovered by the sum over ν

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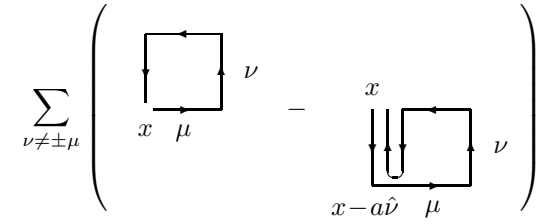
for $\nu < 0$. Equation (6.24) (or (6.25)) is the lattice analog of the non-Abelian Maxwell equation.

In order to show how this equation reproduces the continuum one (5.17) as $a \to 0$, let us rewrite the second term on the LHS of Eq. (6.25) using (6.11):

$$\sum_{\nu \neq \pm\mu} \left(\quad - \quad \right) = 0 \tag{6.26}$$



or, analytically,

$$\sum_{\nu \neq \pm\mu} \Big[U(\partial\{x;\mu,\nu\}) - U_\nu(x - a\hat{\nu})\, U(\partial\{x - a\hat{\nu};\mu,\nu\})\, U_\nu^\dagger(x - a\hat{\nu}) \Big] = 0\,. \tag{6.27}$$

It is now clear that the plaquette boundary in the second term on the LHS, which is the same as the first one but transported by one lattice spacing in the ν-direction, is associated with $\mathcal{F}_{\mu\nu}(x - a\hat{\nu})$. Using Eqs. (6.10) and (6.19), we recover the continuum Maxwell equation (5.17).

Remark on the naive continuum limit

The limit $a \to 0$, when Eqs. (6.10) and (6.19) hold reproducing the continuum action (6.21), is called the *naive continuum limit*. It is assumed in the naive continuum limit that $\mathcal{A}_\mu(x)$ is weakly fluctuating at neighboring lattice links. Fluctuations of the order of $1/a$ are not taken into account, since discontinuities of the vector potential in the continuum theory are usually associated with an infinite action.

Another subtlety with the naive continuum limit is that the next order in a terms of the expansion of the lattice action (6.16), say the term $\propto a^2 \operatorname{tr} \mathcal{F}^3$, are associated with nonrenormalizable interactions and the smallness of a^2 can be compensated, in principle, by quadratic divergences.

The actual continuum limit of lattice gauge theories is, in fact, very similar to the naive one modulo some finite renormalizations of the gauge coupling constant. The large fluctuations of $\mathcal{A}_\mu(x)$ of the order of $1/a$ become frozen when passing to the continuum limit. How one can pass to the continuum limit of lattice gauge theories is explained in Sect. 6.7.

Remark on ambiguities of the lattice action

The Wilson action (6.16) is the simplest one which reproduces the continuum action in the naive continuum limit. One can alternatively use the

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characters of $U(\partial p)$ in other representations of $SU(N)$, e.g. in the adjoint representation

$$\chi_{\rm adj}(U) \;=\; |\mathrm{tr}\, U|^2 - 1\,, \tag{6.28}$$

to construct the lattice action.

The adjoint-representation lattice action is given as

$$S_{\rm adj}[U] \;=\; \sum_p \left[1 - \frac{1}{N^2}\,|\mathrm{tr}\, U(\partial p)|^2\right]. \tag{6.29}$$

The naive continuum limit will be the same as for the Wilson action (6.16).

Moreover, one can define the lattice action as a mixture of the fundamental and adjoint representations [BC81, KM81]:

$$S_{\rm mix}[U] \;=\; \sum_p \left[1 - \frac{1}{N}\mathrm{Re}\,\mathrm{tr}\, U(\partial p)\right] + \frac{\beta_A}{2\beta}\sum_p \left[1 - \frac{1}{N^2}\,|\mathrm{tr}\, U(\partial p)|^2\right]. \tag{6.30}$$

The ratio β_A/β is a constant ~ 1 which does not affect the continuum limit. This action is called the *mixed action*.

The lattice action (6.29) for $N = 2$ is associated with the action of the $SO(3)$ lattice gauge theory. Since algebras of the $SU(2)$ and $SO(3)$ groups coincide, these two gauge theories coincide in the continuum and differ on the lattice.

One more possibility is to use the phase factor associated, say, with the boundary of two plaquettes having a common link, or the phase factors for more complicated closed contours of finite size on the lattice to construct the action. These actions will also reproduce, in the naive continuum limit, the action of the continuum gauge theory.

The independence of the continuum limit of lattice gauge theories on the choice of lattice actions in called the *universality*. We shall say more about this in Sect. 7.4 when discussing the renormalization group on the lattice.

6.3 The Haar measure

The partition function of a pure* lattice gauge theory is defined by

$$Z(\beta) \;=\; \int \prod_{x,\mu} \mathrm{d}U_\mu(x)\; \mathrm{e}^{-\beta S[U]}\,, \tag{6.31}$$

where the action is given by Eq. (6.16).

* Here "pure" means without matter fields.

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This is the analog of a partition function in statistical mechanics at an inverse temperature β given by*

$$\beta \;=\; \frac{N}{g^2}\,. \tag{6.32}$$

This formula results from comparing Eq. (6.21) with the gauge-field part of the continuum action (5.13).**

A subtle question is what is the measure $\mathrm{d}U_\mu(x)$ in Eq. (6.31). To preserve the gauge invariance at finite lattice spacing, the integration is over the *Haar measure* which is an invariant group measure. Invariance of the Haar measure under multiplication by an arbitrary group element from the left or from the right:

$$\mathrm{d}U \;=\; \mathrm{d}(\Omega U) \;=\; \mathrm{d}(U\Omega')\,, \tag{6.33}$$

guarantees the gauge invariance of the partition function (6.31).

This invariance of the Haar measure is crucial for the Wilson formulation of lattice gauge theories.

It is instructive to present an explicit expression for the Haar measure in the case of the $SU(2)$ gauge group. An element of $SU(2)$ can be parametrized using the unit four-vector a_μ ($a_\mu^2 = 1$) as

$$U \;=\; a_4\mathbb{I} + \mathrm{i}\vec{a}\vec{\sigma}\,, \tag{6.34}$$

where $\vec{\sigma}$ are the Pauli matrices. The Haar measure for $SU(2)$ then reads

$$\mathrm{d}U \;=\; \frac{1}{\pi^2}\prod_{\mu=1}^{4}\mathrm{d}a_\mu\,\delta^{(1)}\bigl(a_\mu^2-1\bigr)\,, \tag{6.35}$$

since $\det U = a_\mu^2$.

Problem 6.4 Rewrite the Haar measure on $SU(2)$ via a unit three-vector $\vec{n}$ ($\vec{n}^{\,2}=1$) and an angle φ ($\varphi\in[0,2\pi]$).

Solution An element of $SU(2)$ reads in this parametrization as

$$U \;=\; \mathrm{e}^{\mathrm{i}\varphi\vec{n}\vec{\sigma}/2} \;=\; \cos\frac{\varphi}{2} + \mathrm{i}\vec{n}\vec{\sigma}\sin\frac{\varphi}{2}\,. \tag{6.36}$$

The geometric meaning of this parametrization is simple: the element (6.36) is associated with a rotation through the angle φ around the $\vec{n}$-axis. The Haar measure for the $SU(2)$ group is then

$$\mathrm{d}U \;=\; \frac{\mathrm{d}^2\vec{n}}{4\pi}\frac{\mathrm{d}\varphi}{\pi}\sin^2\frac{\varphi}{2}\,. \tag{6.37}$$

This formula can be obtained from Eq. (6.35) by integrating over $|\vec{a}|$.

* The standard factor of 2 is missing because of the normalization (5.6).

** One has instead $\beta = N/g^2a^{4-d}$ on a d-dimensional lattice since the Yang–Mills coupling g is dimensional for $d\neq 4$.

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Problem 6.5 For the $U(N)$ group represent the Haar measure as a multiple integral over the matrix elements of U.

Solution Elements of a unitary matrix U are complex numbers. The Haar measure can be represented as

$$\int \mathrm{d}U \cdots \;=\; \int\limits_{-\infty}^{+\infty} \prod_{i,j} \mathrm{d}\,\mathrm{Re}\,U_{ij}\,\mathrm{d}\,\mathrm{Im}\,U_{ij}\,\delta^{(N^2)}\bigl(UU^\dagger - \mathbb{I}\bigr)\cdots . \tag{6.38}$$

The integral in this formula goes over unrestricted U_{ij} as if U were a general complex matrix while the delta-function restricts U to be unitary.

The partition function (6.31) characterizes vacuum effects in the quantum theory. Physical quantities are given by the averages of the same type as Eq. (2.6):

$$\langle F[U]\,\rangle \;=\; Z^{-1}(\beta)\int \prod_{x,\mu} \mathrm{d}U_\mu(x)\,\mathrm{e}^{-\beta S[U]}\,F[U]\,, \tag{6.39}$$

where $F[U]$ is a gauge-invariant functional of the link variable $U_\mu(x)$. The averages (6.39) become the corresponding expectation values in the continuum theory as $a \to 0$ and β is related to g^2 by Eq. (6.32).

Remark on the lattice quantization

On a lattice of finite size, the integral over the gauge group in Eq. (6.39) is finite since the integration is over a compact group manifold, in contrast to the continuum case, where the volume of the gauge group is infinite. Therefore, the expression (6.39) is a constructive method for calculating averages of gauge-invariant quantities, though the gauge is not fixed.

The gauge can be fixed on the lattice in the standard way by the Faddeev–Popov method. This procedure involves extracting a (finite) common factor, which equals the volume of the gauge group, from the numerator and denominator on the RHS of Eq. (6.39). Therefore, the averages of gauge-invariant quantities coincide for a fixed and unfixed gauge, while the average of a functional which is not gauge invariant vanishes when the gauge is not fixed.

The fixing of gauge is convenient (though not necessary) for calculations in a lattice perturbation theory. A Lorentz gauge cannot be fixed, however, outside perturbation theory because of Gribov copies [Gri78]. In contrast, the lattice path integral (6.39) with an unfixed gauge is a method of nonperturbative quantization.

A price for the compactness of the group manifold on the lattice is the presence of fluctuations $\mathcal{A}_\mu(x) \sim 1/a$ which do not occur in the continuum (say, the values of the vector potential A_μ and $A_\mu + 2\pi/ae$ are identified

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for the Abelian $U(1)$ group). However, these fluctuations become unimportant when passing to the continuum limit.

6.4 Wilson loops

As has already been mentioned in Sect. 6.2, lattice phase factors are associated with contours which are drawn on the lattice.

In order to write down an explicit representation of the phase factor on the lattice via the link variables, let us specify the (lattice) contour C by its initial point x and by the directions (some of which may be negative) of the links from which the contour is built:

$$C \;=\; \{x;\mu_1,\ldots,\mu_n\}\,. \tag{6.40}$$

Then the lattice phase factor $U(C)$ is given by

$$U(C) \;=\; U_{\mu_n}(x+a\hat{\mu}_1+\cdots+a\hat{\mu}_{n-1})\cdots U_{\mu_2}(x+a\hat{\mu}_1)\,U_{\mu_1}(x)\,. \tag{6.41}$$

For the links with a negative direction it is again convenient to use Eq. (6.11).

A closed contour has $\hat{\mu}_1+\cdots+\hat{\mu}_n=0$. The trace of the phase factor for a closed contour, which is gauge invariant, is called the *Wilson loop*.

The average of the Wilson loop is determined by the general formula (6.39) to be

$$\begin{aligned} W(C) \;&\equiv\; \left\langle \frac{1}{N}\mathrm{tr}\,U(C)\right\rangle \\ &=\; Z^{-1}(\beta)\int\prod_{x,\mu}\mathrm{d}U_\mu(x)\,\mathrm{e}^{-\beta S[U]}\,\frac{1}{N}\mathrm{tr}\,U(C)\,. \end{aligned} \tag{6.42}$$

This average is often called the Wilson loop average.

A very important role in lattice gauge theories is played by the averages of the Wilson loops associated with rectangular contours. Such a contour lying in the (x,t)-plane is depicted in Fig. 6.6.

The Wilson loop average is related for $\mathcal{T}\gg R$ to the energy of the interaction of the static (i.e. infinitely heavy) quarks, which are separated by a distance R, by the formula

$$W(R\times\mathcal{T}) \quad\overset{\mathcal{T}\gg R}{=}\quad \mathrm{e}^{-E_0(R)\cdot\mathcal{T}}\,. \tag{6.43}$$

Problem 6.6 Derive Eq. (6.43) by fixing the gauge $\mathcal{A}_4=0$.

Solution In the axial gauge $\mathcal{A}_4=0$, we have $U_4(x)=1$ so that only vertical segments of the rectangle in Fig. 6.6 contribute to $U(R\times\mathcal{T})$. Denoting

$$\Psi_{ij}(t) \;\equiv\; \left[\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_0^R \mathrm{d}z_1\,A_1(z_1,\ldots,t)}\right]_{ij}, \tag{6.44}$$

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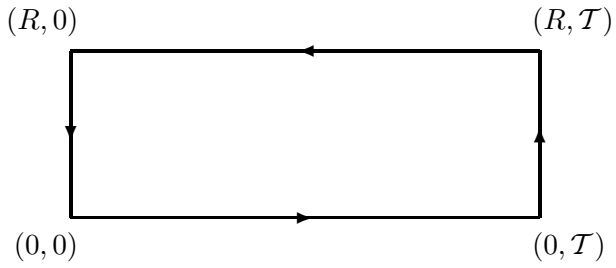

Fig. 6.6. Rectangular loop of size $R \times \mathcal{T}$.

we then have

$$W(R\times\mathcal{T}) \;=\; \left\langle \frac{1}{N}\mathrm{tr}\,\Psi(0)\,\Psi^{\dagger}(\mathcal{T})\right\rangle . \tag{6.45}$$

Inserting in Eq. (6.45) a sum over a complete set of intermediate states

$$\sum_n |n\rangle\langle n| \;=\; 1\,, \tag{6.46}$$

we obtain

$$\begin{aligned} W(R\times\mathcal{T}) &= \sum_n \frac{1}{N}\langle\Psi_{ij}(0)\mid n\rangle\langle n\mid \Psi^{\dagger}_{ji}(\mathcal{T})\rangle \\ &= \sum_n \frac{1}{N}\big|\langle\Psi_{ij}(0)\mid n\rangle\big|^2\,\mathrm{e}^{-E_n\mathcal{T}}, \end{aligned} \tag{6.47}$$

where E_n is the energy of the state $|n\rangle$. As $\mathcal{T}\to\infty$, only the ground state with the lowest energy survives in the sum over states and finally we find

$$W(R\times\mathcal{T}) \;\overset{\text{large }\mathcal{T}}{\longrightarrow}\; \mathrm{e}^{-E_0\mathcal{T}}, \tag{6.48}$$

which results in Eq. (6.43).

Note that nothing in this derivation relies on the lattice. Therefore, Eq. (6.43) holds for a rectangular loop in the continuum theory as well.

Equation (6.43) can also be understood as follows. Let us consider the Abelian case when the interaction is described by Coulomb's law. The contour integral can then be rewritten as the integral over the whole space

$$e\oint\limits_C \mathrm{d}z^{\mu}A_{\mu}(z) \;=\; \int \mathrm{d}^dx J^{\mu}(x)A_{\mu}(x)\,, \tag{6.49}$$

where

$$J^{\mu}(x) \;=\; e\oint\limits_C \mathrm{d}z^{\mu}\delta^{(d)}(x-z) \tag{6.50}$$

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is a four-vector current of a classical particle moving along the trajectory C which is described by the function $z_\mu(t)$.

It is clear that

$$-\ln W(C) \;=\; -\ln\left\langle \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x J^\mu(x)A_\mu(x)}\right\rangle \tag{6.51}$$

determines the change of action of the classical particle arising from the electromagnetic interaction in accordance with Eq. (6.43). How one may obtain Coulomb's law in this language is shown later in Problem 12.3.

A similar interpretation of Eq. (6.43) in the non-Abelian case is somewhat more complicated. For a heavy particle moving along some trajectory in space-time, color degrees of freedom are quantum and easily respond to changes of the gauge field $\mathcal{A}_\mu(x)$, which interacts with them. Let us suppose that a quark and an antiquark are created at the same space-time point in some color state. Then this state must be a singlet with respect to color (or *colorless*) since the average over the gauge field would vanish otherwise. When the quarks separate, their color changes from one point to another simultaneously with the change of color of the gauge field, in order for the system of the quarks plus the gauge field to remain colorless. Therefore, the averaging over the gauge field leads to an averaging over fluctuations of quark color degrees of freedom. $E_0(R)$ in Eq. (6.43) is associated with the interaction energy averaged over color in this way.

Problem 6.7 Derive a non-Abelian analog of Eq. (6.50).

Solution The proper non-Abelian extension of Eq. (6.50) is given by [Won70]

$$\mathcal{J}^a_\mu(x) \;=\; g\int\limits_0^\tau \mathrm{d}t\,\dot z_\mu(t)\,\delta^{(d)}(x-z(t))\,I^a(t)\,, \tag{6.52}$$

where $I^a(t)$, which describes the color state of a classical particle moving along the trajectory $z^\mu(t)$ in an external Yang–Mills field $A_\mu(z)$, is a solution of the equation

$$\dot I^a(t) + g f^{abc}\dot z^\mu(t)\,A^b_\mu(z(t))\,I^c(t) \;=\; 0\,. \tag{6.53}$$

It is convenient to use Grassmann variables again to describe color degrees of freedom as in Problem 5.3 on p. 90. Then [BCL77, BSS77]

$$\mathcal{J}^a_\mu(x) \;=\; \bar\psi(t)t^a\psi(t) \tag{6.54}$$

and $\psi(t)$ is a solution of

$$\dot\psi(t) - \mathrm{i}\,\dot z^\mu(t)\,\mathcal{A}_\mu(z(t))\,\psi(t) \;=\; 0. \tag{6.55}$$

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Remark on mass renormalization

By definition, $E_0(R)$ in Eq. (6.43) includes a renormalization of the mass of a heavy quark owing to the interaction with the gauge field and which is thus independent of R. To the first order in g^2, it is the same as in QED and is given by

$$\Delta E_{\text{mass}} \;=\; \frac{g^2}{4\pi a}\frac{N^2-1}{N} \tag{6.56}$$

as $a \to 0$. The calculation is presented later in Problem 12.2.

The potential energy of the interaction between the static quarks is therefore defined as the difference

$$E(R) \;=\; E_0(R) - \Delta E_{\text{mass}}\,. \tag{6.57}$$

If $g^2/4\pi a$ in ΔE_{mass} did not become infinite as $a \to 0$, the term resulting from the mass renormalization would not have to be subtracted, since it simply changes the reference level for the potential energy.

6.5 Strong-coupling expansion

We already mentioned in Sect. 6.3 that the path integral (6.39) can be calculated by the lattice perturbation theory in g^2. As was pointed out by Wilson [Wil74], there exists an alternative way of evaluating the same quantity on a lattice by an expansion in $1/g^2$ or in β since they are related by Eq. (6.32). This expansion is called the *strong-coupling* expansion. It is an analog of the high-temperature expansion in statistical mechanics since β is the analog of an inverse temperature.

In order to perform the strong-coupling expansion, we expand the exponential of the lattice action, say in Eq. (6.42), in β. Then the problem is to calculate the integrals over the unitary group of the form

$$I^{\,i_1\cdots i_m,k_1\cdots k_n}_{\,j_1\cdots j_m,l_1\cdots l_n} \;=\; \int \mathrm{d}U\, U^{i_1}_{j_1}\cdots U^{i_m}_{j_m} U^{\dagger k_1}_{\;\;l_1}\cdots U^{\dagger k_n}_{\;\;l_n}\,, \tag{6.58}$$

where the Haar measure (given for $SU(2)$ by Eq. (6.35)) is normalized as

$$\int \mathrm{d}U \;=\; 1\,. \tag{6.59}$$

It is clear from general arguments that the integral (6.58) is nonvanishing only if $n = m \pmod N$, i.e. only if $n = m + kN$, where k is integer.

For the simplest case $m = n = 1$, the answer can easily be found by using the unitarity of U and the orthogonality relation:

$$\int \mathrm{d}U\, U^i_j\, U^{\dagger}{}^k_{\;l} \;=\; \frac{1}{N}\delta^i_l\delta^k_j\,. \tag{6.60}$$

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Problem 6.8 Prove Eq. (6.60) for the $U(N)$ group.

Solution From the general arguments we obtain

$$\int \mathrm{d}U\, U^i_j\, U^{\dagger}{}^k_l \;=\; A\,\delta^i_l\delta^k_j + B\,\delta^i_j\delta^k_l\,. \tag{6.61}$$

Contracting by δ^l_i, using the unitarity of U, and Eq. (6.59), we have

$$AN + B \;=\; 1\,. \tag{6.62}$$

One more relation between A and B arises from the fact that the character in the adjoint representation is given by Eq. (6.28). Contracting Eq. (6.61) by δ^j_i and δ^l_k, and using the orthogonality of the characters which states

$$\int \mathrm{d}U \left(|\mathrm{tr}\, U|^2 - 1\right) \;=\; 0\,, \tag{6.63}$$

we find

$$AN + BN^2 \;=\; 1\,. \tag{6.64}$$

Therefore, $A = 1/N$ and $B = 0$ which proves Eq. (6.60).

The simplest Wilson loop average, which is nonvanishing in the strong-coupling expansion, is that for the loop which coincides with the boundary of a plaquette (see Fig. 6.5). It is called the plaquette average and is denoted by

$$W(\partial p) \;=\; \left\langle \frac{1}{N}\mathrm{tr}\, U(\partial p)\right\rangle. \tag{6.65}$$

In order to calculate the plaquette average to order β, it is sufficient to retain only the terms $\mathcal{O}(\beta)$ in the expansion of the exponentials in Eq. (6.42):

$$W(\partial p) \;=\; \frac{\displaystyle\int \prod_{x,\mu} \mathrm{d}U_\mu(x) \Big[1 + \beta \sum_{p'} \frac{1}{N}\mathrm{Re}\,\mathrm{tr}\, U(\partial p')\Big] \frac{1}{N}\mathrm{tr}\, U(\partial p)}{\displaystyle\int \prod_{x,\mu} \mathrm{d}U_\mu(x) \Big[1 + \beta \sum_{p'} \frac{1}{N}\mathrm{Re}\,\mathrm{tr}\, U(\partial p')\Big]} + \mathcal{O}\big(\beta^2\big)\,. \tag{6.66}$$

The group integration can then be performed by remembering that

$$\int \mathrm{d}U_\mu(x)\, [U_\mu(x)]^i_j\, [U^\dagger_\nu(y)]^k_l \;=\; \frac{1}{N}\delta_{xy}\,\delta_{\mu\nu}\,\delta^i_l\,\delta^k_j \tag{6.67}$$

at different links.

Using this property of the group integral in Eq. (6.66), we immediately see that the denominator is equal to 1 (each link is encountered no more than once), while the only nonvanishing contribution in the numerator

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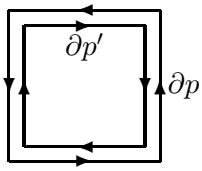

Fig. 6.7. Boundaries of the plaquettes p and p' with opposite orientations ∂p and $\partial p'$, respectively.

is from the plaquette p', which coincides with p but has the opposite orientation as is depicted in Fig. 6.7.

It is convenient to use the graphical notation* for Eq. (6.60) at each link of ∂p:

$$\begin{matrix} i \\ l \end{matrix} \rightleftharpoons \begin{matrix} j \\ k \end{matrix} \;=\; \frac{1}{N} \times \left(\begin{matrix} i \\ l \end{matrix} \right) \qquad \left(\begin{matrix} j \\ k \end{matrix} \right), \tag{6.68}$$

where the semicircles are associated with the Kronecker symbols:

$$\begin{matrix} i \\ l \end{matrix} \big) \;=\; \delta^i_l \,. \tag{6.69}$$

This notation is convenient since the lines which denote the Kronecker symbols in the latter equation can be associated with propagation of the color indices. Analogously a closed line represents the contracted Kronecker symbol, which is summed over the color indices,

$$\bigcirc \;=\; \delta^i_i \;=\; N \,. \tag{6.70}$$

Using the graphical representation (6.68) for each of the four links depicted in Fig. 6.7, we obtain

$$\int \prod_{x,\mu} \mathrm{d}U_\mu(x)\, \mathrm{tr}\, U(\partial p)\, \mathrm{tr}\, U^\dagger(\partial p') \;=\; \frac{1}{N^4} \times \begin{matrix} \bigcirc & \bigcirc \\ & \\ \bigcirc & \bigcirc \end{matrix} \;=\; 1 \,, \tag{6.71}$$

where the contracted Kronecker symbols are associated with the four sites of the plaquette.

* A calculation of more complicated group integrals (6.58) using the graphical notation is discussed in the lectures by Wilson [Wil75] and in Chapter 8 of the book by Creutz [Cre83]. An alternative method of calculating the group integrals using the character expansion is described in the review by Drouffe and Zuber [DZ83].

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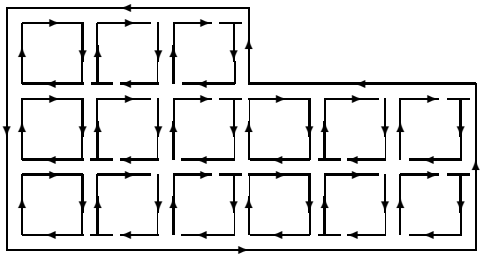

Fig. 6.8. Filling of a loop with elementary plaquettes.

The final answer for the plaquette average is

$$\left.\begin{aligned} W(\partial p) &= \frac{\beta}{2N^2} \qquad \text{for } SU(N) \text{ with } N \geq 3\,, \\ W(\partial p) &= \frac{\beta}{4} \qquad\quad \text{for } SU(2)\,. \end{aligned}\right\} \tag{6.72}$$

The result for $SU(2)$ differs by a factor of $1/2$ because $\operatorname{tr} U(\partial p)$ is real for $SU(2)$ so that the orientation of the plaquettes can be ignored.

The graphical representation (6.68) is useful for evaluating the leading order of the strong-coupling expansion for more complicated loops. According to Eq. (6.67), a nonvanishing result emerges only when plaquettes, arising from the expansion of the exponentials of Eq. (6.42) in β, completely cover a surface enclosed by the given loop C as depicted in Fig. 6.8. In this case each link is encountered twice (or never), once in the positive direction and once in the negative direction, so that all the group integrals are nonvanishing. The leading order in β corresponds to filling a *minimal surface*, whose area takes on the smallest possible value. This yields

$$W(C) \;=\; [W(\partial p)]^{A_{\min}(C)}\,, \tag{6.73}$$

where $W(\partial p)$ is given by Eq. (6.72) and $A_{\min}(C)$ is the area (in units of a^2) of the minimal surface.

For the rectangular loop, which is depicted in Fig. 6.6, the minimal surface is just a piece of the plane bounded by the rectangle. Therefore, we find

$$W(R \times \mathcal{T}) \;=\; [W(\partial p)]^{R\mathcal{T}} \tag{6.74}$$

to the leading order in β.

More complicated surfaces, which do not lie in the plane of the rectangle, will give a contribution to $W(C)$ of the order of β^{area}. They are suppressed at small β since their areas are larger than $A_{\min}$.

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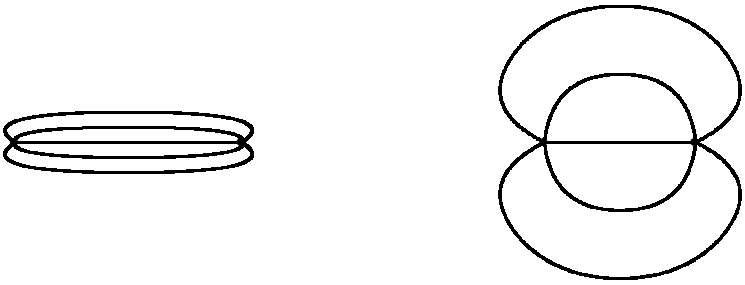

(a) (b)

Fig. 6.9. Lines of force between static quarks for (a) linear and (b) Coulomb interaction potentials. For the linear potential the lines of force are contracted into a tube, while they are distributed over the whole space for the Coulomb one.

6.6 Area law and confinement

The exponential dependence of the Wilson loop average on the area of the minimal surface (as in Eq. (6.73)) is called the *area law*. It is customarily assumed that if an area law holds for loops of large area in pure gluodynamics (i.e. in the pure $SU(3)$ gauge theory) then quarks are confined. In other words, there are no physical $|in\rangle$ or $\langle out|$ quark states. This is the essence of Wilson's *confinement criterion*. The argument is that physical amplitudes (for example, the polarization operator) do not have quark singularities when the Wilson criterion is satisfied. I refer the reader to the well-written original paper by Wilson [Wil74], where this point is clarified.

Another, somewhat oversimplified, justification for the Wilson criterion is based on the relationship (6.43) between the Wilson loop average and the potential energy of interaction between static quarks. When the area law

$$W(C) \quad \overset{\text{large } C}{\longrightarrow} \quad \mathrm{e}^{-KA_{\min}(C)} \tag{6.75}$$

holds for large loops, the potential energy is a linear function of the distance between the quarks:

$$E(R) \;=\; KR\,. \tag{6.76}$$

The coefficient K in these formulas is called the *string tension* because the gluon field between quarks contracts to a tube or string, whose energy is proportional to its length, as is depicted in Fig. 6.9a. The value of K

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is the energy of the string per unit length. This string is stretched with the distance between quarks and prevents them from moving apart to macroscopic distances.

Equation (6.74) gives

$$K \;=\; \frac{1}{a^2}\ln\frac{2N^2}{\beta} \;=\; \frac{1}{a^2}\ln\left(2Ng^2\right) \tag{6.77}$$

for the string tension to the leading order of the strong-coupling expansion. The next orders of the strong-coupling expansion result in corrections in β to this formula.

Therefore, confinement holds in the lattice gauge theory to any order of the strong-coupling expansion.

Remark on the perimeter law

For the Coulomb potential

$$E(R) \;=\; -\frac{g^2}{4\pi R}\frac{N^2-1}{N}\,, \tag{6.78}$$

the gauge field between quarks would be distributed over the whole space as is depicted in Fig. 6.9b. The Wilson loop average would have the behavior

$$W(C) \;\overset{\text{large } C}{\longrightarrow}\; \mathrm{e}^{-\text{const}\cdot L(C)}, \tag{6.79}$$

where $L(C)$ denotes the length (or perimeter) of the closed contour C.

This behavior of the Wilson loops is called the *perimeter law*. To each order of perturbation theory, it is the perimeter law (6.79), rather than the area law (6.75), that holds for the Wilson loop averages. A perimeter law corresponds to a potential which cannot confine quarks.

Remark on the Creutz ratio

To distinguish between the area and perimeter law behavior of the Wilson loop averages, Creutz [Cre80] proposed to consider the ratio

$$\chi(I,J) \;=\; -\ln\frac{W(I\times J)\,W((I-1)\times(J-1))}{W((I-1)\times J)\,W(I\times(J-1))}\,, \tag{6.80}$$

where $W(I\times J)$ is as before the average of a rectangular Wilson loop of size $I\times J$. The exponentials of the perimeter, which is equal to

$$L(I\times J) \;=\; 2I+2J\,, \tag{6.81}$$

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cancel out in the ratio (6.80). In particular, the mass renormalization (6.56) cancels out, which is essential for the continuum limit.

The Creutz ratio (6.80) has the meaning of an interaction force between quarks, which can be seen by stretching the rectangle along the "temporal" axis (as illustrated by Fig. 6.6). If the area law (6.75) holds for asymptotically large I and J, then

$$\chi(I,J) \quad \stackrel{\text{large } I,J}{\longrightarrow} \quad a^2 K\,, \tag{6.82}$$

i.e. it does not depend on I or J and coincides with the string tension. This property of the Creutz ratio was used for numerical calculations of the string tension.

6.7 Asymptotic scaling

Equation (6.77) establishes the relationship between values of the lattice spacing a and the coupling g^2 as follows. Let us set K to be equal to its experimental value*

$$K \;=\; (400\ \text{MeV})^2 \;\approx\; 1\ \text{GeV/fm}\,. \tag{6.83}$$

Then the renormalizability prescribes that variations of a, which plays the role of a lattice cutoff, and of the bare charge g^2 should be made simultaneously in order that K does not change.

Given Eq. (6.77), this procedure calls for $a \to \infty$ as $g^2 \to \infty$. In other words, the lattice spacing is large in the strong-coupling limit, compared with 1 fm – the typical scale of the strong interaction. This is a situation of the type shown in Fig. 6.4a. Such a coarse lattice cannot describe the continuum limit correctly and, in particular, the rotational symmetry.

In order to pass to the continuum, the lattice spacing a should be decreased to have a picture like that in Fig. 6.4b. Equation (6.77) shows that a decreases with decreasing g^2. However, this formula ceases to be applicable in the intermediate region of $g^2 \sim 1$ and, therefore, $a \sim 1$ fm.

The recipe for further decreasing a is the same as in the strong-coupling region, further decreasing g^2. While no analytic formulas are available at intermediate values of g^2, the expected relation between a and g^2 for small g^2 is predicted by the known two-loop Gell-Mann–Low function of QCD.

* This value results from the string model of hadrons where the slope of the Regge trajectory α' and the string tension K are related by $K = 1/2\pi\alpha'$. This formula holds even for a classical string. The slope $\alpha' = 1\ \text{GeV}^{-2}$ say from the $\rho - A_2 - g$ trajectory. A similar value of K is found from the description of mesons made out of heavy quarks using a nonrelativistic potential model.

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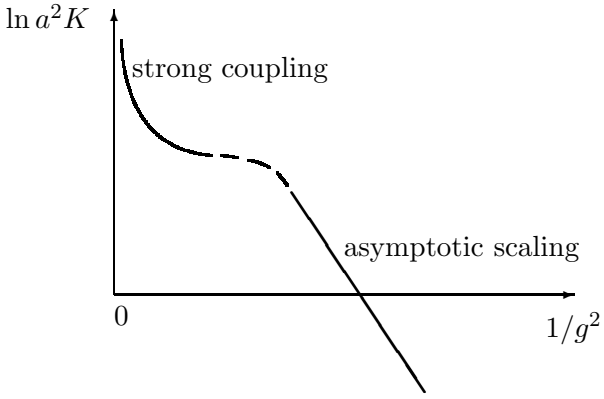

Fig. 6.10. The dependence of the string tension on $1/g^2$. The strong-coupling formula (6.77) holds for small $1/g^2$. The asymptotic-scaling formula (6.84) sets in for large $1/g^2$. Both formulas are not applicable in the intermediate region of $1/g^2 \sim 1$ which is depicted by the dashed line.

For pure $SU(3)$ gluodynamics, Eq. (6.77) is replaced at small g^2 by

$$K \quad = \quad \text{const} \cdot \frac{1}{a^2} \left(\frac{8\pi^2}{11g^2} \right)^{\frac{102}{121}} \mathrm{e}^{-8\pi^2/11g^2}, \tag{6.84}$$

where we have used the two-loop Gell-Mann–Low function.

The exponential dependence of K on $1/g^2$ is called *asymptotic scaling*. Asymptotic scaling sets in for some value of $1/g^2$ as depicted in Fig. 6.10. For such values of g^2, where asymptotic scaling holds, the lattice gauge theory has a continuum limit.

The knowledge of the two asymptotic behaviors says nothing about the behavior of a^2K in the intermediate region of $g^2 \sim 1$. There can be either a smooth transition between these two regimes or a phase transition. Numerical methods were introduced to study this problem, some of which are described in the next chapter.

Remark on dimensional transmutation

The QCD action (5.13) does not contain a dimensional parameter of the order of hundreds MeV. The masses of the light quarks are of the order of a few MeV and can be disregarded. The only parameter of the action is the dimensionless bare coupling constant g^2. At the classical level, there is no way to obtain a dimensional parameter of the order of hundreds MeV.

In quantum theory, these is always a dimensional cutoff (such as a for the lattice regularization). The renormalizability says that a and g^2 are

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not independent but are related by the Gell-Mann–Low equation (3.72). It can be integrated to give the integration constant

$$\Lambda_{\mathrm{QCD}} \;=\; \frac{1}{a}\exp\left[-\int\frac{\mathrm{d}g^2}{\mathcal{B}(g^2)}\right]. \tag{6.85}$$

Up to this point there has been no difference between QCD and QED. The difference stems from the fact that the Gell-Mann–Low function $\mathcal{B}(g^2)$ is positive for QED and negative for QCD. In QED $e^2(a)$ increases with decreasing a, while in QCD $g^2(a)$ decreases with decreasing a. The latter behavior of the coupling constant is called *asymptotic freedom*. In both cases the Gell-Mann–Low function vanishes when the coupling constant tends to zero. Such values of coupling constants where the Gell-Mann–Low function vanishes are called the *fixed point*. Since the infrared behavior of e^2 in QED is interchangeable with the ultraviolet behavior of g^2 in QCD, the origin is an infrared-stable fixed point in QED and an ultraviolet-stable fixed point in QCD. In QED the fine-structure constant ($\approx 1/137$) is measurable in experiments, while in QCD the constant Λ_{QCD} is measurable.

This phenomenon of the appearance of a dimensional parameter in QCD, which remains finite in the limit of vanishing cutoff, is called *dimensional transmutation*. All observable dimensional quantities, such as the string tension or hadron masses, are proportional to the corresponding powers of Λ_{QCD}. Therefore, their dimensionless ratios, such as the ratio of $\sqrt{K}$ to the hadron masses, are universal numbers which do not depend on g^2. The goal of a nonperturbative approach in QCD is to calculate these numbers but not the overall dimensional parameter.

Remark on second-order phase transition

In statistical physics it is usually said that the continuum limits of a lattice system are reached at the points of second-order phase transitions when the correlation length becomes infinite in lattice units. This statement is in perfect agreement with what has been said above concerning the continuum limit of lattice gauge theories.

A correlation length is inversely proportional to Λ_{QCD} given by Eq. (6.85). The only chance for the RHS of Eq. (6.85) to vanish is to have a zero of the Gell-Mann–Low function $\mathcal{B}(g^2)$ at some fixed point $g^2 = g_*^2$. Therefore, the bare coupling should approach the fixed-point value g_*^2 to describe the continuum.

As we have discussed, $\mathcal{B}(0) = 0$ for a non-Abelian gauge theory so that $g_*^2 = 0$ is a fixed-point value of the coupling constant. Therefore, the continuum limit is associated with $g^2 \to 0$ as mentioned above.

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