# DEDEKIND COMPLETENESS AND A FIXED-POINT THEOREM

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**1. Introduction.** McShane **(5, 6)** has introduced the concept of "Dedekind completeness" for partially ordered sets, which seems to be a natural generalization of the usual concept of completeness for lattices. It is the purpose of this paper to discuss some of the properties of Dedekind completeness, particularly with respect to a rather natural class of partially ordered sets which we call "uniform." Among our results we obtain an analogue of MacNeille's "completion by cuts." We also extend the well-known fixed-point theorem, due to Tarski **(7)**, and then generalize the characterization of a complete lattice due to Davis **(3)**.

**2. Dedekind completeness.** Let $P$ be a partially ordered set (poset) with respect to a relation $\leqslant$. We assume that $P$ has a greatest element $I$ and a least element $O$.

Definition 1. We say that a set $S \subset P$ is *up-directed* if and only if for each $a \in S$, $b \in S$, there exists $c \in S$ with $a \leqslant c$, $b \leqslant c$. Dually, $S$ is *down-directed* if and only if for each $a \in S$, $b \in S$, there exists $c \in S$ with $c \leqslant a$, $c \leqslant b$.

Thus, any subset of $P$ which has a greatest element is up-directed, and dually. The following definition is essentially that of McShane.

Definition 2. A poset $P$ is *Dedekind complete* if and only if every up-directed subset of $P$ has a least upper bound in $P$ and every down-directed subset has a greatest lower bound in $P$.

*Example* 1. It is clear that the concepts of Dedekind completeness and ordinary completeness coincide if $P$ is a lattice. A simple example of a Dedekind complete poset, which is not a lattice, is provided by the set $C$ of all closed disks in the Euclidean plane $E_2$, partially ordered by set inclusion, and with $O$ and $I$ elements adjoined. To show that $C$ is Dedekind complete, let $A$ be an up-directed subset of $C$, and let

$$X = \{x | x \in E_2 \text{ and } x \in a \text{ for some } a \in A\}.$$

If $X$ is an unbounded subset of $E_2$, then clearly l. u. b. $(A) = I$. If $X$ is bounded choose two points $x$ and $y$ in the closure of $X$ such that the distance from $x$ to $y$ is equal to the diameter of $X$. Let $m$ be a closed disk with the line seg-

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ment connecting $x$ and $y$ as its diameter. Straightforward arguments then show that

(i) no point of $X$ is exterior to $m$, and

(ii) every interior point of $m$ is a point of $X$. Thus $m$ = l. u. b. $(A)$. The obvious dual argument will then show that any down-directed subset of $C$ has a g. l. b.

If $A \subseteq P$, let

$$A^* = \{x | x \in P \text{ and } x \geqslant a \text{ for all } a \in A\},$$
$$A^+ = \{x | x \in P \text{ and } x \leqslant a \text{ for all } a \in A\}.$$

We shall write $A^{*+}$ for the set $(A^*)^+$. We shall make important use of the following concept:

DEFINITION 3. A poset $P$ is *uniform* if and only if $A^*$ is a down-directed set for every up-directed subset $A$, and dually, $B^+$ is up-directed for every down-directed subset $B$.

Any lattice is obviously a uniform poset. As an example of a uniform poset, which is not a lattice and not Dedekind complete, we may take the set of all closed disks in the plane with rational radii, partially ordered by set inclusion, and with $O$ and $I$ elements adjoined.

We have the following trivial lemma:

LEMMA 1. *A uniform poset $P$ is Dedekind complete if and only if every up-directed subset of $P$ has a* l.u.b. *in $P$ (or every down-directed subset of $P$ has a* g. l. b. *in $P$).*

We shall also use a strong form of Zorn's lemma due to Bourbaki **(2)**:

LEMMA (Bourbaki). *If every well-ordered chain in a poset $S$ has an upper bound in $S$, then $S$ has a maximal element.*

As a consequence of the above lemma the reader may easily deduce

LEMMA 2. *If $Z$ is any chain in a poset $P$, then there exists a well-ordered chain $C \subset Z$ with $C^* = Z^*$.*

We now have the following theorem:

THEOREM 1. *A poset $P$ is Dedekind complete if and only if $P$ is uniform and every well-ordered chain in $P$ has a* l. u. b.

*Proof.* If $P$ is Dedekind complete, and $S$ is up-directed in $P$, then $S^*$ has a least element and hence is down-directed. The obvious dual statement also holds: thus $P$ is uniform and the conclusion follows. Conversely, let $P$ be uniform and suppose that every well-ordered chain in $P$ has a l. u. b. Let $A$ be any down-directed subset of $P$, and let $Z$ be a maximal chain in $A^+$. We assert that $Z$ has a l. u. b., $m$, for otherwise Lemma 2 would provide us with a contradiction of our hypothesis. If $a \in A$, we have $a \geqslant z$ for all $z \in Z$;

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hence $a \geqslant m$ and $m \in A^+$. By maximality of $Z$, $m$ is a maximal element of $A^+$. We assert that $m$ is the greatest element of $A^+$. For suppose that there exists $c \in A^+$ with $c > m$. Since $A^+$ is up-directed, there exists $x \in A^+$ with $x \geqslant m$, $x \geqslant c$, contradicting the maximality of $m$. Thus $m =$ g. l. b. $(A)$, and $P$ is Dedekind complete by Lemma 1.

As a corollary we have the following known result, for which a proof seems to have thus far been lacking in the literature:

COROLLARY. *A lattice $L$ is complete if and only if every well-ordered chain in $L$ has a* l. u. b.

Let us call a chain $Z$ in $P$ *inversely well-ordered* if and only if every subset of $Z$ has a greatest element. We then have obvious dual formulations of Lemma 2 and Theorem 1. We shall also need the following lemma, which extends a result of Davis (**3**, Lemma 1, p. 311); our proof of it becomes trivial by employing Zorn's lemma (rather than transfinite induction as in (**3**)):

LEMMA 3. *Let $P$ be a uniform poset, and let $Z$ be an inversely well-ordered chain in $P$ with no* g. l. b. *in $P$. Then there exists a well-ordered chain $Y$ in $P$ such that*

(i) *$y \in Y$ implies $y < z$ for all $z \in Z$, and*

(ii) *$Y^* \cap Z^+$ is empty.*

*Proof.* $Z^+$ is up-directed, by our hypothesis of uniformity; hence $Z^+$ has no maximal elements. Then by the lemma of Bourbaki there exists a well-ordered chain $Y$ in $Z^+$ such that $Y^* \cap Z^+$ is empty.

**3. Imbedding of a uniform poset in a Dedekind complete poset.** We shall now obtain an analogue of MacNeille's well-known imbedding of a poset in a complete lattice (**4**; also see **1**, p. 58).

DEFINITION 4. A subset $J$ of a poset $P$ is a *normal ideal* in $P$ ("closed ideal" in the terminology of Birkhoff) if and only if $J^{*+} = J$. A subset of $P$ of the form

$$J_a = \{x | x \in P \text{ and } x \leqslant a\}$$

is called a *principal ideal.*

LEMMA 4. *A subset of $P$ is a normal ideal if and only if it is the intersection of a set of principal ideals* (cf. **1**; p. 62, problem 4).

*Proof.* Let $S \subset P$ and let

$$A = \bigcap_{x \in S} J_x.$$

Then $A = S^+$. In general we have $S \subset S^{+*}$; hence $S^+ \supset (S^{+*})^+$, or $A \supset A^{*+}$. Since in general $A \subset A^{*+}$, it follows that $A$ is a normal ideal. Conversely, if $A$ is a normal ideal in $P$, then

$$A = (A^*)^+ = \bigcap_{x \in A^*} J_x$$

For uniform posets we now have another characterization of Dedekind completeness, which generalizes a known result for complete lattices (**1**; p. 59, exercise 2):

THEOREM 2. *A uniform poset P is Dedekind complete if and only if every up-directed normal ideal in P is principal.*

*Proof.* Let $P$ be Dedekind complete, and let $J$ be an up-directed normal ideal in $P$. Then $J$ has a l. u. b. $m$, and $m \in J^{*+} = J$. It follows that $J$ is principal. To prove the converse, let $A$ be a down-directed subset of $P$. By Lemma 4,

$$A^+ = \bigcap_{a \in A} J_a$$

is a normal ideal, which by hypothesis is up-directed. Hence $A^+$ has a l. u. b., which is the g. l. b. of $A$. Thus $P$ is Dedekind complete by Lemma 1.

Now let $N(P)$ be the set of all up-directed normal ideals of $P$, partially ordered by inclusion. The correspondence $x \leftrightarrow J_x$ is a one-to-one order-preserving mapping of $P$ into a subset of $N(P)$. Furthermore, we have

THEOREM 3. *If P is a uniform poset, then N(P) is Dedekind complete.*

*Proof.* Let $\Sigma$ be an up-directed subset of $N(P)$, and let

$$A = \bigcup_{J \in \Sigma} J$$

(where $\bigcup$ denotes set union). It is easily seen that $A$ is an up-directed subset of $P$. Hence $A^*$ is down-directed, and $A^{*+}$ is up-directed. Since $A^{*+}$ is the smallest normal ideal containing $A$, we have $A^{*+} =$ l. u. b. $(\Sigma)$. Now let $\Omega$ be a down-directed subset of $N(P)$. We first show that

$$B = \bigcup_{J \in \Omega} J^*$$

is a down-directed subset of $P$. Let $a$ and $b$ be arbitrary elements of $B$; then there exist $J_1, J_2 \in \Omega$ with $a \in J_1^*$, $b \in J_2^*$. By our hypothesis on $\Omega$, there exists $J_3 \in \Omega$ with $J_3 \subset J_1 \cap J_2$. Then $J_3^* \supset (J_1 \cap J_2)^* \supset J_1^* \cup J_2^*$. But $J_3^*$ is down-directed, by uniformity of $P$: hence there exists $c \in J_3^*$ with $c \leqslant a$, $c \leqslant b$, and thus $B$ is down-directed. Now let

$$K = \bigcap_{J \in \Omega} J.$$

But

$$\bigcap_{J \in \Omega} J = \bigcap_{J \in \Omega} J^{*+} = \Big(\bigcup_{J \in \Omega} J^*\Big)^+ = B^+.$$

Hence $K$ is an up-directed normal ideal, and $K =$ g. l. b. $(\Omega)$.

*Example* 2. Let $P$ be the set of all closed disks in the plane with rational radii, ordered by inclusion. If $z$ is an arbitrary closed disk in the plane, then the set

$$S(z) = \{a | a \in P \text{ and } a \subset z\}$$

is an up-directed normal ideal in $P$; and conversely, the reader may verify that every such ideal is of the form $S(z)$ for some disk $z$. Hence the "Dedekind completion" $N(P)$ is isomorphic to the set of all closed disks in the plane.

*Example* 3. If $P$ is not uniform, then $N(P)$ may fail to be Dedekind complete. We construct an example of such a poset $P$ as follows. Let $A = \{a_{ij}\}$ $(i = 1, 2, \ldots ; j = 1, 2, \ldots)$ be an infinite rectangular array, in which $i$ denotes the column index, $j$ the row index. We partially order $A$ by defining $a_{ij} < a_{mn}$ if and only if $i < m$ or $j < n$. We surmount this array with a sequence $\{z_i\}$ of mutually incomparable elements (with respect to our ordering) such that $a_{ij} < z_i$ for each $i$ and each $j$. We adjoin two more incomparable elements $x$ and $y$ which are upper bounds for the set $\{z_i\}$. We then let $P$ be the set consisting of the array $A = \{a_{ij}\}$, the set $\{z_i\}$, the elements $x$ and $y$, and $O$ and $I$ elements; and let $P$ be partially ordered as described above. Thus we have $a_{ij} < z_k$ if and only if $i \leqslant k$. We see that $A$ is an up-directed subset of $P$. (Note, however, that $A^{*+}$ is the union of $A$ and the set $\{z_i\}$, and hence is not up-directed. Thus $A^{*+}$ is not an element of $N(P)$). Hence

$$\Sigma = \{J_a | a \in A\}$$

is an up-directed subset of $N(P)$. But the set $\Sigma^*$ contains $J_x$ and $J_y$ as minimal elements; hence $\Sigma$ has no l. u. b. in $N(P)$.

**4. The fixed-point theorem.** If $f$ is a function mapping a poset $P$ into itself, we say that $f$ is *isotone* if and only if $x \leqslant y$ implies $f(x) \leqslant f(y)$. $x$ is a *fixed-point* of $f$ if and only if $x = f(x)$. For any isotone function $f$ on $P$ let us write $H(f) = \{x | x \in P \text{ and } x \leqslant f(x)\}$.

DEFINITION 5. An isotone function $f$ on a poset $P$ is *directable* if and only if $H(f)$ is an up-directed subset of $P$.

The reader may verify that any isotone function on a lattice is directable. Thus the following theorem generalizes the fixed-point theorem of Tarski (**7**, Theorem 1):

THEOREM 4. *If every up-directed subset of a poset $P$ has a* l. u. b. *in $P$, then every directable function on $P$ has a fixed-point.*

*Proof.* Let $f$ be a directable function on $P$ and let $u$ = l. u. b. $[H(f)]$. We easily prove, precisely as in the proof of Theorem 1 of (**7**), that $u$ is a fixed-point of $f$. We omit the details.

We now obtain a generalization of the result of Davis (**3**, Theorem 2) :

THEOREM 5. *If every directable function on a uniform poset $P$ has a fixed-point, then $P$ is Dedekind complete.*

*Proof.* Assume $P$ is not Dedekind complete. Applying the dual formulation of Theorem 1 and then Lemma 3, we infer that there exist two chains $Y$ and $Z$ in $P$ such that

(i) $Y$ is well-ordered and $Z$ is inversely well-ordered,

(ii) $y \in Y$ implies $y < z$ for all $z \in Z$,

(iii) $Y^* \cap Z^+$ is empty.

We shall proceed to obtain a contradiction by defining a directable function $f$ on $P$ which has no fixed-points. We do this exactly as in **(3**, pp. 313-314**)**. To define $f(x_0)$ for an arbitrary $x_0 \in P$ we distinguish two cases:

$$(1)\ x_0 \in Z^+, \qquad (2)\ x_0 \notin Z^+.$$

In case (1) we have $x_0 \notin Y^*$. Let $Y(x_0) = \{y|y \in Y \text{ and } y > x_0\}$. $Y(x_0)$ has a least element $y_0$, which we define as $f(x_0)$. In case (2), let $Z(x_0) = \{z|z \in Z \text{ and } z < x_0\}$. $Z(x_0)$ has a greatest element $z_0$, which we define as $f(x_0)$. It is clear that $f$ can have no fixed-points. The proof that $f$ is isotone is identical with that in **(3**, p. 314**)**: we therefore omit the details. It remains to show that $f$ is directable. From our definition of $f$ it is clear that $x \in H(f)$ implies that $f$ falls in case (1) above; i.e., $x \in Z^+$. Also it is clear that $Y \subset H(f)$. Now suppose that we have $a \in H(f)$, $b \in H(f)$. Then $a < f(a)$, $b < f(b)$, and $f(a) \in Y$, $f(b) \in Y$. Let $c = \max\ \{f(a), f(b)\}$. We have $c > a$, $c > b$, and $c \in H(f)$, thus completing the proof.

Combining Theorems 4 and 5, we obtain the following characterization of Dedekind completeness:

COROLLARY. *A uniform poset $P$ is Dedekind complete if and only if every directable function on $P$ has a fixed-point.*

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