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CONSTANT-SIGN AND NODAL SOLUTIONS TO A DIRICHLET PROBLEM WITH p-LAPLACIAN AND NONLINEARITY DEPENDING ON A PARAMETER

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Abstract A homogeneous Dirichlet problem with p-Laplacian and reaction term depending on a parameter $\lambda > 0$ is investigated. At least five solutions—two negative, two positive and one sign-changing (namely, nodal)—are obtained for all λ sufficiently small by chiefly assuming that the involved nonlinearity exhibits a concave–convex growth rate. Proofs combine variational methods with truncation techniques.

Keywords: concave–convex nonlinearities; p-Laplacian; constant-sign solutions; nodal solutions

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$ and let $p \in \,]1, +\infty[$. Consider the homogeneous Dirichlet problem

$$\left.\begin{aligned} -\Delta_p u &= f(x, u, \lambda) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{P'_λ}$$

where Δ_p denotes the p-Laplace differential operator, namely, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for all $u \in W_0^{1,p}(\Omega)$, while the reaction term $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ satisfies Carathéodory's conditions. The main result (Theorem 4.1) of [**14**] provides a $\lambda^* > 0$ such that (P'_λ) possesses at least five non-trivial weak solutions belonging to $C_0^1(\bar{\Omega})$, four of which have constant sign, for every $\lambda \in \,]0, \lambda^*[$.

A bifurcation theorem describing the dependence of positive solutions of (P'_λ) on the parameter $\lambda > 0$ was established in [**15**] for the case when the nonlinearity f takes the form

$$f(x, t, \lambda) := \lambda g(x, t) + h(x, t), \quad (x, t, \lambda) \in \Omega \times \mathbb{R} \times \mathbb{R}^+, \tag{1.1}$$

with suitable Carathéodory functions $g, h\colon \Omega \times \mathbb{R} \to \mathbb{R}$.

This paper contains a more precise version of [**14**, Theorem 4.1], which, however, requires that f satisfies (1.1). Thus, here, we deal with the problem

$$\left.\begin{aligned} -\Delta_p u &= \lambda g(x,u) + h(x,u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{P$_\lambda$}$$

A $(p-1)$-sublinear growth rate for $g(x,\cdot)$ is assumed, i.e.

$$\lim_{t\to 0}\frac{g(x,t)}{|t|^{p-2}t} = +\infty, \qquad \lim_{|t|\to+\infty}\frac{g(x,t)}{|t|^{p-2}t} = 0, \tag{1.2}$$

while, roughly speaking, $h(x,\cdot)$ is $(p-1)$-superlinear; namely,

$$\lim_{t\to 0}\frac{h(x,t)}{|t|^{p-2}t} = 0, \qquad \lim_{|t|\to+\infty}\frac{h(x,t)}{|t|^{p-2}t} = +\infty. \tag{1.3}$$

Under these hypotheses, in addition to some further technical conditions, we prove that for each $\lambda \in]0,\lambda^*[$ there exist at least five non-trivial weak solutions of (P_λ): two negative, two positive and one sign-changing (i.e. nodal) (see Theorem 4.3). As in [**14**], proofs combine variational arguments with truncation methods.

Because of (1.2), (1.3), the reaction term that appears in (P_λ) exhibits a concave–convex behaviour. Following the seminal paper [**1**], treating the case $p = 2$, such problems have been thoroughly investigated (see, for example, [**6**, **11**, **14**–**16**] and the references therein).

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space. If V is a subset of X, we write $\bar{V}$ for the closure of V, ∂V for the boundary of V and $\mathrm{int}(V)$ for the interior of V. $(X^*, \|\cdot\|_{X^*})$ denotes the dual space of X, $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X and X^* and $x_n \to x$ (respectively, $x_n \rightharpoonup x$) in X means 'the sequence $\{x_n\}$ converges strongly (respectively, weakly) in X'.

The next elementary but useful result [**15**, Proposition 2.1] will be used in § 4.

Proposition 2.1. *Suppose $(X, \|\cdot\|)$ is an ordered Banach space with order cone K. If $x_0 \in \mathrm{int}(K)$, then to every $z \in K$ there corresponds $t_z > 0$ such that $t_z x_0 - z \in K$.*

A function $\Phi\colon X \to \mathbb{R}$ satisfying

$$\lim_{\|x\|\to+\infty}\Phi(x) = +\infty$$

is called coercive. We say that Φ is weakly sequentially lower semicontinuous when $x_n \rightharpoonup x$ in X implies $\Phi(x) \leqslant \liminf_{n\to\infty}\Phi(x_n)$. Let $\Phi \in C^1(X)$. The classical Palais–Smale condition for Φ reads as follows.

(PS) Every sequence $\{x_n\} \subseteq X$ such that $\{\Phi(x_n)\}$ is bounded and $\|\Phi'(x_n)\|_{X^*} \to 0$ possesses a convergent subsequence.

Define, for any $c \in \mathbb{R}$,

$$\Phi^c := \{x \in X : \Phi(x) \leqslant c\}, \qquad K_c(\Phi) := K(\Phi) \cap \Phi^{-1}(c),$$

where, as usual, $K(\Phi)$ denotes the critical set of Φ, i.e. $K(\Phi) := \{x \in X : \Phi'(x) = 0\}$.

An operator $A\colon X \to X^*$ is said to be of type $(\mathrm{S})_+$ if

$$x_n \rightharpoonup x \text{ in } X, \qquad \limsup_{n\to+\infty} \langle A(x_n), x_n - x\rangle \leqslant 0$$

imply $x_n \to x$. The next simple result is more-or-less known and will be employed in § 4.

Proposition 2.2. *Let X be reflexive and let $\Phi \in C^1(X)$ be coercive. Assume $\Phi' = A + B$, where $A\colon X \to X^*$ is of type $(\mathrm{S})_+$, while $B\colon X \to X^*$ is compact. Then Φ satisfies (PS).*

Proof. Pick a sequence $\{x_n\} \subseteq X$ such that $\{\Phi(x_n)\}$ turns out to be bounded and

$$\lim_{n\to+\infty} \|\Phi'(x_n)\|_{X^*} = 0. \tag{2.1}$$

By the reflexivity of X, in addition to the coercivity of Φ, we may suppose, up to subsequences, $x_n \rightharpoonup x$ in X. Since B is compact, using (2.1) and taking a subsequence when necessary, one has

$$\lim_{n\to+\infty} \langle A(x_n), x_n - x\rangle = \lim_{n\to+\infty} (\langle \Phi'(x_n), x_n - x\rangle - \langle B(x_n), x_n - x\rangle) = 0.$$

This forces $x_n \to x$ in X, because A is of type $(\mathrm{S})_+$, as desired. □

Given a topological pair (A, B) satisfying $B \subset A \subseteq X$, the symbol $H_k(A, B)$, $k \in \mathbb{N}_0$, indicates the kth relative singular homology group of (A, B) with integer coefficients. If $x_0 \in K_c(\Phi)$ is an isolated point of $K(\Phi)$, then

$$C_k(\Phi, x_0) := H_k(\Phi^c \cap U, \Phi^c \cap U \setminus \{x_0\}), \quad k \in \mathbb{N}_0,$$

are the critical groups of Φ at x_0. Here, U stands for any neighbourhood of x_0 such that $K(\Phi) \cap \Phi^c \cap U = \{x_0\}$. By excision, this definition does not depend on the choice of U. The monograph [**3**] is a general reference on the subject.

Throughout the paper, Ω denotes a bounded domain of the real Euclidean N-space $(\mathbb{R}^N, |\cdot|)$ with a smooth boundary $\partial\Omega$, $p \in \,]1, +\infty[$, $p' := p/(p-1)$, $\|\cdot\|_p$ is the usual norm of $L^p(\Omega)$ and $W_0^{1,p}(\Omega)$ indicates the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. On $W_0^{1,p}(\Omega)$ we introduce the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p \,\mathrm{d}x\right)^{1/p}, \quad u \in W_0^{1,p}(\Omega).$$

Write p^* for the critical exponent of the Sobolev embedding $W_0^{1,p}(\Omega) \subseteq L^q(\Omega)$. Recall that $p^* = Np/(N-p)$ if $p < N$, $p^* = +\infty$ otherwise and the embedding is compact whenever $1 \leqslant q < p^*$.

Let $W^{-1,p'}(\Omega)$ be the dual space of $W_0^{1,p}(\Omega)$ and let $A\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ be the nonlinear operator stemming from the negative p-Laplacian, i.e.

$$\langle A(u), v\rangle := \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x, \quad \forall u, v \in W_0^{1,p}(\Omega).$$

Denote by λ_1 the first eigenvalue of the operator $-\Delta_p$ in $W_0^{1,p}(\Omega)$. It is known [**13**, **16**] that

(p_1) $\|u\|_p^p \leqslant \lambda_1^{-1}\|u\|^p$ for all $u \in W_0^{1,p}(\Omega)$ and

(p_2) $A\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is bijective and of type $(\mathrm{S})_+$.

Define $C_0^1(\bar{\Omega}) := \{u \in C^1(\bar{\Omega})\colon u = 0 \text{ on } \partial\Omega\}$. Obviously, $C_0^1(\bar{\Omega})$ is an ordered Banach space with order cone

$$C_0^1(\bar{\Omega})_+ := \{u \in C_0^1(\bar{\Omega})\colon u(x) \geqslant 0,\ \forall x \in \bar{\Omega}\}.$$

Moreover, one has

$$\mathrm{int}(C_0^1(\bar{\Omega})_+) = \left\{u \in C_0^1(\bar{\Omega})\colon u > 0 \text{ in } \Omega,\ \frac{\partial u}{\partial n} < 0 \text{ on } \partial\Omega\right\},$$

where $n(x)$ denotes the outward unit normal vector to $\partial\Omega$ at the point $x \in \partial\Omega$ (see, for example, [**8**, Remark 6.2.10]).

On account of (p_2), we can find a function $e \in W_0^{1,p}(\Omega)$ such that

$$-\Delta_p e = 1 \quad \text{in } \Omega. \tag{2.2}$$

Theorems 1.5.6 and 1.5.7 of [**7**] then give $e \in \mathrm{int}(C_0^1(\bar{\Omega})_+)$.

Finally, 'measurable' always signifies Lebesgue measurable, while $m(E)$ indicates the Lebesgue measure of E. Provided $t \in \mathbb{R}$, we can set

$$t^- := \max\{-t, 0\}, \qquad t^+ := \max\{t, 0\}.$$

If $u, v\colon \Omega \to \mathbb{R}$ belong to a given function space X and $u(x) \leqslant v(x)$ for almost every $x \in \Omega$, then we set

$$[u, v] := \{w \in X\colon u(x) \leqslant w(x) \leqslant v(x) \text{ almost everywhere in } \Omega\}.$$

3. Basic assumptions and auxiliary results

To avoid unnecessary technicalities, 'for every $x \in \Omega$' will take the place of 'for almost every $x \in \Omega$' and the variable x will be omitted when no confusion can arise.

Let $g, h\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be two Carathéodory functions such that $g(x, 0) = h(x, 0) = 0$ for all $x \in \Omega$. Write, as usual,

$$G(x, z) := \int_0^z g(x, t)\,\mathrm{d}t, \quad H(x, z) := \int_0^z h(x, t)\,\mathrm{d}t, \quad \forall (x, z) \in \Omega \times \mathbb{R}.$$

The hypotheses below will be posited later.

(a_{11}) There exist $c_1 > 0$, $q \in]1, p^*[$ satisfying

$$|g(x,t)| \leqslant c_1(1 + |t|^{q-1}) \quad \text{in } \Omega \times \mathbb{R}.$$

(a_{12}) $\lim_{|z|\to+\infty} G(x,z)/|z|^p = 0$ uniformly with respect to $x \in \Omega$.

(a_{13}) To every $\rho > 0$ there corresponds $\mu'_\rho > 0$ such that the function

$$t \mapsto g(x,t) + \mu'_\rho |t|^{p-2}t$$

is non-decreasing in $[-\rho, \rho]$ for all $x \in \Omega$.

(a_{14}) $g(x,t)t \geqslant 0$, $(x,t) \in \Omega \times \mathbb{R}$. Moreover, for every $x \in \Omega$, the function

$$t \mapsto \frac{g(x,t)}{|t|^{p-2}t}$$

turns out to be non-decreasing in $]-\infty, 0[$ and non-increasing in $]0, +\infty[$.

(a_{15}) $0 < g(x,z)z \leqslant \theta G(x,z)$ provided $x \in \Omega$ and $0 < |z| \leqslant \delta$, where $\theta \in]1, p[$, while $\delta > 0$. Further, $\operatorname{ess\,inf}_{x\in\Omega} G(x,\delta) > 0$.

(a_{21}) There exist $c_2 > 0$, $r \in]\max\{p,q\}, p^*[$ satisfying

$$|h(x,t)| \leqslant c_2|t|^{r-1} \quad \text{in } \Omega \times \mathbb{R}.$$

(a_{22}) $\lim_{|z|\to+\infty} H(x,z)/|z|^p = +\infty$ uniformly with respect to $x \in \Omega$.

(a_{23}) To every $\rho > 0$ there corresponds $\mu''_\rho > 0$ such that the function

$$t \mapsto h(x,t) + \mu''_\rho |t|^{p-2}t$$

is non-decreasing in $[-\rho, \rho]$ for all $x \in \Omega$.

(a_{24}) $h(x,t)t \geqslant 0$, $(x,t) \in \Omega \times \mathbb{R}$.

(a_{25}) $h(x,t) \leqslant \theta H(x,t)$, provided $x \in \Omega$ and $0 < |z| \leqslant \delta$, where θ, δ come from (a_{15}).

Finally, let $\lambda > 0$ and let

$$\xi_\lambda(x,z) := z[\lambda g(x,z) + h(x,z)] - p[\lambda G(x,z) + H(x,z)], \quad (x,z) \in \Omega \times \mathbb{R}.$$

The next assumption, involving both nonlinearities, will also be adopted.

(a_{31}) For every $\lambda > 0$ there exists $\alpha_\lambda \in L^1(\Omega)$ such that

$$\alpha_\lambda(x) \geqslant 0, \quad \xi_\lambda(x,z') \leqslant \xi_\lambda(x,z'') + \alpha_\lambda(x) \quad \text{in } \Omega$$

whenever $z', z'' \in \mathbb{R}$, $|z'| \leqslant |z''|$ and $z'z'' \geqslant 0$.

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Throughout the paper, we shall write

$$f(x,t,\lambda) := \lambda g(x,t) + h(x,t), \quad \forall (x,t,\lambda) \in \Omega \times \mathbb{R} \times \mathbb{R}^+, \tag{3.1}$$

as well as

$$F(x,z,\lambda) := \int_0^z f(x,t,\lambda)\,\mathrm{d}t, \quad (x,z,\lambda) \in \Omega \times \mathbb{R} \times \mathbb{R}^+. \tag{3.2}$$

Remark 3.1. An elementary verification shows that if (a_{ij}), $i = 1,2$, $j = 1,\dots,5$, and (a_{31}) hold true then f satisfies (f_1)–(f_5) of [**14**]. Hence, all the results in that paper can be exploited here.

Remark 3.2. Due to (a_{12}) and (a_{15}) the function $G(x,\cdot)$ is p-sublinear; namely,

$$\lim_{z\to 0} \frac{G(x,z)}{|z|^p} = +\infty, \qquad \lim_{|z|\to+\infty} \frac{G(x,z)}{|z|^p} = 0.$$

Likewise, due to (a_{21}) and (a_{22}), the function $H(x,\cdot)$ turns out to be p-superlinear, i.e.

$$\lim_{z\to 0} \frac{H(x,z)}{|z|^p} = 0, \qquad \lim_{|z|\to+\infty} \frac{H(x,z)}{|z|^p} = +\infty.$$

Consequently, the reaction term in problem (P_λ) exhibits a growth rate of concave–convex type.

Example 3.3. A simple but meaningful situation when all the hypotheses stated above are satisfied is the following:

$$g(x,t) := |t|^{q-2}t, \quad h(x,t) := |t|^{r-2}t, \quad (x,t) \in \Omega \times \mathbb{R},$$

where $1 < q < p < r < p^*$. The same conclusion holds if

$$h(x,t) := |t|^{p-2}t \log(1 + |t|^p).$$

However, in such a case, the nonlinearity f given by (3.1) does not comply with the well-known Ambrosetti–Rabinowitz condition; namely,

(AR) there exist $\sigma > p$, $M > 0$ such that

$$0 < \sigma F(x,z,\lambda) \leqslant z f(x,z,\lambda)$$

for every $x \in \Omega$, $|z| \geqslant M$.

To simplify notation, define $X := W_0^{1,p}(\Omega)$ and $C_+ := C_0^1(\bar{\Omega})_+$. Let F be as in (3.2) and let

$$\varphi_\lambda(u) := \frac{1}{p}\|u\|^p - \int_\Omega F(x,u(x),\lambda)\,\mathrm{d}x, \quad u \in X. \tag{3.3}$$

Obviously, one has $\varphi_\lambda \in C^1(X)$. Theorem 3.1 in [**14**] directly yields the next result.

Lemma 3.4. *Suppose* (a_{i1}), (a_{i3}) *and* (a_{i5}), $i = 1, 2$, *hold true. Then there exists* $\lambda^* > 0$ *such that, for all* $\lambda \in]0, \lambda^*[$, (P_λ) *possesses two solutions* $u_0 \in \mathrm{int}(C_+)$, $v_0 \in -\mathrm{int}(C_+)$, *which are local minima of* φ_λ.

Actually, the proof of [**14**, Theorem 3.1] guarantees that

$$u_0 \in \mathrm{int}(C_+) \cap [0, \bar{u}], \qquad v_0 \in -\mathrm{int}(C_+) \cap [-\bar{u}, 0], \tag{3.4}$$

where $\bar{u} := t_\lambda e$, with e given by (2.2) and $t_\lambda > 0$ a suitable constant.

Lemma 3.5. *Under assumptions* (a_{1j}), $j = 1, 2, 4, 5$, *there correspond to every* $\lambda > 0$ *a unique* $\tilde{u} \in \mathrm{int}(C_+)$ *and a unique* $\tilde{v} \in -\mathrm{int}(C_+)$ *solving the equation*

$$-\Delta_p u = \lambda g(x, u) \quad \text{in } \Omega. \tag{3.5}$$

Proof. Fix $\lambda > 0$. Set $g_+(x, t) := g(x, t^+)$,

$$G_+(x, z) := \int_0^z g_+(x, t)\,\mathrm{d}t$$

and

$$\psi_{\lambda,+}(u) := \frac{1}{p}\|u\|^p - \int_\Omega G_+(x, u(x))\,\mathrm{d}x, \quad \forall u \in X. \tag{3.6}$$

On account of (a_{11}) and (a_{12}), given any $\varepsilon > 0$, we can find $c_3 > 0$ such that

$$G_+(x, z) < \frac{\varepsilon}{p}|z|^p + c_3, \quad (x, z) \in \Omega \times \mathbb{R}.$$

This implies that

$$\psi_{\lambda,+}(u) > \frac{1}{p}\left(1 - \frac{\lambda\varepsilon}{\lambda_1}\right)\|u\|^p - \lambda c_3 m(\Omega) \quad \text{in } X.$$

Hence, the functional $\psi_{\lambda,+}$ turns out to be coercive. A simple argument, based on the compact embedding $X \subseteq L^p(\Omega)$, shows that it is also weakly sequentially lower semicontinuous. So, there exists $\tilde{u} \in X$ satisfying

$$\psi_{\lambda,+}(\tilde{u}) = \inf_{u \in X} \psi_{\lambda,+}(u). \tag{3.7}$$

Let us verify that $\tilde{u} \neq 0$. If $u \in C_+ \setminus \{0\}$, then $tu(x) \leqslant \delta$, $x \in \Omega$, for every sufficiently small $t > 0$. Through (a_{15}) we infer that

$$\psi_{\lambda,+}(tu) = \frac{t^p}{p}\|u\|^p - \lambda \int_\Omega G_+(x, tu(x))\,\mathrm{d}x \leqslant \frac{t^p}{p}\|u\|^p - c_4 t^\theta \|u\|^\theta,$$

where $c_4 > 0$. Since $\theta < p$, fixing $t > 0$ small enough yields $\psi_{\lambda,+}(tu) < 0$. Therefore,

$$\psi_{\lambda,+}(\tilde{u}) = \inf_{u \in X} \psi_{\lambda,+}(u) < 0 = \psi_{\lambda,+}(0),$$

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which clearly means $\tilde{u} \neq 0$, as desired. Now, from (3.7), it follows that $\psi'_{\lambda,+}(\tilde{u}) = 0$; namely,

$$\langle A(\tilde{u}), v\rangle = \lambda \int_{\Omega} g_+(x, \tilde{u}(x))v(x)\,\mathrm{d}x, \quad \forall v \in X. \tag{3.8}$$

By (3.8) for $v := -\tilde{u}^-$, one has $\|\tilde{u}^-\|^p = 0$. Thus, $\tilde{u} \geqslant 0$ in Ω and, *a fortiori*, the function $\tilde{u}$ solves (3.5). Standard regularity results [**7**, Theorems 1.5.5 and 1.5.6] then give $\tilde{u} \in C_+$. Since, by (a_{14}), $\Delta_p \tilde{u}(x) \leqslant 0$ for almost every $x \in \Omega$, [**18**, Theorem 5] ensures that $\tilde{u} \in \mathrm{int}(C_+)$. Finally, the uniqueness of $\tilde{u}$ is an immediate consequence of [**4**, Theorem 1]. Similar reasoning produces a function $v \in -\mathrm{int}(C_+)$ with the asserted properties. □

4. Nodal solutions

The main purpose of this section is to find a sign-changing (i.e. nodal) solution of (P_λ). We start with the following.

Lemma 4.1. *Let hypotheses* (a_{ij}), $i = 1, 2$, $j = 1, \dots, 5$, *be satisfied and let* $\lambda \in\,]0, \lambda^*[$. *Then* ($\mathrm{P}_\lambda$) *has a biggest non-trivial negative solution* $\hat{v} \in -\mathrm{int}(C_+)$ *and a smallest non-trivial positive solution* $\hat{u} \in \mathrm{int}(C_+)$.

Proof. Assume that $u \in X$ is a non-trivial positive solution of (P_λ). Arguing as in the proof of Lemma 3.5, we obtain $u \in \mathrm{int}(C_+)$. Hence, due to Proposition 2.1, there exists $t > 0$ such that

$$t\tilde{u}(x) \leqslant u(x), \quad \forall x \in \Omega, \tag{4.1}$$

where $\tilde{u}$ comes from Lemma 3.5. Denote by $t_0 > 0$ the biggest positive constant for which (4.1) holds true. We claim that $t_0 \geqslant 1$. Indeed, set $\rho := \|u\|_\infty$. Conditions (a_{13}) and (a_{23}) provide $\mu_\rho > 0$ such that

$$z \mapsto \lambda g(x, z) + h(x, z) + \mu_\rho |z|^{p-2} z$$

turns out to be non-decreasing in $[-\rho, \rho]$ for all $x \in \Omega$. If the assertion were false then, on account of (a_{14}), (a_{24}) and (4.1),

$$\begin{aligned}
-\Delta_p(t_0\tilde{u}) + \mu_\rho(t_0\tilde{u})^{p-1} &= t_0^{p-1}[\lambda g(x, \tilde{u}) + \mu_\rho \tilde{u}^{p-1}] \\
&< \lambda g(x, t_0\tilde{u}) + \mu_\rho(t_0\tilde{u})^{p-1} \\
&\leqslant \lambda g(x, t_0\tilde{u}) + h(x, t_0\tilde{u}) + \mu_\rho(t_0\tilde{u})^{p-1} \\
&\leqslant \lambda g(x, u) + h(x, u) + \mu_\rho u^{p-1} \\
&= -\Delta_p u + \mu_\rho u^{p-1}.
\end{aligned}$$

So, by [**2**, Proposition 2.6], we would have $u - t_0\tilde{u} \in \mathrm{int}(C_+)$, against the maximality of t_0. Now, since $t_0 \geqslant 1$ while u was arbitrary, from (4.1) it results in

$$\tilde{u} \leqslant u \text{ in } \Omega \quad \text{for every non-trivial positive solution of } (\mathrm{P}_\lambda). \tag{4.2}$$

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Define

$$S_{\lambda,+} := \{u \in [0, \bar{u}] \colon u \neq 0 \text{ and satisfies } (\mathrm{P}_\lambda)\}.$$

Lemma 3.4 guarantees that $S_{\lambda,+} \neq \emptyset$, because $u_0 \in S_{\lambda,+}$. Reasoning as before, we get $S_{\lambda,+} \subseteq \operatorname{int}(C_+)$. Moreover, $S_{\lambda,+}$ turns out to be downward directed (see [**9**, Lemma 4.2]). By the Kuratowski–Zorn lemma, a smallest non-trivial positive solution $\hat{u} \in \operatorname{int}(C_+)$ of (P_λ) exists once we know that each chain $C \subseteq S_{\lambda,+}$ is bounded below. Using [**5**, p. 336] one has

$$\inf C = \inf\{u_k \colon k \in \mathbb{N}\} \tag{4.3}$$

for some $\{u_k\} \subseteq C$, while [**10**, Lemma 1.1.5] allows this sequence to be decreasing. Since

$$u_k \in [0, \bar{u}] \text{ and } A(u_k) = \lambda g(\cdot, u_k) + h(\cdot, u_k) \text{ in } W^{-1,p'}(\Omega),\ \forall k \in \mathbb{N}, \tag{4.4}$$

$\{u_k\}$ is bounded in $W_0^{1,p}(\Omega)$. Passing to a subsequence when necessary, we may thus suppose $u_k \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ as well as $u_k \to u$ in $L^q(\Omega)$, with

$$u = \inf\{u_k \colon k \in \mathbb{N}\}. \tag{4.5}$$

This forces

$$\lim_{k \to +\infty} \int_\Omega [\lambda g(x, u_k(x)) + h(x, u_k(x))](u_k(x) - u(x))\,\mathrm{d}x = 0.$$

Therefore, on account of (4.4),

$$\lim_{k \to +\infty} \langle A(u_k), u_k - u \rangle = 0.$$

Property (p_2) yields $u_k \to u$ in $W_0^{1,p}(\Omega)$. From (4.4), letting $k \to +\infty$ it follows that

$$u \in [0, \bar{u}], \quad A(u) = \lambda g(\cdot, u) + h(\cdot, u) \quad \text{in } W_0^{-1,p'}(\Omega);$$

namely, $u \in S_{\lambda,+}$ because, by (4.2), $\tilde{u} \leqslant u$ in Ω. Now, (4.3) and (4.5) lead to $\inf C \in S_{\lambda,+}$, as desired. Finally, due to (4.2) again, $\tilde{u}(x) \leqslant \hat{u}(x)$ for all $x \in \Omega$. The construction of a biggest non-trivial negative solution $\hat{v} \in -\operatorname{int}(C_+)$ of (P_λ) such that $\hat{v} \leqslant \tilde{v}$ in Ω is analogous. □

We are now in a position to find a sign-changing solution of (P_λ).

Theorem 4.2. *Under hypotheses* (a_{ij}), $i = 1, 2$, $j = 1, \dots, 5$, *and* (a_{31}), *if* $\lambda \in \,]0, \lambda^*[$, *then* (P_λ) *possesses a nodal solution* $w \in C_0^1(\bar{\Omega})$.

Proof. Define, for every $(x, t) \in \Omega \times \mathbb{R}$,

$$\hat{f}(x, t, \lambda) := \begin{cases} \lambda g(x, \hat{v}(x)) + h(x, \hat{v}(x)) & \text{if } t < \hat{v}(x), \\ \lambda g(x, t) + h(x, t) & \text{if } \hat{v}(x) \leqslant t \leqslant \hat{u}(x), \\ \lambda g(x, \hat{u}(x)) + h(x, \hat{u}(x)) & \text{if } \hat{u}(x) < t, \end{cases} \tag{4.6}$$

$$\hat{f}_+(x, t, \lambda) := \hat{f}(x, t^+, \lambda), \qquad \hat{f}_-(x, t, \lambda) := \hat{f}(x, -t^-, \lambda). \tag{4.7}$$

Moreover, provided $u \in X$, set

$$\hat{\varphi}_\lambda(u) := \frac{1}{p}\|u\|^p - \int_\Omega \hat{F}(x, u(x), \lambda)\,\mathrm{d}x,$$
$$\hat{\varphi}_{\lambda,\pm}(u) := \frac{1}{p}\|u\|^p - \int_\Omega \hat{F}_\pm(x, u(x), \lambda)\,\mathrm{d}x,$$

where

$$\hat{F}(x, z, \lambda) := \int_0^z \hat{f}(x, t, \lambda)\,\mathrm{d}t \quad \text{and} \quad \hat{F}_\pm(x, z, \lambda) := \int_0^z \hat{f}_\pm(x, t, \lambda)\,\mathrm{d}t.$$

By (4.6), (4.7), one has

$$K(\hat{\varphi}_\lambda) \subseteq [\hat{v}, \hat{u}], \qquad K(\hat{\varphi}_{\lambda,-}) \subseteq [\hat{v}, 0], \qquad K(\hat{\varphi}_{\lambda,+}) \subseteq [0, \hat{u}]. \tag{4.8}$$

We may assume that

$$K(\hat{\varphi}_{\lambda,-}) = \{\hat{v}, 0\}, \qquad K(\hat{\varphi}_{\lambda,+}) = \{0, \hat{u}\}. \tag{4.9}$$

Indeed, if, for example, $u \in K(\hat{\varphi}_{\lambda,+}) \setminus \{0, \hat{u}\}$, then (4.8) forces $u \in [0, \hat{u}] \setminus \{0, \hat{u}\}$. Thanks to (4.6) we thus obtain $u \in K(\varphi_\lambda)$, with φ_λ given by (3.3). Hence, on account of (4.2), u is a non-trivial positive solution of (P_λ) and, like before, $u \in \mathrm{int}(C_+)$. However, this is impossible because of the minimality of $\hat{u}$ (see Lemma 4.1).

Let us next verify that $\hat{u}$, $\hat{v}$ are local minima for $\hat{\varphi}_\lambda$. Due to (4.7), the functional $\hat{\varphi}_{\lambda,+}$ is weakly sequentially lower semicontinuous and coercive. Thus, there exists $\bar{u} \in X$ such that

$$\hat{\varphi}_{\lambda,+}(\bar{u}) = \inf_{u \in X} \hat{\varphi}_{\lambda,+}(u). \tag{4.10}$$

Arguing as in the proof of Lemma 3.5 produces

$$\hat{\varphi}_{\lambda,+}(\bar{u}) < 0 = \hat{\varphi}_{\lambda,+}(0), \quad \text{i.e. } \bar{u} \neq 0. \tag{4.11}$$

By (4.9), this implies $\bar{u} = \hat{u} \in \mathrm{int}(C_+)$. Since $\hat{\varphi}_\lambda|_{X_+} = \hat{\varphi}_{\lambda,+}|_{X_+}$, where

$$X_+ := \{u \in X : u \geqslant 0 \text{ in } \Omega\}, \tag{4.12}$$

$\hat{u}$ turns out to be a $C_0^1(\bar{\Omega})$-local minimum for $\hat{\varphi}_\lambda$. Theorem 1.1 of [**6**] guarantees that the same is true with X in place of $C_0^1(\bar{\Omega})$. A similar reasoning then holds for $\hat{v}$.

Now, observe that $\hat{\varphi}_\lambda$ is coercive and if

$$\langle B(u), v\rangle := -\int_\Omega \hat{f}(x, u(x), \lambda)v(x)\,\mathrm{d}x, \quad \forall u, v \in X,$$

then

$$\langle \hat{\varphi}_\lambda'(u), v\rangle = \langle A(u), v\rangle + \langle B(u), v\rangle.$$

The operator A is of type $(\mathrm{S})_+$ (see (p_2)), while $B\colon X \to X^*$ turns out to be compact, because (a_{i1}), $i = 1, 2$, hold true and X embeds compactly in $L^p(\Omega)$. Therefore, Proposition 2.2 guarantees that $\hat{\varphi}_\lambda$ satisfies (PS). Through [**17**, Corollary 1] we thus obtain

$$K(\hat{\varphi}_\lambda) \setminus \{\hat{v}, \hat{u}\} \neq \emptyset.$$

Let $w \in K(\hat{\varphi}_\lambda) \setminus \{\hat{v}, \hat{u}\}$ be a critical point of mountain pass type. From (4.8) and (4.6) it follows that

$$A(w) = \lambda g(\cdot, w) + h(\cdot, w) \quad \text{in } W^{-1,p'}(\Omega);$$

namely, w solves (P_λ), while standard regularity results [**7**, Theorems 1.5.5 and 1.5.6] produce $w \in C_0^1(\bar{\Omega})$. We may assume that

$$C_1(\hat{\varphi}_\lambda, w) \neq 0 \tag{4.13}$$

(see [**3**, pp. 89–90]). By [**12**, Proposition 2.1] one has

$$C_k(\hat{\varphi}_\lambda, 0) = 0, \quad \forall k \in \mathbb{N}_0. \tag{4.14}$$

Comparing (4.13) with (4.14) yields $w \neq 0$. Now, since $w \in [\hat{v}, \hat{u}] \setminus \{\hat{v}, 0, \hat{u}\}$, Lemma 4.1 immediately leads to the conclusion. $\square$

Through Lemma 3.4, Theorem 3.2 in [**14**] and Theorem 4.2 we easily infer the next multiplicity result.

Theorem 4.3. *If* (a_{ij}), $i = 1, 2$, $j = 1, \dots, 5$, *and* (a_{31}) *hold true, then for every* $\lambda \in \,]0, \lambda^*[$ *problem* (P_λ) *has at least four constant-sign solutions,* $v_0, v_1 \in -\mathrm{int}(C_+)$, $u_0, u_1 \in \mathrm{int}(C_+)$, *and a nodal solution,* $w \in C_0^1(\bar{\Omega})$. *Moreover,* $v_1 \leqslant v_0 < 0 < u_0 \leqslant u_1$ *in* Ω.

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