'STATISTICAL BOOTSTRAP' EQUATION OF STATE FOR COLD ULTRA-DENSE MATTER

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Abstract. The statistical bootstrap theory of hadrons predicts a particle level density which increases with mass like $\varrho(m) = cm^a \exp bm$. The motivation for this level density is explored and then it is used to derive an equation of state for zero-temperature ultra-dense ($> 10^{17}$ g cm^{-3}) matter. The nature, uses, and limitations of the equation of state are discussed.

1. Introduction

Developments in the theory of particle physics over the last several years have had surprising implications concerning the multiplicity of particle states at energies far beyond the reach of present day experiments. A natural impulse is to apply this new found knowledge to physically interesting situations involving high temperatures and/or high densities. In particular, knowledge of the variety of particle species and their interactions under extreme conditions can be used to construct an equation of state applicable to the early stages of the universe in the context of a big bang cosmology or to the structure and stability of hadron stars.

This paper is oriented toward conditions of high density and negligible temperature and so is most directly concerned with the subject of hadron stars. While the equation of state itself follows straightforwardly from two key assumptions regarding the density of particle states and the manner in which their interactions are treated, the interesting physics is involved in the motivation of these assumptions and the degree of their validity. Thus these assumptions will be explored here even though they are of a much more general nature than the specific topic of the equation of state for cold ultra-dense matter. The derivation of the equation of state will then be sketched and its nature, uses, and limitations will be discussed.

2. The Density of Particle States

On aesthetic grounds one might like to think that there were some finite number of fundamental building blocks out of which all strongly interacting particles (hadrons) could be constructed. In fact some recent work indicates a quite different situation. Rather than there being any truly fundamental particles the hadrons may be their own constituents. In this picture not only will hadrons not be broken down into a few prime constituents at high energies but, in addition, the number of different particles will increase exponentially with mass-energy.

That the density of particle states would behave like

$$\varrho(m) \equiv \frac{\mathrm{d}n}{\mathrm{d}m} \underset{m \to \infty}{\longrightarrow} cm^a e^{bm}, \tag{1}$$

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was first predicted by Hagedorn (1965). Hagedorn used a bootstrap type argument of the sort illustrated below in which it is supposed that each hadron is composed of other hadrons in some statistical sense. An independent suggestion that something like (1) is true came with the establishment of the concept of duality – that the Regge-pole contribution is in itself an average description of the full scattering amplitude. Dolen *et al.* (1968) argued that Regge-pole contributions could not be added coherently to resonance contributions to the scattering amplitude, as in the interference model, without resulting in multiple counting. Duality says that resonances and Regge-poles are related by finite-energy sum rules so that one formalism is equivalent to the other.

Krzywicki (1969) and Fubini and Veneziano (1969) used duality models to derive the degeneracy of particle states as a function of energy. They found the same result as did Hagedorn, the number of particle states increases drastically with mass, agreeing asymptotically with (1). For a recent review of duality and the Veneziano model see Kaĭdalov (1972).

The derivation of the level density based on statistical bootstrap arguments is conceptually much simpler than that based on duality models and so will be used here to illustrate the origin of (1). The following discussion closely follows that of Frautschi (1971).

The current crude bootstrap model assumes that an effective potential serves to confine the constituents of a hadron in a volume with radius of order 10^{-13} cm. Inside this box the constituents are presumed to circulate freely. The density of levels in the box is then identified with the density of hadron states. This treatment assumes that all attractive interactions are accounted for in a statistical sense by counting all the resonance states in the box. In reality we know that a pair of particles in the box will interact *via* some potential. For an attractive potential, as the particles get closer their wave function oscillates more rapidly and for every extra oscillation corresponding to a phase shift of 180° the effect is to create another state in the box. Thus while an exact treatment would count states of motion of all the particles in their mutual potential, an approximate treatment is to count states of motion of the original particles plus the states of motion of all resonances, all considered to be non-interacting. Just such a treatment is shown to be rigorously true in certain idealized cases by Dashen *et al.* (1969) and Veneziano (1971). They show that narrow resonances in the scattering amplitude are manifested as free particles in the corresponding statistical ensemble.

What of repulsive interactions? Leung and Wang (1971) outline a possible theoretical argument that indicates that any repulsive baryon-baryon interaction would be roughly cancelled by a corresponding attractive interaction. The idea is that while interactions mediated by mesons of the ω family would be repulsive, interactions would also occur with coupling constants of the same order mediated by members of the f family which would provide an attractive interaction. Thus Leung and Wang argue that on the average not only repulsive but all baryon-baryon interactions might be ignored in certain cases.

In a more general way Hamer and Frautschi (1971) argue that repulsive interactions just serve to eliminate possible resonances from the system leaving the attractive interactions to generate the multiplicity of resonances which dominate the physics of the situation. More specifically, they point out that there are no limits to the rate of increase with energy of the number of states which attraction can generate whereas, while repulsion can serve to decrease the number of states up to a point, causality puts a finite limit on the rate of decrease. Therefore as long as channels exist which provide attractive interactions there will be an exponential increase in the density of states which will swamp any effect of repulsion in other channels.

Thus the assumption that the hadron constituents can be treated as moving freely in a box as long as all possible states are counted can be seen to be a fair first approximation. This model should be particularly satisfactory under proper conditions, for instance a very dilute collection of hadrons. On the other hand, in the context we have in mind, dense zero temperature hadron matter, we will require the hadrons to be closely packed and baryons to be degenerate. The possible effects of interactions will then have to be re-examined.

The problem, therefore, is to compute all the states in the box which composes the hadron volume. The number of states of one particle in a box of volume V with momentum between p and $p+\mathrm{d}p$ is

$$\frac{V\,\mathrm{d}^3p}{h^3}. \tag{2}$$

By extension, the density of states for n independent particles with total energy m is

$$\varrho_n(m) = \left(\frac{V}{h^3}\right)^{n-1} \prod_{i=1}^{n} \int \mathrm{d}^3p_i \delta\left(\sum_{i=1}^{n} E_i - m\right) \delta^3\left(\sum_{i=1}^{n} \mathbf{p}_i\right). \tag{3}$$

The momentum delta function and the omitted factor of V/h^3 correspond to taking the center of mass at rest.

For the bootstrap model of hadrons, the hadrons are assumed to be compounds of other hadrons, for instance, schematically,

$$\pi = (\pi\pi + \pi K + KK + \cdots) + (\pi\pi\pi + \pi\pi K + \cdots) + \cdots.$$

To compute the level density corresponding to this case two conditions must be met. The statistical condition is that the level density of hadrons $\varrho_{out}(m)$ is given by the phase space volume of an arbitrary number of noninteracting internal constituents confined in the volume V which have level density $\varrho_{in}(m_i)$. This level density is

$$\varrho_{out}(m) = \sum_{n=2}^{\infty} \frac{1}{n!}\left(\frac{V}{h^3}\right)^{n-1} \prod_{i=1}^{n} \int \mathrm{d}^3p_i \int \mathrm{d}m_i \varrho_{in}(m_i) \times$$
$$\times\, \delta\left(\sum_{i=1}^{n} E_i - m\right) \delta^3\left(\sum_{i=1}^{n} \mathbf{p}_i\right). \tag{4}$$

Here the integral over mass sums the contribution to the phase space volume of each mass interval. The factor $1/n!$ eliminates the effect of double counting (cf.

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Frautschi, 1971). Equation (4) is incorrect in that it does not take into account the Pauli principle and hence two or more fermions are allowed in the same state. Such states are statistically insignificant and do not contribute to the asymptotic level density in the non-degenerate case. The effects of the Pauli principle on the level density have not yet been included in the computation of the present equation of state. The possible effects of this correction will be discussed.

The second condition necessary to compute the hadron level density is the bootstrap condition, that the internal constituents are just the hadrons themselves, i.e.

$$\varrho_{\text{out}}(m) = \varrho_{\text{in}}(m). \tag{5}$$

Frautschi originally applied this relation only in the asymptotic region of large m. Hamer and Frautschi (1971) call that relation the 'strong asymptotic bootstrap condition.' They discovered that (5) could be applied at *all* masses above a certain threshold which they take to be the two pion threshold in the simplest case. Hamer and Frautschi call (5) the 'strong bootstrap condition.' The 'weak asymptotic bootstrap condition' is that originally used by Hagedorn (1965) where ϱ_{out} is consistent with ϱ_{in} only to within a power of the mass.

Frautschi (1971) then shows that the solution to (4) and (5) is of the form

$$\varrho = cm^a e^{bm}. \tag{6}$$

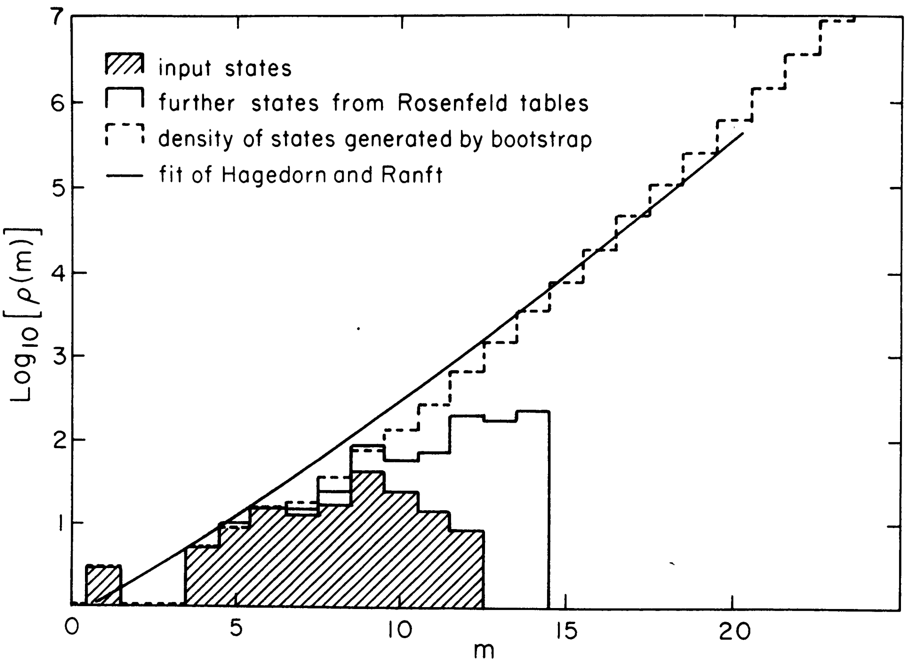

Fig. 1. The bootstrap density of all hadron states compared with experiment. Also shown is the fit of Hagedorn and Ranft (1968). This figure is from Hamer and Frautschi (1971). (Reprinted with the kind permission of the authors and *Physical Review*.)

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He also shows that if one considers only hadrons of a particular baryon number B, change Q, strangeness S, etc., the parameters a and b are unique, i.e.,

$$\varrho_{BQS\ldots} = c_{BQS\ldots} m^a e^{bm} . \tag{7}$$

Figure 1 shows the results of the statistical bootstrap model as computed by Hamer and Frautschi (1971) and the theoretical solution as given by Hagedorn and Ranft (1968). The experimental data can not yet distinguish an exponential mass spectrum from a power law. With a level density of the form of (6) Hamer and Frautschi can fit their results down to $m \sim 10\, m_\pi$.

The various theories give estimates of the values of the parameters as indicated in in Table I. Frautschi (1971) finds that the requirement of a consistent solution to (4) and (5) imposes an upper limit to a,

$$a < -5/2 . \tag{8}$$

Frautschi's analysis shows that it is not necessary to discuss thermodynamics in order to derive the level density; however, Hagedorn's (1965) thermodynamic analysis indicates that for local thermodynamical equilibrium to hold in high energy collisions a lower limit is placed on a,

$$a > -7/2 . \tag{9}$$

TABLE I

Parameter values for the statistical bootstrap level density, $\varrho(m) = cm^a \exp bm$

$a > -7/2$	Hagedorn
$a < -5/2$	Frautschi
$a = -5/2, -3, -7/2\ldots$	Veneziano Model
$a \simeq -3$	Hamer and Frautschi
$a = -3$	Nahm
$b^{-1} \sim 160$ MeV	Hagedorn
$b^{-1} \sim 174$ MeV	Veneziano Model ($a = -3$)
$b^{-1} \sim m_\pi \sim 140$ MeV	Hamer and Frautschi
$c \sim (0.9 \text{ BeV})^{3/2}$	Hagedorn ($a = -5/2$)
$c \sim (1\text{–}4)\, m_\pi^2 \sim (0.2 \text{ BeV})^2$	Hamer and Frautschi ($a = -3$)

The Veneziano model (Fubini and Veneziano, 1969; Bardakci and Mandelstam, 1969; Huang and Weinberg, 1970) allows a series of discrete values

$$a = -5/2, -3, -7/2\ldots . \tag{10}$$

Consistent with all these results Hamer and Frautschi (1971) compute that the value of a does not differ significantly from $a = -3$ and report that W. Nahm has established this unique value analytically by extending the work of Frautschi.

Hamer and Frautschi have shown that if *all* exotic resonances ($|B| > 1$) are included in the bootstrap spectrum the value of a decreases by one half for each extra internal quantum number included, i.e., Q, S etc. In the present work all exotics will be excluded.

The value of b was first estimated by Hagedorn (1968) and Hagedorn and Ranft

(1968) to be

$$b^{-1} \sim 160 \text{ MeV}. \tag{11}$$

They obtained this number by noticing that the transverse momentum distribution of secondaries in very high energy collisions goes as $\exp(-|p_\perp|/160 \text{ MeV})$. Hagedorn's statistical model interprets the constant in the denominator as b^{-1}. The Veneziano model gives $b^{-1} = 174$ MeV for $a = -3$ (cf. Huang and Weinberg, 1970) and Hamer and Frautschi (1971) find, as expected on dimensional grounds, $b^{-1} \sim m_\pi \simeq 140$ MeV with the value roughly linearly proportional to the radius of the hadron volume assumed.

Hagedorn (1967) estimates a value of the parameter c for the complete hadron spectrum of $(0.9 \text{ GeV})^{3/2}$ for a value of $a = -5/2$. He does this by fitting the derived level density to the known hadron distribution, as far as it goes. Hamer and Frautschi find a value of $c \sim m_\pi^2$ for $a = -3$ from their numerical computations.

The statistical bootstrap model outlined here has some advantages over the Veneziano model. The latter strictly treats only multi-particle interactions among mesons while the bootstrap model incorporates baryons as well. The bootstrap model also gives an interesting constraint on the parameters a, b, c, V namely

$$1 = \sum_{n=2}^{\infty} \frac{x^{n-1}}{(n-1)!}, \tag{12}$$

where

$$x = \frac{(2\pi)^{3/2} \, c V m_\pi^{a+5/2}}{-(a+5/2)\, h^3 b^{3/2}}. \tag{13}$$

The solution to (12) is just $x = \ln 2$ so that

$$V = \frac{\ln 2 h^3 b^{3/2} m_\pi^{1/2}}{2(2\pi)^{3/2} \, c}, \tag{14}$$

for $a = -3$. Equation (14) implies that some of the parameters of the model will change if the volume of the hadron is altered as it could conceivably be under conditions of high density. In fact Hamer and Frautschi find that, in general, a = constant but that

$$b \propto V^{1/3}, \tag{15a}$$

$$c \propto V^{(a+1)/3}, \tag{15b}$$

for given input physics. The nth term in (12) gives the probability that a given break-up channel will have n components. Thus 69% of the resonances will couple to only two components, 24% to three components and 7% to four or more components, with the average number being 2.4.

Chiu and Heimann (1971) have incorporated the effects of spin on the hadron spectrum. For the most part this extra sophistication does not alter the basic results of the present work.

3. The Equation of State

With the level density of hadron states at hand, it is straightforward to calculate the equation of state. To be consistent we must assume that hadron interactions in the macroscopic physical system of interest can be treated in the same manner as were the internal constituents of each hadron. That is, we must assume that the level density implicitly accounts for all hadron interactions and hence that the hadrons can be formally treated as free particles. The validity of the level density of (6) or (7) in the context in which it is used here will be examined later.

In general the partition function for a non-interacting ensemble of particles whose mass distribution is given by the level density $\rho(m)$ is (taking $c=k=1$),

$$\ln Z = \frac{4\pi V}{h^3}\left[\int \varrho_{\text{bosons}}(m)\,\mathrm{d}m \int \ln\left(\frac{1}{1-\Lambda}\right) p^2\,\mathrm{d}p + \right.$$
$$\left. + \int \varrho_{\text{fermions}}(m)\,\mathrm{d}m \int \ln(1+\Lambda)\,p^2\,\mathrm{d}p\right], \tag{16}$$

where

$$\Lambda = \exp\{[\mu - (p^2+m^2)^{1/2}]/T\}. \tag{17}$$

The parameters of the equation of state, pressure, number density and mass-energy density are obtained from the partition function,

$$P = T\left.\frac{\partial \ln Z}{\partial V}\right|_{T,\,\mu}, \tag{18}$$

$$n_i = T\left.\frac{\partial \ln Z}{\partial \mu_i}\right|_{T,\,V}, \tag{19}$$

$$\varepsilon = T^2\left.\frac{\partial \ln Z}{\partial T}\right|_{V,\,\mu} + \mu T\left.\frac{\partial \ln Z}{\partial \mu}\right|_{T,\,V} + \int \varrho_{\text{bosons}}(m)\,n(m)\,m\,\mathrm{d}m. \tag{20}$$

Note that for temperatures much greater than the Fermi energy all particles will obey Maxwell-Boltzmann statistics so that using the level density of (6),

$$\ln Z \propto \int_0^\infty m^{3/2}\varrho(m)\,e^{-m/T}\,\mathrm{d}m \propto \int_m^\infty m^{3/2+a}e^{(b-1/T)m}\,\mathrm{d}m. \tag{21}$$

Thus the partition function does not converge and thermodynamics is not valid if $T>b^{-1}$. This is the source of Hagedorn's (1965) by now famous statement that the exponential level density implies a finite limiting temperature to any thermodynamical system of order 160 MeV. As energy is put into the system it goes into the rest mass of new particles rather than into the kinetic energy of existing particles.

For application to hadron star matter we must make the opposite assumption, namely that the temperature is much less than the Fermi energy, which we idealize by specifically assuming $T=0$. In this case we find

$$P = \frac{\pi}{6h^3}\int_0^\infty \varrho_{\text{fermion}}(m)\,m^4 f(x)\,\mathrm{d}m, \tag{22}$$

$$\varepsilon = \int_0^\infty \varrho_{\text{bosons}}(m)\, n(m)\, m\, \mathrm{d}m + \frac{\pi}{6h^3} \int_0^\infty \varrho_{\text{fermions}}(m)\, m^4 g'(x)\, \mathrm{d}m\,, \tag{23}$$

where

$$x(m) = \frac{[\mu^2(m) - m^2]^{1/2}}{m}\,, \tag{24}$$

and f and g' are the standard Chandrasekhar functions (the g' contains the rest mass contribution to the energy density),

$$f = x(1 + x^2)^{1/2}(2x^2 - 3) + 3\ln[x + (1 + x^2)^{1/2}]\,, \tag{25}$$

$$g'(x) = g(x) + 8x^3 = 3x(1 + x^2)^{1/2}(2x^2 + 1) - 3\ln[x + (1 + x^2)^{1/2}]\,. \tag{26}$$

The formalism of Ambartsumyan and Saakyan (1960) then gives the chemical potentials as

$$\mu_{\text{lepton}} = \mu_{e^-} = \mu_{\mu^-} = m_\pi \tag{27}$$

$$\mu_{\text{meson}} = -\, m_\pi Q_{\text{meson}} \tag{28}$$

$$\mu_{\text{baryon}} = \mu_{BQ} = \mu_n B_{\text{baryon}} - m_\pi Q_{\text{baryon}} \tag{29}$$

in terms of the charge (Q) and baryon number (B) where the chemical potential of the neutron is the same as for all $B=1$, $Q=0$ particles. Since baryons with mass greater than their chemical potential μ_{BQ} are prohibited and since there is a lowest mass particle for given B and Q, the integrals for the equation of state variables split into a sum over B and Q of integrals with finite limits. The lepton contribution is fixed at a negligible level and dropping it allows algebraic elimination of the meson contribution giving the following expressions for pressure, energy density and baryon number density;

$$P = \frac{\pi}{6h^3} \Sigma_{B,Q} \int_{m^0_{BQ}}^{\mu_{BQ}} \varrho_{BQ}(m)\, m^4 f(x)\, \mathrm{d}m\,, \tag{30}$$

$$\varepsilon = \frac{4}{3}\frac{\pi m_\pi}{h^3} \Sigma_{B,Q} Q \int_{m^0_{BQ}}^{\mu_{BQ}} \varrho_{BQ}(m)\, m^3 x^3\, \mathrm{d}m + \frac{\pi}{6h^3} \Sigma_{B,Q} \int_{m^0_{BQ}}^{\mu_{BQ}} \varrho_{BQ}(m)\, m^4 g'(x)\, \mathrm{d}m\,, \tag{31}$$

$$n_B = \frac{4\pi}{3h^3} \Sigma_{B,Q} B \int_{m^0_{BQ}}^{\mu_{BQ}} \varrho_{BQ}(m)\, m^3 x^3\, \mathrm{d}m\,. \tag{32}$$

Using the level density

$$\varrho_{BQ} = c_{BQ} m^a e^{bm}\,, \tag{33}$$

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gives a 'peak' in the integrands as the combined result of the exponential rise and the cutoff at μ_{BQ}. Thus the main contribution to the integrals always comes from masses within about b^{-1} of the cutoff. The ratio of the (Fermi) kinetic energy to the mass is then of order $1/b\mu_{BQ}$. The smallest μ_{BQ} can be is about m_n in which case $1/b\mu_{BQ} \sim 6$. Assuming that μ_{BQ} is always much greater than b^{-1} implies the particles will always be non-relativistic which means $f(x) \simeq 8/5\, x^5$ and $g'(x) \simeq 8\, x^3$.

The 'peak' in the integrands also allows the integrals to be approximated by the asymptotic value of a modified Bessel function of argument $b\mu_{BQ}$ which is again taken to be $b\mu_{BQ} \gg 1$.

Equations (30–32) then become

$$P = \frac{1}{b\mu_n} \Sigma_{B,Q} \frac{\varepsilon_{BQ}}{B}, \tag{34}$$

$$\varepsilon = \Sigma_{B,Q} \varepsilon_{BQ}, \tag{35}$$

$$n_B = \frac{\varepsilon}{\mu_n}, \tag{36}$$

where the partial density $\varepsilon_{B,Q}$ is given by

$$\varepsilon_{B,Q} = \frac{(2\pi)^{3/2}}{h^3} \mu_n c_{BQ} B b^{-5/2} \mu_{BQ}^{3/2+a} \exp b\mu_{BQ}. \tag{37}$$

The exponential

$$e^{b\mu_{BQ}} = e^{b(\mu_n B - m_\pi Q)} \tag{38}$$

dominates ε_{BQ} and so the class of baryons with highest B will dominate the sum in (35) giving

$$\varepsilon \simeq \Sigma_Q \varepsilon_{B_{max},Q}. \tag{39}$$

Using Hagedorn's (1965) interpretation of b^{-1} as the limiting temperature T_0 then gives

$$P = n_B T_0, \tag{40}$$

$$= \frac{\varepsilon}{\mu_n} T_0, \tag{41}$$

taking, in the absence of any evidence to the contrary, $B_{max} = 1$. See Wheeler (1971) for the details of this derivation.

The form of (40a) is not surprising since the assumptions made here imply that noninteracting nonrelativistic particles with Fermi kinetic energy $\sim b^{-1} = T_0$ dominate the equation of state. Note from (37) and (38) that the chief dependence is $\varepsilon \propto e^{\mu_n}$ so that asymptotically (40b) will become

$$P \propto \frac{\varepsilon}{\ln \varepsilon}. \tag{42}$$

This result has been derived independently by Hagedorn according to Rhoades and

Ruffini (1971). A thorough discussion of the physics associated with the exponential hadron level density has been given by Lee *et al.* (1971) followed by discussions of the zero temperature case by Leung and Wang (1971, 1972). See also the contribution by Wang in this volume.

In particular by taking $a=-3$, neglecting $m_\pi Q$ with respect to μ_n in the polynomial of μ_{BQ} and eliminating μ_n by (40b), the expression for the equation of state becomes

$$P = \frac{(2\pi)^{3/2}}{h^3} \frac{T_0^2}{(\varepsilon/P)^{3/2}} c_{\text{eff}} e^{\varepsilon/P}, \tag{43}$$

where

$$c_{\text{eff}} = \Sigma_Q c_{1,Q} e^{-m_\pi Q}, \tag{44}$$

which has dimension (mass)^2.

Hamer and Frautschi (1971) find a value for the parameter c of between $(1-4)\, m_\pi^2$. A value for c_{eff} of m_π^2 implies that $\varepsilon \sim 5\times 10^{14}$ g cm^{-3}, a typical nuclear density, when $\mu_n = m_n$.

Using $c_{\text{eff}} = T_0^2 = m_\pi^2$, the approximate numerical representation of (43) is

$$P \simeq 5\times 10^{33} (\varepsilon/P)^{-3/2} \exp(\varepsilon/P) \text{ dyne cm}^{-2}, \tag{45}$$

where for convenience of expression the units of ε are taken to be erg cm^{-3}. Figure 2

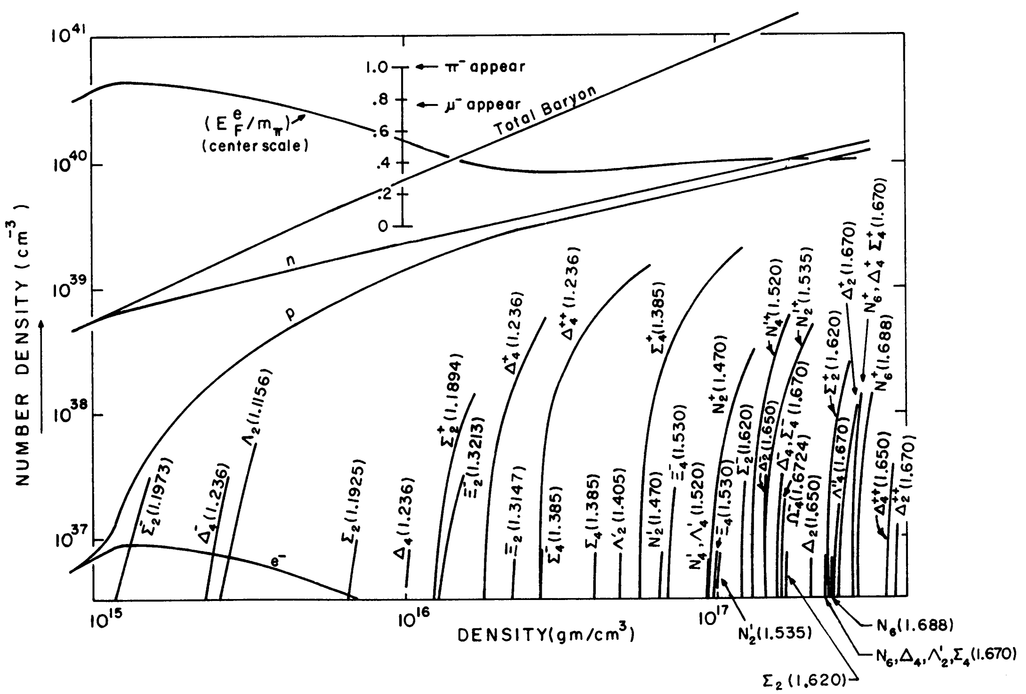

Fig. 2. The number densities of all known baryons with mass less than 1.7 BeV as a function of density when treated as independent particles, as given by Leung and Wang (1971). (Courtesy C. G. Wang.)

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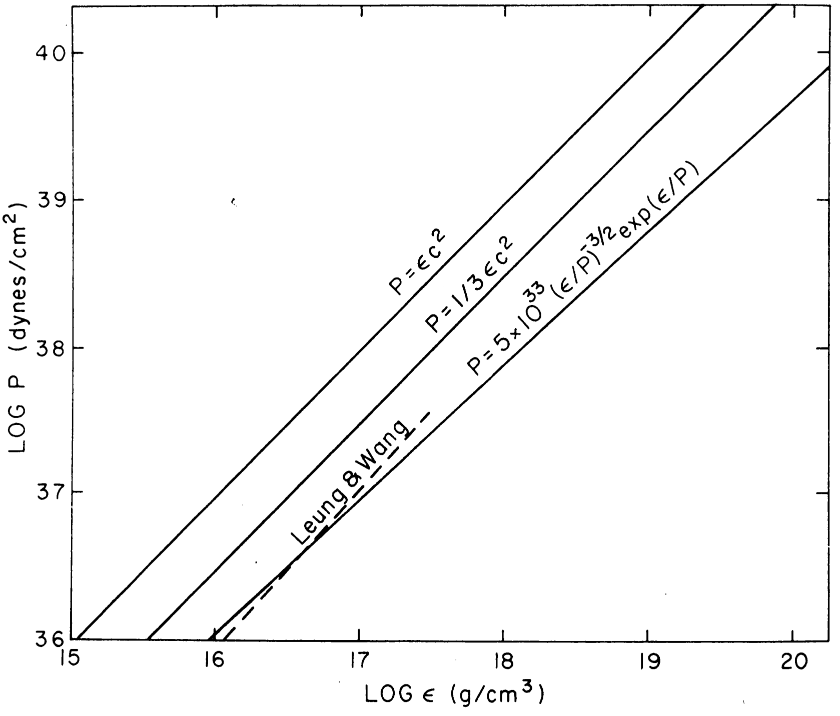

Fig. 3. The statistical bootstrap equation of state is shown in comparison with two common limiting cases, that for free relativistic particles ($P = 1/3\ \varepsilon$), and the causality limit ($P = \varepsilon$). Also shown are the numerical results of Leung and Wang (1971) based on the known hadron spectrum.

shows the number densities of the known resonances as a function of density as given by Leung and Wang (1971) who incorporated all known particles in a noninteracting gas in the spirit of the statistical bootstrap treatment. Equation (45) is plotted in Figure 3 along with the results of the calculations of Leung and Wang (1971).

As mentioned previously Hamer and Frautschi (1971) find that the form of the level density in (6) gives a good fit at masses as low as 10 m_π, corresponding to a density of about 10^{17} g cm^{-3}. Leung and Wang (1972) have come to the same conclusion in fitting the known hadron spectrum with a power law. Thus, while there may be criticisms of the applicability of the level density of (6) at any density, there is no question that the equation of state (45) becomes inapplicable at densities less than about 10^{17} g cm^{-3}. Fortunately, this is close to the regime where the known resonances can be employed as indicated in Figure 2 and hence the present analytic result can be used to extrapolate the numerical study of Leung and Wang (1971). Some adjustment, for instance of the parameter c_{eff}, might smooth the extrapolation. No such attempt has been made at present.

4. Discussion

The direct use of the equation of state derived here is limited by its region of applicability. Although there are still uncertainties in the details, it is generally agreed that the maximum central density expected in stable hadron stars is about 10^{15} g cm^{-3}. Thus a central density greater than 10^{17} g cm^{-3} would imply a mass far greater than the stability limit.

The value of the statistical bootstrap concept to hadron stars is that it provides a high density boundary condition on the equation of state. The equation of state at sub-nuclear densities is by now fairly well established (cf. the contributions by Negele and Buchler in this volume). Unfortunately the region around the critical central density where the maximum allowable mass of a hadron star occurs is still plagued by insufficient knowledge of the nuclear physics. Thus until these problems are solved the best one can do is to extrapolate to possible equations of state at much higher densities. The statistical bootstrap equation of state gives a serious alternative high density limit to be considered along with those used in the past, mainly on the basis of simplicity, i.e. $p=1/3\ \varepsilon$ for a strictly non-interacting gas of relativistic particles or $p=\varepsilon$, the causality limit. Both the latter limits tend to imply a relatively 'stiffer' equation of state than the statistical bootstrap case in the region around the critical density.

Leung and Wang (1971) have used their bootstrap equation of state to extrapolate in a straightforward way from sub-nuclear densities and have computed a maximum neutron star mass of around one half solar mass. Their result is all the more intriguing in light of the fact that a recent estimate of a neutron star mass, that of Wilson (1972) for the X-ray pulsar Centaurus X-3, is very low, one tenth of a solar mass. On the other hand Börner, in this volume, argues that the mass of the Crab pulsar is $\sim 1.2\ M_{\odot}$. In any case the extrapolation of the equation of state is not unique since a 'kink' in the equation of state at just the right density due to some quirk of the nuclear potential might raise the limiting neutron star mass considerably from the result of Leung and Wang and still be consistent with the statistical bootstrap results at higher densities. The limiting mass of a neutron star is ultimately a problem for nuclear physics, not hadron physics.

Some cosmologists, noting the difficulty in forming galaxies in the context of a hot big bang model have considered a cold big bang which expands from the singularity at zero temperature (cf. Layzer, 1969). The equation of state discussed here would seem to be relevant in such a situation.

The work outlined here can only be regarded as a first step. Many potential difficulties can be discovered. A few of them and their possible effects will be mentioned.

As noted previously the expression for the level density (4) does not incorporate the Pauli principle and is thus actually inappropriate to problems dealing with a degenerate Fermi gas. Frautschi (1971) has checked that inclusion of the Pauli principle does not significantly alter the asymptotic expression for the level density for cases where degeneracy is not important. Thus there is certainly hope that the exponential level density which is the mainstay of the present work will be retained when the degenerate case is handled more rigorously.

Sawyer (1972) has discussed one possible effect of the Pauli principle. This is that due to the high Fermi level of its decay products the mass of a given resonance may be shifted upward from the mass determined in the laboratory where the resonance is free to decay with no phase space restrictions. Leung and Wang (1972) consider this point and argue that the exponential level density could still be expected to result although perhaps based on an 'effective' baryon mass spectrum. The basic idea is that, although the mass of each resonance might be shifted upward by some fraction, this effect would not have any ultimate bearing on the basic form of the level density since an exponential increase will always eventually dominate a geometric increase. The question of whether the level spacing between baryons, i.e., T_0, could change with density is examined in a somewhat different context below.

Another serious point is that in practice the hadron spatial distribution is anything but that of a dilute gas. Frautschi *et al.* (1972) have worked out the effective volume of the hadrons in the limit as $T \to T_0$ and find that it is somewhat larger than the actual volume assumed for the hadrons. Thus, although the situation is marginal the results are reasonably consistent with the assumptions of the statistical bootstrap model.

The situation is more severe in the zero temperature case. Writing out (36) again neglecting $m_\pi Q$ in the power of μ_{BQ} gives

$$n_B = \frac{(2\pi)^{3/2}}{h^3} \frac{T_0^{5/2}}{\mu_n^{3/2}} \Sigma_{B,Q} c_{BQ} B^{-1/2} \exp(\mu_n B - m_\pi Q)/T_0. \tag{46}$$

Eliminating a weighting factor B gives the number density of hadrons (with $B_{max} = 1$)

$$n_h = \frac{(2\pi)^{3/2}}{h^3} \frac{T_0^{5/2}}{\mu_n^{3/2}} e^{\mu_n/T_0} c_{eff}, \tag{47}$$

with c_{eff} as in (43). Frautschi's constraint (14) gives an expression for the assumed hadron volume

$$V = \frac{\ln 2 \, h^3 m_\pi^{1/2}}{2(2\pi)^{3/2} T_0^{3/2} c}. \tag{48}$$

The ratio of the actual assumed volume to the effective volume $V_{eff} \equiv 1/n_h$, taking $m_\pi \sim T_0$ and $c \sim c_{eff}$, is then

$$\frac{V}{V_{eff}} = \frac{\ln 2}{2} \left(\frac{\mu_n}{T_0}\right)^{-3/2} e^{\mu_n/T_0}. \tag{49}$$

This number should be less than unity for self-consistency. However, even at $\mu_n = m_n$ (49) gives $V/V_{eff} \simeq 9$. In actuality there are more than 9 hadrons in the volume assumed for one.

At this point the pessimist will argue that application of the exponential level density simply makes no sense. A more optimistic outlook would be to say that the basic results are reasonable even though there is some scrambling of construct with constituent. The constituent states were, after all, assumed to be completely overlapping within the hadron volume. A third alternative would be to take advantage of the result of Hamer and Frautschi (1971) that the parameter T_0 actually scales with the hadron volume V. A consistent scheme can be constructed in which the basic results given here hold true but where one regards $T_0 = T_0(V)$.

Suppose, for instance, that in the degenerate case the hadron volume continually adjusts itself to the condition which holds naturally in the high temperature case, namely $V \sim V_{\text{eff}}$. By requiring

$$V_{\text{eff}} = \alpha V , \tag{50}$$

where $\alpha \geqslant 1$ and employing the empirical scaling law of Hamer and Frautschi (1971)

$$T_0 = T_{0,0} \left(\frac{V}{V_0} \right)^{-1/3} , \tag{51}$$

where $T_{0,0}$ and V_0 are the standard values, one finds the following relation

$$\frac{\ln 2}{2} \alpha \left(\frac{T_{0,0}}{\mu_n} \right)^{3/2} = \left(\frac{V}{V_0} \right)^{1/2} \exp \left[- \frac{\mu_n}{T_{0,0}} \left(\frac{V}{V_0} \right)^{1/3} \right]. \tag{52}$$

Solving (52) gives

$$T_0 \simeq \left(1 - 2 \ln \frac{\alpha}{1.03} \right) \mu_n . \tag{53}$$

Thus (40b) becomes

$$P = \left(1 - 2 \ln \frac{\alpha}{1.03} \right) \varepsilon . \tag{54}$$

That is, the equation of state again becomes $P \propto \varepsilon$, with causality intact if $\alpha = V_{\text{eff}}/V >$ > 1.03, and positive pressure if $\alpha < 1.03\, e^{1/2} \simeq 1.7$.

Perhaps the real value of the study of the statistical bootstrap model in the context of cold ultradense matter is not that it will teach us more about the nature of hadron stars but that it will lead us to new insights into the nature of hadron physics itself.

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