FINITENESS AT INFINITY

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1. Introduction and preliminaries

If X is a Tychonoff topological space, and if βX is the Stone-Cech compactification of X, then $\beta X \backslash X$ will denote the complement of X in βX. If A is a subset of X, then cl $[A: X]$ will denote the closure of A in X, and int $[A: X]$ will denote the interior of A in X. In Isbell ((3), p. 119) a property of $\beta X \backslash X$ is called a *property which X has at infinity*, and it is the aim of this paper to give necessary and sufficient conditions for X to be finite at infinity. Since βX is T_1 we can say that if X is finite at infinity, then $\beta X \backslash X$ is closed in βX. So we lose nothing by restricting our attention to locally compact, Tychonoff spaces, and for the remainder of the paper X will denote such a space.

The closed sets A and B are said to be *completely separated* if and only if there exists a bounded, continuous, real-valued function f on X taking the value 0 on A and the value 1 on B. From the description of βX in terms of such functions we see that A and B are completely separated in X if and only if cl $[A: \beta X] \cap$ cl $[B: \beta X] = \emptyset$.

Unless otherwise stated, uniformities on X are assumed to be compatible with the topology of X.

The following result is due to Doss ((1), p. 20): *there is just one uniform structure on X if and only if, of any two completely separated closed sets in X, one is compact.*

Let $C^*(X)$ be the algebra of bounded, continuous real-valued functions on X with the topology of uniform convergence on the space X. Let $A(X)$ be the subalgebra of $C^*(X)$ consisting of those functions each of which is constant on the complement of some compact subset of X. Gál ((2), p. 1053) has proved: *there is just one uniform structure on X if and only if $A(X)$ is dense in $C^*(X)$.*

Now there is just one uniform structure on X if and only if βX is the Alexandroff compactification of X, so we can write the preceding results in the form

The following statements are equivalent:

(*a*) $|\beta X \backslash X| = 1$.

(*b*) *Of any two closed, completely separated sets in X, at least one is compact.*

(*c*) *$A(X)$ is dense in $C^*(X)$.*

In this form, we extend these results in the next three sections.

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2. Conditions on X for X to have power n at infinity

Lemma 2.1. *If $|\beta X \backslash X| = m$ (m any cardinal), then any collection C of closed sets in X, completely separated in pairs and such that $|C| > m$, contains a compact member.*

Proof. Suppose $C = \{C_\lambda: \lambda \in \Lambda\}$ where $|\Lambda| > m$. Then

$$\{\mathrm{cl}\,[C_\lambda: \beta X]: \lambda \in \Lambda\}$$

is a collection of sets disjoint in pairs, and so no element of $\beta X \backslash X$ belongs to more than one member of this collection. Since $|\Lambda| > |\beta X \backslash X|$ there exists $\mu \in \Lambda$ such that $\mathrm{cl}\,[C_\mu: \beta X] = C_\mu$.

Then C_μ is compact as required.

We can now give the required condition on X.

Theorem 2.2. $|\beta X \backslash X| \leqq n$ *(n finite) if and only if, of any $n+1$ closed sets in X, completely separated in pairs, at least one is compact.*

Proof. Suppose $|\beta X \backslash X| > n$ and choose $n+1$ distinct points $a_1, \ldots, a_{n+1}$ in $\beta X \backslash X$. Let $U_1, \ldots, U_{n+1}$ be neighbourhoods of $a_1, \ldots, a_{n+1}$ respectively. Since βX is T_2 and compact, we can assume these neighbourhoods to be compact and disjoint in pairs. Now let $B_i = U_i \cap X \neq \varnothing$, so that $\mathrm{cl}\,[B_i: \beta X] \subseteq U_i$ ($i = 1, \ldots, n+1$). Thus the B_i's are completely separated in pairs and each B_i is closed in X. On the other hand, each B_i fails to be compact, since it contains a net converging to $a_i \in \beta X \backslash B_i$ ($i = 1, \ldots, n+1$).

The reverse implication follows from Lemma 2.1.

Note that the inequality can be removed from the statement of Theorem 2.2 to give:

Theorem 2.2′. $|\beta X \backslash X| = n$ *(n finite) if and only if there exist n closed, non-compact sets in X which are completely separated in pairs, but no such collection of $n+1$ sets exists.*

3. Conditions on $C^*(X)$ for X to have power n at infinity

Define $B_n(X)$ to be the subset of $C^*(X)$ consisting of those functions which take at most n values (n finite) on the complement of a compact subset of X.

Theorem 3.1. *If $|\beta X \backslash X| = n$ (n finite), then $B_n(X)$ is uniformly dense in $C^*(X)$.*

Proof. Let $f: X \to [0, 1]$ be continuous and denote by $\bar{f}$ the continuous extension of f to βX. Suppose $\beta X \backslash X = \{a_1, \ldots, a_n\}$ and $\bar{f}(a_i) = b_i$ ($i = 1, \ldots, n$). If necessary, relabel $b_1, \ldots, b_n$ as the distinct numbers $c_1, \ldots, c_m$ with $m \leqq n$.

Let $\varepsilon > 0$ be given, and let δ be equal to the smallest member of

$$\{\{\varepsilon\} \cup \{|c_i - c_j|: i \neq j\}\}.$$

Put $U_i = \{x \in \beta X: |\bar{f}(x)-c_i| < \frac{1}{2}\delta\}$ $(i = 1, 2, \ldots, m)$. For each $i \in \{1, \ldots, m\}$ choose a closed neighbourhood (in βX) V_i of $\{a_j: \bar{f}(a_j) = c_i, j = 1, \ldots, n\}$ such that $V_i \subseteq U_i$.

Since βX is normal, given $i \in \{1, \ldots, m\}$ there exists $\bar{g}_i: \beta X \to [0, 1]$ such that $\bar{g}_i$ is continuous and

$$\begin{cases} \bar{g}_i(x) = 0 & (x \in V_i) \\ \bar{g}_i(x) = 1 & (x \in \beta X \backslash U_i). \end{cases}$$

Denote by g_i the restriction of $\bar{g}_i$ to X $(i = 1, \ldots, m)$ and define

$$h_1 = c_1 + (f - c_1)g_1,$$

$$h_i = c_i + (h_{i-1} - c_i)g_i \quad (i = 2, \ldots, m).$$

Then for each $i \in \{1, \ldots, m\}$ the following properties (*a*), (*b*), (*c*) hold:

(*a*) $h_i: X \to [0, 1]$ is continuous.

Clearly, h_i is continuous. Also, $h_i(x) = c_i[1 - g_i(x)] + h_{i-1}(x).g_i(x)$, which is a number between c_i and $h_{i-1}(x)$. A simple finite induction argument shows that $h_i(x) \in [0, 1]$ $(x \in X, i \in \{1, \ldots, m\})$, establishing (*a*).

(*b*) $h_i(x) = c_j$ $(x \in X \cap \text{int}[V_j: \beta X])$ for each $j \in \{1, \ldots, i\}$.

To prove this let $W_j = X \cap \text{int}[V_j: \beta X] (j \in \{1, \ldots, m\})$. Clearly, $h_1(x) = c_1$ $(x \in W_1)$. Suppose then that $h_k(x) = c_j$ $(x \in W_j, 1 \leqq j \leqq k)$ for each $k \in \{1, \ldots, i-1\}$. Consider $h_i(x) = c_i + (h_{i-1}(x) - c_i)g_i(x)$.

Then $g_i(x) = 1$ $(x \in W_j, 1 \leqq j < i)$, and so $h_i(x) = h_{i-1}(x) = c_j$ $(x \in W_j, 1 \leqq j < i)$. Also $g_i(x) = 0$ $(x \in W_i)$, whence $h_i(x) = c_i$ $(x \in W_i)$. This completes the proof of (*b*).

(*c*) $$h_i(x) = f(x) \quad \left(x \in X \backslash \left[X \cap \bigcup_{j=1}^{i} U_j\right]\right).$$

The proof is straightforward.

In particular, (*a*), (*b*) and (*c*) hold for h_m.

We now show that $|h_m(x) - f(x)| < \varepsilon$ $(x \in X)$. In fact, from (*b*) and (*c*), the inequality need only be proved for x in $T_i = X \cap (U_i \backslash V_i)$ $(i \in \{1, \ldots, m\})$. Clearly,

$$|h_1(x) - f(x)| \leqq \tfrac{1}{2}\delta < \varepsilon \quad (x \in T_1).$$

Suppose then that $|h_i(x) - f(x)| < \varepsilon$ $(x \in T_j, j \in \{1, \ldots, i\})$ for each $i = 1, \ldots, k-1$. Consider h_k on T_j $(j \in \{1, \ldots, k-1\})$. Since $j < k$, $g_k(x) = 1$ $(x \in T_j)$. Thus $h_k(x) = c_k + (h_{k-1}(x) - c_k)g_k(x) = h_{k-1}(x)$. Next, on T_k, $h_{k-1}(x) = f(x)$, so that

$$|h_k(x) - f(x)| = |c_k + [f(x) - c_k]g_k(x) - f(x)|$$
$$= |c_k - f(x)|.|1 - g_k(x)| \leqq \tfrac{1}{2}\delta < \varepsilon.$$

By the principle of finite induction, the required inequality follows.

Finally, for each $i \in \{1, \ldots, m\}$, $\text{int}[V_i: \beta X]$ is an open neighbourhood of a_i in βX having empty intersection with $Y = X \backslash \left(\left[\bigcup_{i=1}^{m} \text{int}[V_i: \beta X]\right] \cap X\right)$ and

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so Y is compact. But from (*b*) we see that h_m takes m values on $X\backslash Y$, and so

$$f \in \mathrm{cl}\,[B_m(X)\colon\ C^*(X)] \subseteq \mathrm{cl}\,[B_n(X)\colon\ C^*(X)].$$

It follows that $B_n(X)$ is uniformly dense in $C^*(X)$.

Lemma 3.2. *If $B_n(X)$ is uniformly dense in $C^*(X)$ then $|\beta X\backslash X| \leqq n$.*

Proof. Suppose $|\beta X\backslash X| > n$ and let $a_1, \ldots, a_{n+1}$ be distinct elements of $\beta X\backslash X$. Choose, for $i = 1, \ldots, n+1$, mutually disjoint open neighbourhoods U_i, in βX, of a_i, together with closed neighbourhoods V_i of a_i such that $V_i \subseteq U_i$. Since βX is normal, given $i \in \{1, \ldots, n+1\}$, there exists a continuous function $f_i\colon \beta X \to [0, i]$ such that $f_i(V_i) = i$ and $f_i(\beta X\backslash U_i) = 0$.

Now define $f\colon \beta X \to [0, n+1]$ by $f = \sum_{i=1}^{n+1} f_i$. Then f is continuous and $f(a_i) = i$ $(i = 1, \ldots, n+1)$. Let h denote the restriction of $(n+2)^{-1}f$ to X. Then

$$h(x) = (n+2)^{-1}i \quad (x \in X \cap \mathrm{int}\,[V_i\colon\ \beta X],\ i = 1, \ldots, n+1).$$

Let $g \in B_n(X)$ be such that $\sup\{|h(x) - g(x)|\colon x \in X\} < \tfrac{1}{2}(n+2)^{-1}$ and let K be a compact set such that outside K, g assumes at most n distinct values. Since K is compact, for each $i \in \{1, \ldots, n+1\}$ there is a neighbourhood W_i of a_i in βX such that $K \cap W_i = \varnothing$. Let $A_i = X \cap \mathrm{int}\,[V_i\colon\ \beta X] \cap W_i$. Then $A_i \neq \varnothing$, and $h(x) = (n+2)^{-1}i$ $(x \in A_i;\ i = 1, \ldots, n+1)$. Clearly, we cannot have

$$|h(x) - g(x)| < \tfrac{1}{2}(n+2)^{-1} \quad \left(x \in \bigcup_{i=1}^{n+1} A_i\right)$$

unless g takes at least $n+1$ distinct values on the complement of K. This implies that $B_n(X)$ is not uniformly dense in $C^*(X)$.

From Theorem 3.1 and Lemma 3.2 we then get

Theorem 3.3. *$|\beta X\backslash X| \leqq n$ (n finite) if and only if $B_n(X)$ is uniformly dense in $C^*(X)$.*

Once again we note that the inequality in Theorem 3.3 can be removed to give

Theorem 3.3′. *$|\beta X\backslash X| = n$ (n finite) if and only if $B_n(X)$ is uniformly dense in $C^*(X)$ and $B_{n-1}(X)$ is not.*

We note that if aX is a compactification of X (we consider X to be a subspace of any compactification of X) and if $C_a(X)$ denotes the subset of $C^*(X)$ consisting of those functions having a continuous extension to aX, then the proofs of this section carry through, with slight modifications, to give

Theorem 3.4. *$|aX\backslash X| = n$ and aX is the unique (up to equivalence of compactifications) n-point compactification of X if and only if $B_n(X)$ is uniformly dense in $C_a(X)$ and $B_{n-1}(X)$ is not.*

4. Conditions on X for X to be finite at infinity

Theorem 4.1. *X is finite at infinity if and only if every infinite collection of closed sets in X, completely separated in pairs, contains a compact member.*

Proof. Suppose X is infinite at infinity and let (x_n) be an infinite sequence of distinct points of $\beta X \backslash X$. Let x be a cluster point of (x_n) in

$$\beta X \backslash \{x_n \colon n = 1, 2, \ldots\}.$$

We define a sequence (y_n) recursively as follows. Put $y_1 = x_1$. Then there exist disjoint closed neighbourhoods U_1, V_1 of x and y_1 respectively in βX. Since x is a cluster point of (x_n) there is x_{n_2} in U_1. Set $y_2 = x_{n_2}$. Then choose a closed neighbourhood V_2 of y_2 and a closed neighbourhood U_2 of x such that $V_2 \cap V_1 = V_2 \cap U_2 = \varnothing$. Now suppose $y_1, \ldots, y_k$ have been chosen with

$$y_i \in U_{i-1},\ V_i \cap U_i = \varnothing,\ V_i \cap \bigcup_{j=1}^{i-1} V_j = \varnothing \quad (i = 1, \ldots, k).$$

(This is possible since the V_i are compact.) Since x is a cluster point of (x_n), there is $x_{n_{k+1}}$ in U_k. Set $y_{k+1} = x_{n_{k+1}}$. Choose neighbourhoods V_{k+1} of y_{k+1} and U_{k+1} of x such that $V_{k+1} \cap \bigcup_{i=1}^{k} V_i = \varnothing$ and $V_{k+1} \cap U_{k+1} = \varnothing$.

This process defines a sequence (V_k) of closed (compact) subsets of βX which are mutually disjoint. Hence $(X \cap V_k)$ is a sequence of closed subsets of X, completely separated in pairs. Since y_k in V_k is adherent to $(X \cap V_k)$ in βX, no set in this collection is compact. The desired implication follows.

On the other hand, if X is finite at infinity, Lemma 2.1 applies, and every infinite collection of closed subsets of X, completely separated in pairs, contains a compact member.

From Theorems 4.1 and 2.2 we then get the following result within X.

Corollary 4.2. *If every infinite collection of closed sets in X, completely separated in pairs, contains a compact member, then there exists an integer n such that every collection of n closed sets in X, completely separated in pairs, contains a compact member.*

Now, using Theorem 4.1 we obtain the following necessary condition for a normal space X to be finite at infinity.

Corollary 4.3. *If X is normal, then X finite at infinity implies that X is countably compact.*

Proof. Suppose X fails to be countably compact. Then there is a sequence (U_n) of open sets in X such that $U_n \subset U_{n+1}$ $(n = 1, 2, \ldots)$, $\bigcup_n U_n = X$, and no finite collection of U_n's covers X. Then choose $x_{n+1} \in U_{n+1} \backslash U_n$ $(n = 1, 2, 3, \ldots)$. This defines an infinite sequence (x_n) of distinct points in X. For $i = 1, 2, \ldots$, let $s_i = 2^{i-1}$ and put $A_i = \{x_{s_i(2n+1)} \colon n = 0, 1, 2, \ldots\}$.

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Now, if $x \in X \backslash A_i$, then for some integer N, $x \in U_N$. Then U_N is a neighbourhood of x containing only a finite number of members of A_i, and so, since X is T_2, $x \notin \mathrm{cl}\,[A_i\colon X]$. It follows that, for each i, A_i is closed in X.

We see next that the members of $\{A_i\colon i = 1, 2, \ldots\}$ are disjoint in pairs, for if $i<j$ and $x_{s_i(2n+1)} = x_{s_j(2m+1)}$ for some m, n, then since the members of (x_n) are distinct, $s_i(2n+1) = s_j(2m+1)$ and so $2n+1 = 2^{j-i}(2m+1)$.

So we have an infinite collection of closed sets in X, disjoint in pairs. Since X is normal we see from Urysohn's Lemma that the members of this collection are completely separated in pairs, and so from Theorem 4.1 we see that X is infinite at infinity.

Note that if Ω_0 denotes the set of all ordinal numbers less than the first uncountable ordinal, with the order topology, then $\beta(\Omega_0)$ is the set of ordinals less than or equal to the first uncountable ordinal i.e. $|\beta(\Omega_0)\backslash\Omega_0| = 1$. So by taking suitable disjoint unions of the space Ω_0 with itself we obtain examples of spaces having any given finite power at infinity.

Note also that $\Omega_0 \times \Omega_0$ is normal, T_2, countably compact and locally compact, and so, since $\beta(\Omega_0 \times \Omega_0) = \beta(\Omega_0) \times \beta(\Omega_0)$, we have a counterexample to the converse implication to Corollary 4.3.

Finally, we note that since $\beta X \backslash X$ is finite if and only if $\beta X \backslash X$ is discrete, we can replace " finite at infinity " throughout by " discrete at infinity ".

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