A comparison theorem and
the forced oscillation

Hiroshi Onose

Consider the n-th order delay differential equation

(A) $$x^{(n)}(t) + f\big(t, \tilde{x}(\tilde{g}_0(t))\big) = q(t) \ .$$

In the last few years, the oscillatory behavior of delay differential equations has been the subject of intensive investigations. But much less is known about the equation (A) with small forcing term $q(t)$. The only papers devoted to this problem are by Kartsatos, Kusano, and the present author. The purpose of this paper is to prove some new oscillation theorems which contain the previous results.

1. Introduction

We are concerned with the oscillatory behavior of solutions of the retarded differential equation

(A) $$x^{(n)}(t) + f\big(t, \tilde{x}(\tilde{g}_0(t))\big) = q(t) \ ,$$

where $\tilde{x}(\tilde{g}_0(t)) \equiv \big(x(g_{01}(t)), \ldots, x(g_{0(n-1)}(t))\big)$

We only consider solutions of (A) which are defined and nontrivial for all large t . A solution $x(t)$ of (A) is called oscillatory if there exists a sequence $\{t_m\}_{m=1}^{\infty}$ such that $\lim_{m\to\infty} t_m = \infty$ and $x(t_m) = 0$ for all m . Otherwise the solution is called nonoscillatory. Equation (A) itself

---

is called oscillatory if all solutions of (A) are oscillatory. R^d denotes the d-dimensional space of vectors $\tilde{x} = (x_1, \ldots, x_d)$. The zero vector in R^d is denoted by $\tilde{0}$. Inequality between vectors $\tilde{x}$ and $\tilde{y} = (y_1, \ldots, y_d)$ is defined as

$\tilde{x} \geq \tilde{y}$ $[\tilde{x} > \tilde{y}]$ if and only if $x_j \geq y_j$ $[x_j > y_j]$ for $j = 1, \ldots, d$.

2. Comparison theorem

First we mention Kiguradze's [2] result:

LEMMA. *If* $x(t) \in C^n[a, \infty)$, $x(t) \geq 0$ *and* $x^{(n)}(t)$ *is nonnegative on* $[a, \infty)$, *then either*

> (I) $x'(t), \ldots, x^{(n-1)}(t)$ *tend monotonically to zero as* $t \to \infty$; *or*

> (II) *there exists an odd integer* k , $1 \leq k \leq n-1$, *for* n *even and an even integer* k *for* n *odd, such that*
> $$\lim_{t \to \infty} x^{(n-j)}(t) = 0 \quad for \quad 1 \leq j \leq k-1 \ , \quad \lim_{t \to \infty} x^{(n-k)}(t) \geq 0 \ ,$$
> $$\lim_{t \to \infty} x^{(n-k-1)}(t) > 0 \quad and \quad x(t), x'(t), \ldots, x^{(n-k-2)}(t)$$
> *tend to* ∞ *as* $t \to \infty$.

For simplicity, we mention the following condition:

> (a) $f(t, \tilde{y}_0) > \tilde{0}$ for $\tilde{y}_0 > \tilde{0}$, $f(t, \tilde{y}_0) < \tilde{0}$ for $\tilde{y}_0 < \tilde{0}$,
> $f(t, \tilde{y}_0) \geq f(t, \tilde{z}_0)$ for $\tilde{y}_0 \geq \tilde{z}_0 \geq \tilde{0}$,
> $g_{0j}(t) \in C[[0, \infty), R]$, $g_{0j}(t) \leq t$, $\lim_{t \to \infty} g_{0j}(t) = \infty$,
> $j = 1, \ldots, n-1$, $q(t)$ is continuous, and $f(t, \tilde{x})$ is continuous.

THEOREM 2. *Consider the equation*

(1)
$$x^{(n)}(t) + f(t, \tilde{x}(\tilde{g}_0(t))) = 0$$

and the inequality

(2)
$$x^{(n)}(t) + f(t, \tilde{x}(\tilde{g}_0(t))) \leq 0 \ .$$

Assume that (a) *holds. If, for n even, equation* (1) *is oscillatory, then inequality* (2) *is oscillatory, while for n odd, if every solution $x(t)$ of equation* (1) *is oscillatory, or* $\lim\limits_{t\to\infty} x(t) = 0$ *, then every solution of inequality* (2) *is oscillatory, or* $\lim\limits_{t\to\infty} x(t) = 0$.

Proof. Suppose that there exists a nonoscillatory solution $x(t)$ of (2). Without loss of generality we may assume that $x(t) > 0$ for $t \geq t_0$, where t_0 is a sufficiently large constant. Suppose first n to be even. From (a), there is a $t_1 \geq t_0$ such that $x\big(g_{0j}(t)\big) > 0$ $(j = 1, \ldots, n-1)$ for $t \geq t_1$. From (2), we have

$$(3) \qquad x^{(n)}(t) = -f\big(t, \tilde{x}(\tilde{g}_0(t))\big) < 0 .$$

By repeated integration of (3), we obtain

$$(4) \quad x(t) \geq c + \int_{t_2}^{t} \int_{t_2}^{u_{n-1}} \cdots \int_{t_2}^{u_{n-l+1}} \int_{u_{n-l}}^{\infty} \cdots$$

$$\cdots \int_{u_1}^{\infty} f\big(s,\, \tilde{x}(\tilde{g}_0(s))\big)\, ds\, du_1 \, \cdots \, du_{n-1}$$

$$\equiv c + \phi\big(t,\, \tilde{x}(\tilde{g}_0(t))\big) \, , \text{ for } \quad t \geq t_2 ,$$

where c is a positive constant and l is the constant of the lemma.

Now consider the equation

$$(5) \qquad z(t) = \phi\big(t,\, \tilde{z}(\tilde{g}_0(t))\big) + c .$$

If (5) has a solution $z(t) > 0$ for $t \geq t_2$, then $z(t)$ is a positive solution of (1). To prove this, we define $\{z_n(t), n = 0, 1, \ldots\}$ such that

$$z_0(t) = x(t) \qquad\qquad\qquad \text{for } \quad t \geq t_1 ,$$

$$z_{n+1}(t) = \begin{cases} c + \phi\big(t,\, \tilde{z}_n(\tilde{g}_0(t))\big) & \text{for } \quad t \geq t_2 , \\[2mm] c & \text{for } \quad t_1 \leq t \leq t_2 . \end{cases}$$

Then we see that $z_n(t)$ is well-defined, and

(6) $$0 < z_n(t) < x(t) \ , \quad c \leq z_{n+1}(t) \leq z_n(t) \ .$$

If we put

(7) $$z(t) = \lim_{n \to \infty} z_n(t) \ , \text{ for every point of } t \ \big(\geq t_1 \big) \ ,$$

then by (6), (7), and Lebesgue's Theorem, we have

$$z(t) = c + \phi\big(t, \, \tilde{z}\big(\tilde{g}_0(t)\big)\big) \ , \text{ for all } t \geq t_2 \ .$$

Hence we see that (1) has a nonoscillatory solution. Suppose n is odd. By the lemma, if $x'(t) > 0$, then the proof proceeds as in the even case. In case $x'(t) < 0$ and $\lim_{t \to \infty} x(t) \neq 0$, put $L = \lim_{t \to \infty} x(t)$ and consider the following inequality

$$x(t) \geq L/2 + \int_t^\infty \int_{u_{n-1}}^\infty \cdots \int_{u_1}^\infty f\big(s, \, \tilde{x}\big(\tilde{g}_0(s)\big)\big) ds du_1 \cdots du_{n-1} \ .$$

Then the rest of the proof is the same as in the case of even n .   //

   REMARK.   Theorem 1 shows that it is sufficient to consider only equation (1) for the oscillation theorem.

3.   Forced oscillation

   THEOREM 2. *Consider the equations*

(8) $$x^{(n)}(t) + f\big(t, \, \tilde{x}\big(\tilde{g}_0(t)\big)\big) = 0$$

and

(9) $$x^{(n)}(t) + f\big(t, \, \tilde{x}\big(\tilde{g}_0(t)\big)\big) = q(t) \ .$$

Assume that (a) *and the following condition hold:*

(10) *there exists $Q(t)$ such that $Q^{(n)}(t) = q(t)$, $\lim_{t \to \infty} Q(t) = 0$,*

      and $Q(t)$ has an unbounded set of zeros.

If, for n even, (8) is oscillatory, then (9) is oscillatory, while for n

odd, if every solution $x(t)$ *of* (8) *is oscillatory, or* $\lim\limits_{t\to\infty} x(t) = 0$,

then every solution $x(t)$ *of* (9) *is oscillatory, or* $\lim\limits_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of (9) and assume $x(t) > 0$ for $t \geq t_0$. Put $y(t) \equiv x(t) - Q(t)$; then $y(t)$ satisfies

(11) $y^{(n)}(t) + f\big(t,\, y\big(g_{01}\big) + Q\big(g_{01}\big),\, \ldots,\, y\big(g_{0(n-1)}\big) + Q\big(g_{0(n-1)}\big)\big) = 0$.

From (11), we see $y^{(n)}(t) < 0$, so that we have

(12) $y'(t)$ and $y(t)$ are of constant sign.

If we suppose $y(t) < 0$, then we have $0 < x(t) < Q(t)$; this leads to a contradiction to the assumption that $Q(t)$ has an unbounded set of zeros. So we have

(13) $y(t) > 0$.

From (12) and (13), we have that $y'(t) < 0$ can happen only for n odd. By (12), there exists a definite limit of $y(t)$,

(14) $\lim\limits_{t\to\infty} y(t) = c$, $(0 \leq c \leq \infty)$,

where $c = 0$ can only happen for the case n odd. For $c = 0$, we have that $\lim\limits_{t\to\infty} x(t) = 0$. For $c > 0$, put $z(t) \equiv y(t) - c/2$; then we have

(15) $0 < c/2 < z(t) < y(t) + Q(t)$, for $t \geq t_2 > t_1$.

From (11) and (15) we have

(16) $z^{(n)}(t) + f\big(t,\, \tilde{z}\big(\tilde{g}_0(t)\big)\big) \leq 0$ for $t \geq t_3 > t_2$.

Now we consider the equation

(17) $z^{(n)}(t) + f\big(t,\, \tilde{z}\big(g_0(t)\big)\big) = 0$ for $t \geq t_3$.

Since equation (16) has a solution $z(t)$ such that $\liminf\limits_{t\to\infty} z(t) > c/2 > 0$ we see, from Theorem 1, that not every solution $z(t)$ of (17) is oscillatory or $\lim\limits_{t\to\infty} z(t) = 0$.   //

THEOREM 3. *Assume that* (a) *and the following condition hold:*

(10') *there exist* $Q(t)$ *such that* $Q^{(n)}(t) = q(t)$ *and*
$$\lim_{t \to \infty} Q(t) = 0 .$$

For n *even, if* (8) *is oscillatory, then every solution* $x(t)$ *of* (9) *is oscillatory or* $\lim_{t \to \infty} x(t) = 0$. *For* n *odd, if every solution* $x(t)$ *of* (8) *is oscillatory or* $\lim_{t \to \infty} x(t) = 0$, *then every solution* $x(t)$ *of* (9) *is oscillatory or* $\lim_{t \to \infty} x(t) = 0$.

Proof. The proof is contained in the proof of Theorem 2.

THEOREM 4. *Assume that* (a) *and the following condition hold:*

(18) *there exist constants* q_1 , q_2 *and sequences* $\{t_s'\}$, $\{t_s''\}$ *such that* $\lim_{t \to \infty} t_s' = \lim_{t \to \infty} t_s'' = \infty$, $Q(t_s') = q_1$, $Q(t_s'') = q_2$ *and*
$$q_1 \leq Q(t) \leq q_2 .$$

For n *even, if* (8) *is oscillatory, then every solution of* (9) *is oscillatory, while for* n *odd, if every solution* $x(t)$ *of* (8) *is oscillatory or* $\lim_{t \to \infty} x(t) = 0$, *then every solution of* (9) *is oscillatory or*
$$\lim_{t \to \infty} \left(x(t) - Q(t) \right) = -q_1 \quad \left(or \quad -q_2 \right).$$

Proof. The same procedure of the proof of Theorem 2 is used. Suppose that $x(t) > 0$ is a nonoscillatory solution of (9). Put $y(t) \equiv x(t) - Q(t)$. Then we see that our proof proceeds until (12). Put $z(t) = y(t) + q_1$. Then, by the fact that $y'(t)$ is monotone, we have

(19) $$\lim_{t \to \infty} z(t) = c \quad (-\infty \leq c \leq +\infty) .$$

If c is negative, then we have $y(t) + q_1 < 0$ for sufficiently large t ; this leads to a contradiction to the fact that
$$y(t_s) + q_1 = y(t_s) + Q(t_s) = x(t_s) > 0 .$$

If $c = 0$, then we have that $\lim_{t \to \infty} \left(x(t) - Q(t) \right) = -q_1$. If c is positive,

then we have

(20) $0 < z(t) = y(t) + q_1 \leq y(t) + Q(t)$.

By (11) and (20), we have

(21) $z^{(n)}(t) + f(t, \tilde{z}(\tilde{g}_0(t))) \leq 0$ for $t \geq t_2$.

Here, we consider the following equation

(22) $z^{(n)}(t) + f(t, \tilde{z}(\tilde{g}_0(t))) = 0$ for $t \geq t_2$.

From (21), (22), and Theorem 1, we have the proof.

REMARK. The results of Kartsatos [1] and Kusano and the present author [3] are contained in Theorem 2, Theorem 3, and Theorem 4.

References

[1] Athanassios G. Kartsatos, "Maintenance of oscillations under the effect of a periodic forcing term", *Proc. Amer. Math. Soc.* 33 (1972), 377-383.

[2] И.Т. Нигурадзе [I.T. Kiguradze], "К вопросу колеблемости решений нелинейных дифференциальных уравнений" [The problem of oscillation of solutions of nonlinear differential equations], *Differencial'nye Uravnenija* 1 (1965), 995-1006; *Differential Equations* 1 (1965), 773-782.

[3] T. Kusano and H. Onose, "Oscillations of functional differential equations with retarded argument", *J. Differential Equations* 15 (1974), 269-277.

Department of Mathematics,
Ibaraki University,
Mito,
Japan.