ON PARTITIONING PLANAR GRAPHS

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In 1879 Kempe [5] presented what has become the most famous of all incorrect proofs of the Four Colour Conjecture, but even though his proof was erroneous his method has become quite useful. In 1890 Heawood [4] was able to modify Kempe's method to establish the Five Colour Theorem for planar graphs. In this article we show that other modifications of Kempe's method can be made which enable one to establish more results about planar graphs. By this process we obtain upper bounds for several parameters which involve partitioning the point set of a graph. In particular, we show that the point set of any planar graph can be partitioned into four or less subsets such that the subgraph induced by each subset is either disconnected or trivial (consists of a single point). We also show that the point set of any planar graph can be partitioned into three or less subsets such that the subgraph induced by each subset contains no cycles.

A graph G consists of a set $V = V(G)$ of points together with a set $E = E(G)$ of unordered pairs $[u, v]$ of distinct points of $V(G)$, called lines of G. A line $[u, v]$ is said to join points u and v. A graph G' is a subgraph of G, denoted $G' \subset G$, if $V' \subset V$ and $E' \subset E$. The subgraph $\langle S \rangle$ induced by a set S of points consists of the set S together with the set of all lines of G which join two points of S. Given a point $u \in V(G)$, by $G-u$ we mean the graph obtained from G by deleting point u and all lines $[u, v]$ incident with it. A complete graph K_p is a graph containing p points in which every pair of distinct points are joined by a line.

Let $\pi = \{V_1, V_2, \ldots, V_m\}$ be a partition of the set of points of a graph G. The factor graph G/π is the graph whose points are the m subsets V_i, and $[V_i, V_j] \in E(G/\pi)$ if and only if there exist points $u \in V_i, v \in V_j$ and $[u, v] \in E(G)$. A partition π is complete if G/π is a complete graph.

Let P denote any property of a graph G. A subset $S \subset V(G)$ is a P-set of G if $\langle S \rangle$ has property P. A P-colouring of G is an assignment of colours to the points of G such that for any given colour the set of points having this colour is a P-set. An (m, P)-colouring of G is a partition $\pi = \{V_1, V_2, \ldots, V_m\}$ of V(G) such that every subset V_i is a P-set. The P-chromatic number $\chi_P(G)$ of G is the smallest integer m for which G has an (m, P)-colouring. In the case where P denotes the property that a graph be totally disconnected, the terms P-colouring, (m, P)-colouring, and P-chromatic number $\chi_p(G)$ coincide with the traditional terms colouring, m-colouring, and chromatic number $\chi(G)$, respectively. For any other definitions not given here, the reader is referred to [3].

Using our terminology the Four Colour Conjecture can be stated as follows: if P_0 denotes the property that a graph be totally disconnected, then for any planar graph G, $\chi_{P_0}(G) \leqq 4$. Even though Kempe's method apparently does not enable one to settle the Four Colour Conjecture, his method does enable one to prove the following modification of this conjecture. Let P_1 denote the property that a graph be either disconnected or trivial.

THEOREM 1. *For any planar graph* G, *we have* $\chi_{P_1}(G) \leqq 4$.

Proof. We proceed by induction on the number p of points in G. Clearly this holds for all graphs having $p \leqq 4$ points. Suppose then that it holds for all planar graphs having p-1 points and let G have p points. Since G is planar it must have at least one point, say u, of degree five or less (cf. Kempe [5]).

Let u have degree $\leqq 3$, and consider the graph G-u. By hypothesis $\chi_{P_1}(G\text{-}u) \leqq 4$. Therefore since u has degree $\leqq 3$, we can always assign a fourth colour to point u given any $(4, P_1)$-colouring of G-u to obtain a $(4, P_1)$-colouring of G, i.e., $\chi_{p_1}(G) \leqq 4$.

Next let u have degree 4, and suppose there exists at least one $(4, P_1)$-colouring of G-u which assigns only three colours to the four points adjacent to u. We can then assign a fourth colour to point u to obtain a $(4, P_1)$-colouring of G. Suppose then that every $(4, P_1)$-colouring of G-u assigns a different colour to each of the four points adjacent to u. This situation is described by Figure 1 in which the points adjacent to u are coloured c_1, c_2, c_3, and c_4.

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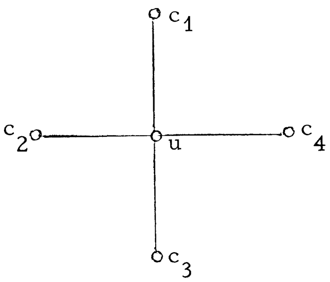

Fig. 1

Consider in this case the number of points in G-u which have colour c_i in any given $(4, P_1)$-colouring of G-u. If for any colour c_i this number is greater than one, then by assigning colour c_i to point u, the subgraph of G induced by the set of points coloured c_i will still be disconnected. Thus we will obtain a $(4, P_1)$-colouring of G. On the other hand, in every $(4, P_1)$-colouring of G-u, the number of points coloured c_1, c_2, c_3, or c_4 is exactly one, i.e., G-u consists only of the four points adjacent to u. But since G is planar at least two of these points are not adjacent. Therefore we can assign the colour c_1 to these two points, colours c_2 and c_3 to the other points adjacent to u, and colour c_4 to point u. This produces a $(4, P_1)$-colouring of G.

Finally, let point u have degree 5. Essentially we only have to consider the case that every $(4, P_1)$-colouring of G-u assigns four colours to the points adjacent to u. In this case any $(4, P_1)$-colouring of G-u must colour the five points adjacent to u in one of the two ways illustrated in Figure 2.

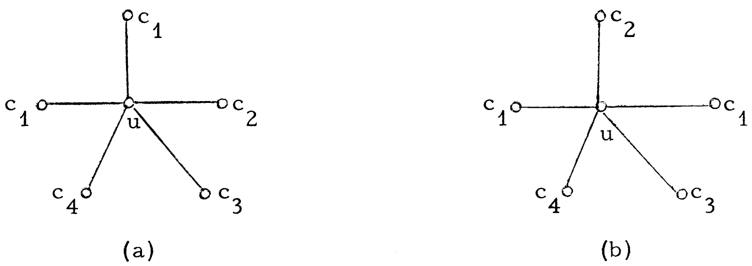

Fig. 2

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As before, if any point adjacent to u is not the only point with colour c_i, i = 2, 3, or 4, in a given $(4, P_1)$-colouring of G-u, then we can assign colour c_i to point u and produce a $(4, P_1)$-colouring of G. Otherwise in every $(4, P_1)$-colouring of G-u, the points with colours c_2, c_3, and c_4 in Figures 2a or 2b are the only points so coloured. Consider then, in either case, whether these three points are mutually adjacent. If they are not, then at least two of them are not adjacent and can be assigned colour c_2, the remaining point adjacent to u can be assigned colour c_3 and point u can be assigned colour c_4. This produces a $(4, P_1)$-colouring of G. Finally, if these three points are mutually adjacent, we have a situation essentially described by Figures 3a and 3b.

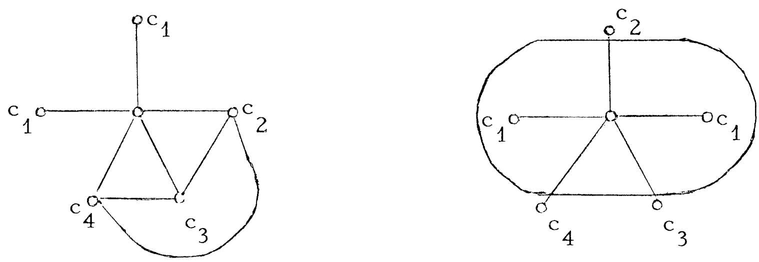

Fig. 3

In the case of Figure 3a the two points coloured c_1 can be assigned colour c_3, and thus u can have colour c_1. In the case of Figure 3b, one point with colour c_1 can be assigned colour c_3, the other point with colour c_1 can be assigned colour c_4, and point u can be assigned colour c_1. This produces a $(4, P_1)$-colouring of G in either case, completing the proof.

An interesting aspect of this theorem is its relation to the famous conjecture of Hadwiger [2]: $\chi(G) = n$ implies G has a connected contraction onto K_n. Relatively few conditions are known which guarantee that a given graph have a connected contraction onto K_n; it readily follows, however, that if $\chi_{P_1}(G) = n$, then the complement of G, $\bar{G}$, has a connected contraction of order n.

A planar graph G is said to be <u>outerplanar</u> (cf. Chartrand and Harary [1]) if G can be drawn in the plane in such a way that every

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point lies on the exterior region. This class of planar graphs has been studied by Tang [7], who showed among other things that if a graph is outerplanar then its chromatic number does not exceed three. A simple proof of this result can be given which is based on the observation in [7] that every outerplanar graph contains at least one point of degree $\leqq 2$. We will now use this observation to obtain another result about outerplanar graphs which closely parallels Theorem 1.

THEOREM 2. For any outerplanar graph G, $\chi_{P_1}(G) \leq 3$.

Proof. We proceed again by induction on the number p of points in G. Clearly this holds for all outerplanar graphs having $p \leq 3$ points. Suppose then that it holds for all outerplanar graphs having $p-1$ points and let G have p points.

Since G is outerplanar it must contain a point, say u, of degree $\leqq 2$. Consider the graph $G-u$. Since $G-u$ is outerplanar, it follows by hypothesis that $\chi_{P_1}(G-u) \leqq 3$. But clearly, since u has degree $\leqq 2$, we can always assign a third colour to u given any $(3, P_1)$-colouring of $G-u$, to obtain a $(3, P_1)$-colouring of G. Thus, $\chi_{P_1}(G) \leqq 3$.

The graphs in Figure 4 illustrate that the bound in Theorem 2 is tight; each outerplanar graph in Figure 4 has P_1-chromatic number equal to 3.

Fig. 4

It is interesting to observe that for any graph G which contains no cycles, i.e., for any forest, $\chi_{P_1}(G) \leqq 2$. This follows immediately from two simple observations: for any graph G, $\chi_{P_1}(G) \leqq \chi(G)$, and for any forest G, $\chi(G) \leqq 2$. Thus we observe that if a graph G is totally disconnected, a forest, outerplanar, or planar, then $\chi_{P_1}(G)$ is less than or equal to 1, 2, 3, or 4, respectively, and each of these bounds is tight.

A striking similarity among the graphs in Figure 4 serves to illustrate the following result. The join $G + G'$ of two graphs G and

G' is the graph for which $V(G+G') = V(G) \cup V(G')$ and $E(G+G') = E(G) \cup E(G') \cup \{[u, v] \mid u \in V(G),\ v \in V(G')\}$.

THEOREM 3. For any graphs G and G'

$$\chi_{P_1}(G+G') = \chi_{P_1}(G) + \chi_{P_1}(G').$$

Proof. Clearly, $\chi_{P_1}(G+G') \leq \chi_{P_1}(G) + \chi_{P_1}(G')$, for if $\chi_{P_1}(G) = m$ and $\chi_{P_1}(G') = n$, then any (m, P_1)-colouring of G combined with any (n, P_1)-colouring of G' can produce an $(m+n, P_1)$-colouring of $G+G'$.

Conversely, let $\chi_{P_1}(G+G') = k$. Then in any (k, P_1)-colouring of $G + G'$ no point of G can be assigned the same colour as any point of G' and conversely. For if a point in G is assigned the same colour, say c_i, as a point in G', then no matter what other points are coloured c_i in $G + G'$, the set of points having c_i will induce a connected subgraph of $G + G'$. Therefore $\chi_{P_1}(G+G') \geq \chi_{P_1}(G) + \chi_{P_1}(G')$.

The arboricity arb (G) of a graph G is the minimum number of line-disjoint, acyclic subgraphs which contain all the lines of G. In [6] Nash-Williams established a formula for the arboricity of any graph, a consequence of which is the result that the arboricity of every planar graph is three or less. We define the point-arboricity of a graph G as the minimum number of induced, point-disjoint, acyclic subgraphs which contain all the points of G. Using our terminology above it can be seen that if P_2 denotes the property that a graph contain no cycles, then the point arboricity of G equals $\chi_{P_2}(G)$.

Again using Kempe's method, we now obtain a result, a corollary of which asserts that the point-arboricity of every planar graph is also three or less. A property P is treelike if K_1 has property P, and whenever a subset S of points is a P-set and there exists a point $v \notin S$ which is adjacent to at most one point of S, then $S \cup \{v\}$ is also a P-set. The property P_2 that a graph contain no cycles is treelike, for example.

THEOREM 4. If P denotes any treelike property of a graph, then for every planar graph G, $\chi_P(G) \leq 3$.

Proof. We proceed in much the same way as in the proof of Theorem 1, and assume that G contains a point u of degree five or less. If u has degree 1 or 2, then since by hypothesis $\chi_P(G-u) \leq 3$, we can always assign a third colour to point u, given any (3, P)-colouring of G-u, to obtain a (3, P)-colouring of G.

If point u has degree 3, 4, or 5, and every (3, P)-colouring of G-u assigns three colours to the points adjacent to u, then in every (3, P)-colouring of G-u, at least one point adjacent to u will have a colour, say c_i, different from all other points adjacent to u. But then since P is treelike, we can assign colour c_i to u and obtain a (3, P)-colouring of G; it will still be the case that the subgraph induced by the set of points having colour c_i will have property P.

COROLLARY 4a. *If P_2 denotes the property that a graph contain no cycles, then for every planar graph* G, $\chi_{P_2}(G) \leq 3$, i.e., *the point-arboricity of* G *is* ≤ 3.

Since the property that a graph be outerplanar is treelike, we obtain

COROLLARY 4b. *The points of every planar graph* G *can be partitioned into three or less subsets such that the subgraph induced by each subset is outerplanar.*

The following result, which follows immediately from Theorem 4, is also a corollary of Nash-William's result on the arboricity of planar graphs.

COROLLARY 4c. *Every planar graph can be decomposed into a line disjoint union of at most three bipartite graphs.*

Proof. From Corollary 4a, let $\pi = \{V_1, V_2, V_3\}$ be a partition of the points of G into three subsets, each of which induces a subgraph of G which contains no cycles. The set of lines of G can be decomposed as follows:

$$E(G) = E(\langle V_1 \rangle) \cup E(\langle V_2 \rangle) \cup E(\langle V_3 \rangle) \cup E_{12}, \cup E_{13} \cup E_{23},$$ where

E_{ij} denotes the set of lines of G connecting points of V_i and V_j. It follows therefore that the partition $\{E(\langle V_1 \rangle) \cup E_{23}, E(\langle V_2 \rangle) \cup E_{13}, E(\langle V_3 \rangle) \cup E_{12}\}$ of E(G) determines a decomposition of G into at most 3 subgraphs each of which contains no odd cycles, and hence is bipartite.

The last result of this paper is a straightforward extension of the proof of Corollary 4c. Consider for any graph G the minimum number of bipartite subgraphs of G whose union is G, and denote this number by $\chi_b'(G)$.

THEOREM 5. *Let* P_2 *denote the property that a graph contain no cycles; then for any graph* $G, \chi_b'(G) \leq \left\{ \frac{\chi_{P_2}(G)}{2} \right\} + 1$.

Proof. Let $\chi_{P_2}(G) = m$ and let $\pi = \{V_1, V_2, \ldots, V_m\}$ be a partition of $V(G)$ such that every subset V_i is a P_2-set. It can be seen that the factor graph G/π is complete and has m points, $V(G/\pi) = \{V_1, \ldots, V_m\}$. Consider then the arboricity of $G/\pi \simeq K_m$.

By the result of Nash-Williams [6], $\text{arb}(K_m) = \{\frac{m}{2}\}$. Thus we can construct $\{\frac{m}{2}\}$ line disjoint subgraphs (forests), say $F_1, F_2, \ldots, F_{\{\frac{m}{2}\}}$, each of which contains no cycles, whose union is G/π. Corresponding to each of these subgraphs F_i of G/π we can uniquely construct a bipartite subgraph G_i of G such that $V(G_i) = V(G)$ and $[u, v] \in E(G_i)$ if and only if $u \in V_i$, $v \in V_j$ and $[V_i, V_j] \in E(F_i)$. Since any two subgraphs F_i, F_j of G/π have no lines in common, it follows that the corresponding subgraphs G_i, G_j of G have no lines in common. Finally consider the subgraph $H = \langle V_1 \rangle \cup \langle V_2 \rangle \cup \ldots \cup \langle V_m \rangle$ i.e., $V(H) = V(G)$ and $[u, v] \in E(G)$ if and only if for some $i, u, v \in V_i$ and $[u, v] \in E(G)$. Since each subset V_i is a P_2-set, H is clearly bipartite. It follows now that $G_1, G_2, \ldots, G\{\frac{m}{2}\}$, H is a collection of $\{\frac{m}{2}\} + 1$ line disjoint bipartite subgraphs whose union is G.

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