28 Beyond weak coupling: non-perturbative string theory

In the previous chapter we were forced to face the fact that on the one hand string theory, if it describes nature, is not weakly coupled. On the other hand, the very formulation of the theory that we have put forward is perturbative. We have described the quantum mechanics of single strings and given a prescription for calculating their interactions order by order in perturbation theory in a parameter g_s. There is a parallel here to Feynman's early work on relativistic quantum theory: Feynman guessed a set of rules for computing the perturbative amplitudes of electrons. In that case, however, one already had a candidate for an underlying description: quantum electrodynamics. It was Dyson who clarified the connection. For Abelian theories a non-perturbative approach probably does not exist, but in the case of non-Abelian gauge theories it does. The field-theoretic formulation provides an understanding of the underlying symmetry principles and access to a treasure trove of theoretical information.

A string field theory would be a complicated object. The string fields themselves would be functionals of the classical two-dimensional fields which describe the string. The quantization of such fields is sometimes called the "third quantization." Much effort has been devoted to writing down such a field theory. For open strings one can obtain relatively manageable expressions which reproduce string perturbation theory. For closed strings, infinite sets of contact interactions are required. But, quite apart from their cumbersome structure, there are reasons to suspect that this is not a useful formulation. There would seem to be, for example, vastly too many degrees of freedom. At one loop we have seen that the string amplitudes are to be integrated only over the fundamental region the moduli space. Naively, a field theory which simply describes all of the states of the string would have amplitudes integrated over the whole region, and the cosmological constant would be extremely divergent. The contact interaction terms mentioned above solve this problem but not in a very satisfying way.

Despite this, there has been great progress in understanding the non-perturbative aspects of the known string theories. Most strikingly, it is now known that all theories with 16 or more supersymmetries are the same. Many tools have been developed to study phenomena beyond string perturbation theory, especially D-branes and supersymmetry. There exist some cases where non-perturbative formulations of string theory are possible, and we will discuss them briefly in this chapter. They are technically and conceptually much simpler than string field theory. They have a puzzling, perhaps disturbing feature, however: they are special to strings propagating in particular backgrounds. It is as if, in Einstein's theory, for each possible geometry one had to give a different Hamiltonian. All these results are "empirical." They have been developed by collecting circumstantial evidence on a

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case-by-case basis. There is still much that is not understood. In Chapter 31, we will discuss how this developing understanding might lead to a closer connection of string theory and nature.

28.1 Perturbative dualities

Before considering examples of weak–strong coupling dualities, we return to the large-radius–small-radius duality (T-duality) we studied in Section 25.3; many dualities that we will study have a similar flavor to this, even though they cannot be demonstrated so directly. Thus we saw that there is an equivalence of the heterotic string theory at small radius to the theory at large radius. By examining the action of these transformations at their fixed points, we saw that these duality symmetries are gauge symmetries. We could ask, as well, the significance of duality transformations in the IIA and IIB theories. As with other closed strings, in addition to transforming the radii, the duality transformation is as follows:

$$\partial X^9 \to -\partial X^9, \quad \bar{\partial} X^9 \to \bar{\partial} X^9. \tag{28.1}$$

Because of the world-sheet supersymmetry, the transformation has the same action on the fermions: $\psi^9 \to -\psi^9$; $\tilde{\psi}^9 \to \tilde{\psi}^9$. But, under this, the chirality operator appearing in the GSO projector is reversed in sign, i.e. duality interchanges the IIA and IIB theories: *the small-radius IIA theory is equivalent to the large-radius IIB theory* and vice versa. There are other weak coupling connections between string theories. For example, the compactified $O(32)$ heterotic string theory is equivalent to the $E_8 \times E_8$ theory.

28.2 Strings at strong coupling: duality

Duality is a term used in physics to label different descriptions of the same physical situation. At the level of perturbation theory we have learned about five apparently different string theories. On the basis of on the perturbative dualities discussed above, we see that there are at most three inequivalent string theories, the Type I, Type II and heterotic theories. But it is tempting to ask whether there are more connections between the theories. In this chapter we will see that all the known string theories are equivalent in a similar way, but these equivalences relate small and large coupling. For example, the strong coupling limit of the $O(32)$ heterotic string theory is the weak coupling limit of the Type I string theory; the strongly coupled limit of the $E_8 \times E_8$, compactified to six dimensions on a torus, is the weakly coupled limit of the Type II theory compactified on a K3 manifold (K3 manifolds are essentially four-dimensional Calabi–Yau spaces); the ten-dimensional Type II theory is self-dual and, perhaps most intriguingly of all, the strong coupling limit of the Type IIA theory in ten dimensions is described, at low energies, by a theory whose low-energy limit is eleven-dimensional supergravity.

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Lacking a non-perturbative formulation of the theory, the evidence for these connections is necessarily circumstantial. While circumstantial, however, it is compelling. All the evidence relies on supersymmetry. We will not be able to review it all here but will try to give the flavor of some of the arguments. Supersymmetry, especially supersymmetry with 16 or 32 supercharges, allows one to write down a variety of exact formulas, for Lagrangians (based on strong non-renormalization theorems) and for spectra (based on BPS formulas), which can be trusted in both weak and strong coupling limits. This allows detailed tests of the various dualities.

28.3 *D*-branes

When we discussed strong–weak (electric–magnetic) dualities in field theory, topological objects played a crucial role. The same is true in string theory, where the solitons are various types of branes. In general, a *p-brane* is a soliton with a $(p+1)$-dimensional world volume, so a 0-brane is a particle, a 1-brane is a string, a 2-brane is a membrane and so on. One might construct these by solving complicated non-linear differential equations. But a large and important class of topological objects can be uncovered in string theory in a different – and much simpler – way. These are the D-branes. These branes fill an important gap in our understanding of the Type I and Type II theories. In these theories we encountered gauge fields in the Ramond–Ramond sectors: two-forms in Type I, one-forms and three-forms in Type IIA, zero-forms, two-forms, and four-forms in Type IIB. One natural question is: what are the charged objects that couple to these fields? They are not within the perturbative string spectrum. The vertex operators for these fields involve the gauge-invariant field strengths only, so in perturbation theory there are no objects with minimal coupling. The answer is that they are D-branes. Their masses (*tensions*) are proportional to $1/g_s$, so at weak coupling they are very heavy. This is why they are not encountered in the string perturbation expansion.

When we discussed open strings we noted that there are two possible choices of boundary condition: Neumann and Dirichlet. At first sight, Neumann boundary conditions appear more sensible; Dirichlet boundary conditions would violate translational invariance, implying that strings end at a particular point or points. But we have already encountered violations of translational invariance within translationally invariant theories: solitons, for example magnetic monopoles or higher-dimensional objects such as cosmic strings or domain walls. Admitting the possibility of Dirichlet boundary conditions for some of or all the coordinates leads to a class of topological objects known as D-branes (for Dirichlet branes). If $d-p-1$ of the boundary conditions are Dirichlet while $p+1$ are Neumann, the system is said to describe a Dp-brane.

We can be quite explicit. We start with the bosonic string. For the Neumann directions we have our previous open-string mode expansion of Eq. (21.16). For the Dirichlet directions we have:

$$X^I = x_0^I + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^I e^{-in\tau} \sin n\sigma, \quad I = 1, d-p-1. \tag{28.2}$$

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Note that there are no momenta associated with the Dirichlet directions. The x_0^Is should be thought of as collective coordinates. We will argue shortly that the tension of the branes is proportional to M_s^{p+1}/g_s.

Consider an extreme case, that of a $D0$-brane. There are 25 collective coordinates and no momenta, so this object is a conventional soliton. In field theory the excitations near the soliton, which describe the scattering of mesons (field theory excitations) from the soliton must be found by studying the eigenfunctions of the quadratic fluctuation operator. But here they are very simple: they are just the excitations of the open string. As a second example, consider a $D3$-brane. Now the momentum has four components, so the excitations which propagate on the brane are four-dimensional fields. These break up into two types. The Neumann fields X^μ give rise to a massless gauge boson, the state $\alpha_{-1}^{\mu}|0\rangle$; the Dirichlet fields X^I give rise to massless scalars on the brane $\alpha_{-1}^{I}|0\rangle$. In the superstring version of this construction there are six scalars, a gauge boson and their superpartners. In $N=1$ language this amounts to a vector multiplet and three chiral multiplets, the content of $N=4$ Yang–Mills theory with gauge group $U(1)$.

Before considering some of these statements in greater detail, let us explore a further aspect of this construction. Suppose that we have several branes, say $D3$-branes, parallel to each other; here, "parallel" just means that the strings which end on these branes have Dirichlet or Neumann boundary conditions for the same coordinate. Now, however, we have the possibility that the strings end on different branes. Take the simplest case of two branes. If the branes are separated by a distance r, in addition to the modes above, labeled by the collective coordinate $x_i^I, i=1,2$, we have to allow for expansions of the form

$$X^I(\sigma,\tau) = x_i^I + \sigma\frac{r}{\pi}\left(x_j^I - x_i^I\right) + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^I e^{-in\tau}\sin n\sigma, \quad I=1,\ldots,d-p-1. \tag{28.3}$$

There are two such configurations, one starting on the first brane and ending on the second and one starting on the second brane and ending on the first. The ground states in these sectors have mass-squared proportional to r^2. For $r\neq 0$, all these states are massive. The massless bosons consist of a $U(1)$ gauge boson on each brane, as well as scalars. As $r\to 0$, we have two additional massless gauge bosons. If we generalize to n branes, we have n massless gauge bosons and $6n$ scalars; as we bring the branes close together, we have n^2 gauge bosons and $6n^2$ scalars.

There is a natural conjecture as to what is going on here. When all the branes coincide we have a $U(n)$ gauge symmetry, with three complex scalars transforming in the adjoint representation of the group. As the branes are separated, the adjoint scalars acquire (commuting) expectation values; these break the gauge symmetry to $U(1)^n$, giving mass to the other gauge bosons. In principle we would like to check that these n^2 gauge bosons interact as required for Yang–Mills theories, as we did for the gauge bosons of the heterotic string. This is more challenging here, since we need vertex operators which connect strings ending on different branes, and we will not attempt this. We will provide further evidence for the correctness of this picture shortly.

The branes break some of the supersymmetry of the Type II theory in infinite space; instead of 32 conserved supercharges there are 16. A simple way to understand this uses

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the light cone gauge construction. There are now open strings ending on the brane. For the world-sheet fermions, the boundary conditions relate the left and right movers on the string. Calling these S_a and $\tilde{S}_a$, we have

$$S^a(\sigma,\tau) = \sum_n S_n^a e^{-in(\tau+\sigma)}, \quad \tilde{S}^a(\sigma,\tau) = \sum_n S_n^a e^{-in(\tau-\sigma)}. \tag{28.4}$$

Recall that half the supercharges have the very simple form

$$Q^a = \int d\sigma\, S^a, \quad \tilde{Q}^a = \int d\sigma\, \tilde{S}^a, \tag{28.5}$$

so $Q^a = \tilde{Q}^a$. This is the structure of a broken supersymmetry generator, with S the goldstino. The other set supercharges is linearly realized. Other configurations, such as non-parallel sets, preserve less supersymmetry. Brane–anti-brane configurations preserve no supersymmetry at all.

We can imagine other sets of branes, which would respect different amounts of supersymmetry. If we have branes which are not parallel, for example, different sets of supersymmetries will be preserved. In order to count supersymmetries we need to compare the supersymmetries on different branes at different angles relative to one another.

28.3.1 Brane charges

We have seen that the simplest D-brane configurations preserve half the supersymmetries. In other words, they are BPS states. Typically BPS states are associated with conserved charges. In the case of the IIA and IIB theories, in the Ramond–Ramond sectors there are gauge fields but, in perturbation theory, no charged objects. Polchinski guessed – and showed – that the objects which carry Ramond–Ramond charges are D-branes. In the IIA case the gauge fields are a one-form and a three-form; in the IIB case they are a zero-form a two-form, and a (self-dual) four-form. In relativistic mechanics, a gauge field couples to a particle – a zero-brane. We have seen that a two-index tensor couples naturally to a string – a one-brane. So, this suggests that, in the IIA theory, there should be Dp-branes with p even, coupling to the corresponding R–R gauge fields, while in the IIB theory there should be Dp-branes with p odd. Polchinski verified this by direct calculation. He computed the one-loop amplitude for two separated branes. For large separations he found the poles associated with exchange of the massless gauge fields (more precisely, for fixed separation r one should see a falloff with powers of $1/r$). His calculation not only yields the brane charges, it also gives the brane tensions.

Consider the case of two branes, separated by a distance y. In empty flat space, the trace over states in the one-loop amplitude for open strings gives a generic scattering amplitude of the form

$$\mathcal{A} = C \int_0^\infty \frac{dt}{t^2}. \tag{28.6}$$

The power of t arises from the momentum integral $\int d^8k \exp(-k^2)$, as well as from manipulation of the oscillator traces. The main difference in the case of two separated

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branes is that the mass-squared has a contribution y^2, from the brane separation y, and $9-p$ coordinates of the brane are fixed, so they do not have associated momenta. So, the result has the form

$$\mathcal{A} = C\int_0^\infty \frac{dt}{t^2}(8\pi^2\alpha' t)^{(9-p)/2}\exp\left(-\frac{ty^2}{2\pi\alpha'}\right)$$
$$\sim y^{-(7-p)} \sim G_{9-p}(y). \tag{28.7}$$

Here $G_d(y)$ is the scalar Green's function in d dimensions. So, one can think of a potential between the branes associated with the exchange of massless states. These massless states include antisymmetric tensor fields and their superpartners as well as gravitons and gravitinos. These contributions can be isolated, and the tensions and charges of the D-branes determined. In the case a the superstring, the full potential vanishes due to boson and fermion cancelations.

28.3.2 Brane actions

We are familiar with the actions for zero-branes and one-branes. The action for a general p-brane is a generalization of these:

$$S_p = -T_p\xi\int d^{p+1}\xi \ \det\left(\frac{\partial X^\mu}{\partial\xi^a}\frac{\partial X^\nu}{\partial\xi^b}\eta_{\mu\nu}\right)^{1/2}, \tag{28.8}$$

where T_p is the brane tension. In the zero-brane case this is the action for a particle; $X^\mu(\tau)$ is the collective coordinate which describes the position of the soliton and T_0 is its mass. For a general background with a bulk metric, a dilaton and an antisymmetric tensor field this generalizes to

$$S_p = -T_p\int d^{p+1}\xi\ e^{-\Phi}[-\det(G_{ab}+B_{ab}+2\pi\alpha' F_{ab})]^{1/2}. \tag{28.9}$$

The terms involving the metric and antisymmetric tensor are similar to those we have encountered elsewhere in string theory, and their form is not surprising. The factor $e^{-\Phi}$ arises because in the open-string sector the coupling constant is the square root of that for the closed-string sector.

28.4 Branes from the T-duality of Type I strings

There is another way to think about D-branes, which provides further insight. We have seen that closed-string theories exhibit a duality between large and small radius. In the heterotic theory there is an exact equivalence of the theories at large and small radius, which can be understood as a gauge symmetry. In Type II theories, T-duality relates two apparently different theories. Therefore, is natural to ask what is the connection between large and small radius in theories with open strings. Open strings have momentum states but no winding states. So, there cannot be a self-duality. Instead we look for an equivalence

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between the open-string theory at one radius and some other theory at the inverse radius. Here we uncover D-branes.

Consider the boundary conditions on the strings in the compactified direction. For the closed-string fields, the effect of the duality transformation in the compactified direction X is:

$$X_{\rm L} \to X_{\rm L}, \quad X_{\rm R} \to -X_{\rm R}. \tag{28.10}$$

In terms of left- and right-moving bosons in open-string theories, the Neumann boundary conditions are

$$\partial_\tau X = (\partial_{\sigma_+} + \partial_{\sigma_-})X = 0. \tag{28.11}$$

So, after a T-duality transformation we would expect that

$$(\partial_{\sigma_+} - \partial_{\sigma_-})X = \partial_\sigma X = 0, \tag{28.12}$$

i.e. we have traded Neumann for Dirichlet boundary conditions. While this follows from simple calculus manipulations, it is instructive to formulate it in terms of the mode expansion for an open string. Prior to the duality transformation, we have

$$X^9 = x_i^9 + \frac{1}{2}p(\tau+\sigma) + \frac{1}{2}p(\tau-\sigma) + i\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^9 e^{-in(\tau+\sigma)} + \alpha_n^9 e^{-in(\tau-\sigma)}\right). \tag{28.13}$$

The effect of the duality transformation is to change the sign of the terms which depend on $\tau - \sigma$. So, instead of an expansion in terms of cosines we have an expansion in terms of sines:

$$X^9 = x_0^9 + p\sigma + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^9 e^{-in\tau}\sin n\sigma. \tag{28.14}$$

These are precisely the Dirichlet branes. Note the role of p: in the T-dual picture it is a sort of winding: it describes strings which start on the brane, wind around the compact dimension some number of times and then end on the brane.

This T-duality of open strings also allows us to understand better the appearance of gauge interactions associated with stacks of branes. In the original open-string picture, gauge degrees of freedom are described by Chan–Paton factors, i.e. charges on the ends of the string. In the case of Type I strings these are described by states of the form $|AB\rangle$, $A, B = 1, \ldots, 32$. Now consider a $U(16)$ subgroup of $O(32)$. The string ends carry labels i, j, within $U(16)$. Taking the diagonal generators of $U(N)$ to be the matrices

$$T_1 = \mathrm{diag}(1, 0, 0, \ldots), \quad T_2 = \mathrm{diag}(0, 1, 0, \ldots) \tag{28.15}$$

etc., the state $(\bar{i}, j)$ carries charge -1 under T_i, $+1$ under T_j and zero under the other generators.

We can consider constant background gauge fields in the 9 direction. We can write these as

$$A = \mathrm{diag}(a_1, a_2, \ldots, a_{16}). \tag{28.16}$$

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This has a gauge-invariant description in terms of the Wilson line:

$$U = \exp\,(i \oint d\vec{x} \cdot \vec{A}), \tag{28.17}$$

where the integral is taken in the periodic directions. Such a background gauge field in general breaks the gauge symmetry to $U(1)^{16}$; the other gauge bosons should gain mass. In field theory the corresponding mass terms are proportional to

$$[A^\mu, A^9]^2, \tag{28.18}$$

so the diagonal gauge bosons are massless and those corresponding to the non-Hermitian generator

$$T^{kl}_{ij} = \delta^k_i \delta^l_j \tag{28.19}$$

have mass-squared

$$m^2 = (a_i - a_j)^2. \tag{28.20}$$

This is similar to the calculations we made of symmetry breaking in grand unified theories.

We would like to understand how this result arises directly in string theory. It is simplest to consider the case of a string which is constant in σ, the space-like world-sheet coordinates. The coupling of the string depends on the Chan–Paton factors; see Section 21.1. In the light cone frame the action in the presence of a gauge field is like that of a particle:

$$\frac{1}{2}\int d\tau \left[\left(\frac{\partial \dot{X}^9}{\partial \tau}\right)^2 + (a_i - a_j)\frac{\partial X^9}{\partial \tau}\right]. \tag{28.21}$$

For a non-constant string the situation is somewhat more complicated, since the gauge fields couple at the string's end points.

The extra term modifies the canonical momenta. These are now

$$P = \frac{n}{R} = \frac{\partial \dot{X}^9}{\partial \tau} + a_i - a_j. \tag{28.22}$$

This means that the leading term in the string mode expansion is

$$X^9 = \left[\frac{n}{R} - (a_i - a_j)\right]\tau. \tag{28.23}$$

This gives an extra contribution to the mass. If $n = 0$, this is *exactly* what we expect from field-theoretic reasoning.

Now we will consider the T-dual picture. Under T-duality the zero-mode part of X transforms into

$$X^9 = x_0 + \left[\frac{n}{R} - (a_i - a_j)\right]\sigma. \tag{28.24}$$

For $i = j$ this corresponds to a string that begins and ends on the same D-brane. For $i \neq j$ the string ends at different points, i.e. on separated D-branes. At least for the Type I theory

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we have *derived* the picture we conjectured earlier: a stack of N coincident branes describes a $U(N)$ gauge symmetry; as the branes are separated, the gauge symmetry is broken by a field in the adjoint representation.

28.4.1 Orientifolds

We have seen that we can understand the appearance of D-branes by considering T-duality transformations of open strings. The Type I theory is a theory of *oriented* strings. In the closed-string sector the action has a parity symmetry which interchanges left and right on the world sheet. Calling the corresponding operator Ω, one keeps only states which are invariant under the action of Ω. This is necessary for the consistency of interactions of open and closed strings. This means that closed-string states like

$$\alpha_{-2}\tilde{\alpha}_{-1}\tilde{\alpha}_{-1}|0\rangle \tag{28.25}$$

are not allowed, but symmetrized combinations such as

$$(\alpha_{-2}\tilde{\alpha}_{-1}\tilde{\alpha}_{-1} + \tilde{\alpha}_{-2}\alpha_{-1}\alpha_{-1})|0\rangle \tag{28.26}$$

are allowed. This projection is similar to the orbifold projections that we have encountered earlier.

Consider the action of Ω in the T-dual theory. We have seen that, in terms of the original fields,

$$(X^9)' = -X^9_{\mathrm{L}} + X^9_{\mathrm{R}}. \tag{28.27}$$

So, the effect of interchanging left and right is to change the sign of X^9, i.e. Ω is a combination of a world-sheet parity transformation and a reflection in space–time.

The effect of this projection on states is similar to a Z_2 orbifold projection. We can combine momentum states to form states with definite transformation properties under the reflection

$$|\tilde{F}\rangle = |p\rangle \pm |-p\rangle. \tag{28.28}$$

Gravitons $G_{\mu\nu}$, for example, with indices in the non-compact directions, must have momentum states which are even; in coordinate space this means that graviton states must be even functions of x. The fields $G_{\mu 9}$ must be odd functions, and so on. It is as if there is an entity, the *orientifold*, sitting at the origin, the fixed point of the reflection. This object in fact has a negative tension. One way to see this is simply to note that the effect of the T-duality transformation is to produce a set of D-branes. These branes have a positive tension. From the point of view of the non-compact dimensions this is a cosmological constant. But the original theory had no such cosmological constant – this must be canceled by the orientifold.

Just as it is not necessary to start from the Type I theory and its dualities in order to encounter D-branes, it is not necessary to start from the Type I theory to consider orientifolds. Starting from Type II theories, in particular, we can perform a projection by world-sheet parity times some Z_2 space–time symmetry. For example, consider a Type II

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theory with a single compact dimension. On this theory, we can make a projection which is a combination of world-sheet parity Ω and a reflection in the compact dimension.

28.5 Strong–weak coupling dualities: the equivalence of different string theories

We have seen that, at weak coupling, there are a variety of connections between different string theories which are surprising from a field-theoretic perspective. The heterotic string, compactified on a circle of very large radius, is equivalent to a string theory compactified at very small radius (with a different coupling). The Type IIA theory at large radius is equivalent to the IIB theory at small radius. The $O(32)$ heterotic string is equivalent to the $E_8 \times E_8$ theory. All of these equivalences involve significant rearrangement of the degrees of freedom. Typically, Kaluza–Klein modes, which are readily understood from a space–time field-theory point of view, must be exchanged with winding modes, which seem inherently stringy-like. So, perhaps it is not surprising that there are other equivalences, involving weak and strong coupling. Again, we have had some inkling of this in field theory, when we studied $N = 4$ Yang–Mills theory. There, the theory at weak coupling is equivalent to a theory at strong coupling. To see this equivalence one needs to significantly rearrange the degrees of freedom. States with different electric and magnetic charges exchange roles as the coupling is changed from strong to weak.

In string theory there is a complex web of dualities. The IIB theory in ten dimensions exhibits a strong–weak coupling duality very similar to that of $N = 4$ Yang–Mills theories; weak and strong coupling are completely equivalent. The $O(32)$ heterotic string theory, in ten dimensions, is equivalent at strong coupling to the weakly coupled Type I theory. These relations are surprising, in that these theories appear to involve totally different degrees of freedom at weak coupling. But there are more surprises still. The strong coupling limit of the IIA theory in ten dimensions is a theory whose low-energy limit is eleven-dimensional supergravity. If we allow for compactifications of the theory, this set of dualities is already enough to establish an equivalence of all string theories as well as some as yet not fully understood theory whose low-energy limit is eleven-dimensional supergravity. But, as we compactify, we find further intricate relations. For example, the Type IIA theory on K3 is equivalent to $E_8 \times E_8$ on T^4. Given that all the sensible theories of quantum gravity we know are equivalent, it is plausible that, in some sense, there is a unique theory of quantum gravity. As we will see, however, we only know this reliably for theories with at least 16 supercharges. For theories with four or fever, the situation is less clear; it is by no means obvious that the statement is even meaningful.

In the sections that follow, we will explore some of these dualities and the evidence for them. We will also discuss two particularly surprising equivalences. We will argue that certain string theories are equivalent to *quantum field theories* – even to quantum mechanical systems. The very notion of space–time in this framework will be a derived concept.

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28.6 Strong–weak coupling dualities: some evidence

In the case of T-dualities, i.e. dualities which relate the behavior of string theories at weak coupling and different radii, it is straightforward to understand the precise mappings between the different descriptions. Lacking a general non-perturbative definition of string theory, it is not possible to do something similar in the case of strong–weak coupling dualities. Instead, one can try to put together compelling circumstantial evidence. Without supersymmetry even this is essentially impossible. But, in the presence of sufficient supersymmetry, one has a high degree of control over the dynamics. Evidence for equivalence can be provided by studying the following.

1. *The effective action* In ten or eleven dimensions the terms in the action with up to two derivatives are uniquely determined by supersymmetry, so they do not receive corrections either perturbatively or non-perturbatively. A similar statement holds for $N \geq 4$ actions in four dimensions (and actions with varying degrees of supersymmetry in between). In some cases one can check higher-derivative terms in the effective action as well.
2. *The spectrum of BPS objects* In many cases the low-lying states are BPS objects. They cannot disappear from the spectrum as the coupling or other parameters are varied. With 16 or more supercharges, they obey *exact* mass formulas. The identity of the BPS states for different theories provides non-trivial evidence for these equivalences.

We will explore only some of the simplest connections here, but it is important to stress that these identifications are often subtle and intricate. In many instances where one might have thought the dualities mentioned above would fail, they do not.

28.6.1 From IIA to eleven-dimensional supergravity (*M* theory)

We will start with the IIA theory, where we can readily access both aspects of the duality. Comparing the actions of eleven-dimensional supergravity and the IIA theory is particularly straightforward, as the Lagrangian of the IIA theory is often obtained by compactifying eleven-dimensional supergravity on a circle, keeping only the zero modes. The basic degrees of freedom in eleven dimensions are the graviton g_{MN}, the antisymmetric tensor gauge field C_{MNO} and the gravitino ψ_M. We are not going to work out the detailed properties of this theory, but it is a useful exercise to check that the numbers of bosonic and fermionic degrees of freedom are the same. As usual, we can count degrees of freedom by going to the light cone (or using the "little group," the group of rotations in $D = 11 - 2 = 9$). The metric is a symmetric traceless tensor; for the gravitino, we need also to impose the constraint $\gamma^I \psi_i = 0$. For the metric, then, we have $((9 \times 10)/2) - 1 = 44$ degrees of freedom while from the three-index antisymmetric tensor we have $(9 \times 8 \times 7)/3! = 84$, giving a total of 128 bosonic degrees of freedom. From the gravitino we have $9 \times 16 - 16 = 128$ degrees of freedom.

If we compactify x^{10} on a circle of radius R, we obtain the following bosonic degrees of freedom in ten dimensions:

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1. the ten-dimensional metric $g_{\mu\nu}$ ($\mu, \nu = 0, \ldots, 9$);
2. from $g_{10\mu}$ we obtain a vector gauge field, which is identified with the Ramond–Ramond vector field of the IIA theory;
3. from $C_{10\mu\nu}$ we obtain an antisymmetric tensor field, which is identified with the antisymmetric tensor $B_{\mu\nu}$ of the NS–NS sector of the IIA theory;
4. from $C_{\mu\nu\rho}$ we obtain the three-index antisymmetric tensor field of the R–R sector of the IIA theory;
5. from $g_{10,10}$ we obtain a scalar field in ten dimensions, the dilaton of the IIA theory; note that this mode corresponds to the radius R of the eleventh dimension.

Now consider the action. We will examine just the bosonic terms. These are constructed in terms of the curvature tensor, the three-index antisymmetric tensor and its corresponding four-index field strength F:

$$\mathcal{L} = \frac{-1}{2\kappa^2}\sqrt{g}\mathcal{R} - \frac{1}{48}\sqrt{g}F^2_{MNPQ} - \frac{\sqrt{2}\kappa}{3456}\epsilon^{M_1\ldots M_{11}}F_{M_1\ldots M_4}F_{M_5\ldots M_8}C_{M_9\,M_{10}\,M_{11}}. \tag{28.29}$$

As we have indicated, the dimensional reduction of this theory gives the Lagrangian of the IIA theory in ten dimensions. It is convenient to parameterize the fields in terms of the vielbein e^A_M. Then

$$e^A_M = \begin{pmatrix} e^A_\mu & A_\mu \\ 0 & R_{11} \end{pmatrix}. \tag{28.30}$$

Correspondingly, the metric has the structure

$$g_{MN} = e^A_M e^B_N \eta_{AB} = \begin{pmatrix} g_{\mu\nu} & R_{11}A_\mu \\ R_{11}A_\nu & R^2_{11} \end{pmatrix}. \tag{28.31}$$

If we simply substitute these expressions into the Lagrangian, the coefficient of the Einstein $\mathcal{R}$ term, will be proportional to R. In order to bring this Lagrangian to the canonical, Einstein, form, it is necessary to perform a Weyl rescaling of the metric. Instead, through we will perform the rescaling in such a way as to bring the action to the "string frame". In this frame, all the NS–NS fields have a factor $e^{-2\phi}$ at the front, where $e^{-2\phi}$ is the string coupling (the dilaton). In ten dimensions, $\sqrt{g} = e$ transforms like $(g_{\mu\nu})^5$ under an overall rescaling of the metric; $\mathcal{R}$ transforms like $(g_{\mu\nu})^{-1}$. So we need to make the rescaling:

$$g_{\mu\nu} \to R_{11}^{-2/3} g_{\mu\nu}. \tag{28.32}$$

The three-form C in Eq. (28.29), upon reduction, leads to various fields in ten dimensions. The components $C_{10\mu\nu}$ give the NS–NS two-form. The fields $C_{\mu\nu\rho}$ give the R–R three-form. The R–R one-form field arises from the $g_{10,\mu}$ components of the metric. The ten-dimensional action becomes

$$S = S_{\rm NS} + S_{\rm R}, \tag{28.33}$$

with

$$S_{\rm NS} = \frac{1}{2}\int d^{10}x\,\sqrt{g}e^{-2\phi}\left(\mathcal{R} + (\nabla\phi)^2 - \frac{1}{2}H^2\right), \tag{28.34}$$

$$I_{\rm R} = -\int d^{10}x\,\sqrt{g}\left(\frac{1}{4}F^2 + \frac{1}{2\times 4!}F_4^2\right) - \frac{1}{4}F_4\wedge F_4\wedge B. \tag{28.35}$$

We have seen that, when the action is written in this way, R is related to the coupling of the ten-dimensional string theory. The Weyl rescaling $g_{\mu\nu}\to R_{11}^{-3/4}g_{\mu\nu}$ gives an action with R^3 at the front, i.e.

$$\mathcal{L} = R_{11}^{-3}\left(-\frac{1}{2}\mathcal{R} - \frac{3}{4}R^{-3/2}H^2_{\mu\nu\rho} - \frac{9}{16}\left(\frac{\partial_\mu R_{11}}{R_{11}}\right)^2\right). \tag{28.36}$$

In this form the unit of length is the string scale $\ell_{\rm s}$. So, loops come with a factor R_{11}^3 (the ultraviolet cutoff is $\ell_{\rm s}^{-1}$). We see that

$$g_{\rm s}^2 = \frac{R_{11}^3}{\ell_{11}^3}. \tag{28.37}$$

We can derive this relation in another way (we will ignore the factor $2\pi s$), which makes a more direct connection between eleven-dimensional supergravity and strings. The eleven-dimensional theory has membrane solutions. We will not exhibit then here, but this fact should not be too surprising since the three-form C_{MNO} couples naturally to membranes. The eleven-dimensional theory has only one scale, ℓ_{11}, so the tension of the membranes is of order ℓ_{11}^{-3}. We can wrap one coordinate of the membrane around the eleventh dimension. If the eleventh dimension is very small, the result is a string propagating in ten dimensions, with tension

$$T = \ell_{11}^{-3}R = \ell_{\rm s}^{-2}. \tag{28.38}$$

Now, again the ten-dimensional gravitational coupling is related to ℓ_{11} by

$$G_{10} = \frac{\ell_{11}^9}{R_{11}}. \tag{28.39}$$

So we find, once more,

$$g_{\rm s}^2 = \frac{R_{11}^3}{\ell_{11}^3}. \tag{28.40}$$

Here we have our first piece of circumstantial evidence for the connection. Let us turn now to the BPS spectrum. Consider, first, the eleven-dimensional supersymmetry algebra. Eleven-dimensional spinors can be decomposed into ten-dimensional spinors of definite chirality, with indices α and $\dot\alpha$. In this basis,

$$\Gamma_{11} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{28.41}$$

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The eleven-dimensional momenta decompose into ten-dimensional momenta and p_{11} in an obvious way:

$$\{Q_\alpha, Q_{\dot\alpha}\} = \not{p}_{\alpha,\dot\alpha} + p_{11}\delta_{\alpha,\dot\alpha}. \tag{28.42}$$

From a ten-dimensional point of view, the last term is a central charge. In the presence of such a central charge, we can prove a BPS bound as we did for the monopole. This bound is saturated by the Kaluza–Klein modes of the graviton and the antisymmetric tensor field. To what charge does this central charge correspond in the IIA theory, and to which states do the momentum states correspond? It is natural to guess that this is an R–R charge. The simplest possibility is the charge associated with the one-form gauge field. The carriers of the one-form charge are $D0$-branes. The $D0$-branes are BPS states – they preserve half the ten-dimensional supersymmetry. So *states of definite eleven-dimensional momentum are states of definite D-brane charge.* More precisely, localized states with N units of Kaluza–Klein momentum correspond to the zero-energy bound states (so-called threshold bound states) of N D-branes.

There are numerous further tests of this duality. For example, if one compactifies the theory further, there are connections to IIB theory. There are also connections involving $M5$-branes. This short discussion should gives some flavor of the duality, and the evidence, for it, however.

28.6.2 IIB self-duality

The IIB theory exhibits an interesting self-duality. We can understand this, first, from the Lagrangian. The Lagrangian for the NS–NS fields is the same as for the IIA theory. For the R–R fields we have now zero- two- and four-form fields. The Lagrangian for these is similar, with appropriate indices, to that for the R–R fields of the IIA case. A careful examination shows that, under the transformation $\phi \to -\phi$, the Lagrangian goes into itself. At the classical level, the action is also invariant under shifts of the axion.

Grouping the dilaton e^ϕ and the Ramond–Ramond scalar θ into a complex field

$$\tau = \frac{4\pi i}{g_s} + \frac{\theta}{2\pi}, \tag{28.43}$$

it is then natural to conjecture that the underlying theory has an $SL(2,Z)$ symmetry similar to that of $N = 4$ Yang–Mills theory:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1. \tag{28.44}$$

Further evidence for this symmetry is obtained by studying BPS objects: the various branes of the theory. In the IIB theory we have fundamental strings and $D1$-branes; we also have $D5$-branes. Under this duality the fundamental strings are mapped into $D1$-branes by the $SL(2,Z)$ transformations. Correspondingly, the H_3-form (which couples to fundamental strings) should be mapped into the F_3-form (which couples to $D1$ strings). The $D3$-branes are associated with the gauge-invariant five-form field strength, which is self-dual, so we might expect the $D3$-branes to be invariant. A study of the BPS formulas for these states lends support to these conjectures.

This leaves the $D5$-branes. These couple to the Ramond–Ramond six-form gauge field, which is associated with a seven-form field strength that is in turn dual to the three-form R–R field strength. In other words the $D5$-brane is a magnetic source for F_3. So, we might expect these to be dual to something which is a magnetic source for the NS three-form. This would be an NS five-brane. Such an object can be constructed as a soliton of the ten-dimensional IIB supergravity theory. It plays an important role in understanding the duality of these theories and also appears in other contexts. For example, in M theory, it is associated with a seven-form field strength, which is dual to the four-form field strength that we have already encountered. The $M5$ solution is

$$g_{mn} = e^{2\phi}\delta_{mn}, \quad g_{\mu\nu} = \eta_{\mu\nu}, \tag{28.45}$$

$$H_{mno} = -\epsilon_{mno}{}^{p}\partial_p\phi, \tag{28.46}$$

$$e^{2\phi} = e^{2\phi(\infty)} + \frac{Q}{2\pi^2 r^2}. \tag{28.47}$$

Here μ, ν are the coordinates tangent to the brane (they are the world-volume coordinates) and $m, n \ldots$ are the coordinates transverse to the brane. The $SL(2, Z)$ duality of the IIB theory is quite intricate and beautiful. There are many subtle and interesting checks.

28.6.3 Duality of Type I and $O(32)$

The duality between the Type I and $O(32)$ theories is particularly intriguing, as it is a duality between a theory with open and closed strings and a theory with closed strings only. It is also puzzling, since the perturbative spectra of these theories, at the level of massive states, are quite different. The $O(32)$ heterotic theory contains towers of massive states in spinor representations; there is nothing like this in the perturbative spectrum of the Type I theory. By way of evidence we can begin, again, with the effective Lagrangian. For the heterotic theory this can be written

$$\int d^{10}x\, e^{-2\phi}(\mathcal{R} + |\nabla\phi|^2 + F^2 + dB^2). \tag{28.48}$$

Here $e^{-2\phi}$ is the dilaton field, and we have written the action in the string frame. Consider, now, the transformation

$$g = e^{\phi}g', \quad \phi = -\phi'. \tag{28.49}$$

This takes the action to

$$\int d^{10}x\,\sqrt{g}[e^{-2\phi'}(R + |\nabla\phi'|^2) + e^{-\phi'}F^2 + dB^2]. \tag{28.50}$$

This is the action for the bosonic fields of the Type I theory. The closed-string fields couple with g^2 while the open-string fields couple with g. In the Type I theory the antisymmetric tensor is an R–R field and, as a result, no factor equal to the coupling (the dilaton) appears out front of its kinetic term.

Now we can ask: how do the hetorotic strings appear in the open-string theory? Here, we might guess that these strings would appear as solitons. More precisely, they are just the

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$D1$-branes of the Type I theory. At weak coupling the tension of these strings will behave as $1/g$, i.e. it will be quite large. In this sector one can find states in spinorial representations of $O(32)$ arising from configurations of ($D1$–$D9$)-branes. Most importantly, the $D1$-branes are BPS. As a result they persist to strong coupling, and in this regime their tension is small. We will not explore the various subtle tests of this correspondence, but other features that one can investigate include the identification of the winding strings of the heterotic theory.

Many other dualities among different string theories have been explored. These include an equivalence between heterotic string theory on a four-torus and Type IIA on K3 and equivalences of Calabi–Yau compactifications of the Type II theory and heterotic theory on K3 $\times$ $T2$.

28.7 Strongly coupled heterotic string

In ten dimensions we have seen that the strong coupling limit of the IIA theory is a theory whose low-energy limit is eleven-dimensional supergravity. The strong coupling limit of the IIB theory is again the IIB theory. The strong coupling limit of the $O(32)$ heterotic string is the Type I string. This still leaves the question: what is the strong coupling limit of the $E_8 \times E_8$ heterotic string? The answer is intriguing. It has some tantalizing connections to facts we see in nature. It also suggests different ways of thinking about compactifications – giving an inkling of the large extra dimension and warped-space pictures which we will discuss in the next chapter.

Horava and Witten recognized that the strong coupling limit of the heterotic string, like the IIA theory, is an eleven-dimensional theory. The theory is defined on an interval of radius R_{11}. The relation of R_{11} to the string tension and coupling are exactly as in the IIA case. This means that as the coupling becomes large the interval becomes large. We will refer to the full eleven-dimensional space as the "bulk." The fields propagating in the bulk are a full eleven-dimensional supergravity multiplet: graviton, gravitino and three-form field. At the end of the interval there are two walls (Fig. 28.1). These walls are similar to orientifolds in that they are not dynamical (there are no degrees of freedom corresponding to motion of the walls). The low-lying degrees of freedom on each wall are those of a

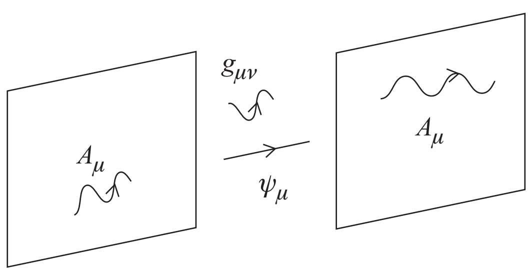

Fig. 28.1 The strongly coupled heterotic string is described by an eleven-dimensional bulk theory and two segregated walls, on which gauge degrees of freedom propagate.

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supersymmetric E_8 gauge theory: gauge bosons and gauginos in the adjoint representation. The Lagrangian has the structure of a bulk plus a boundary term:

$$S = -\frac{1}{2\kappa^2}\int d^{11}x\,\sqrt{g}\,\mathcal{R} - \sum_{i=1}^{2}\frac{1}{8\pi}(4\pi\kappa^2)^{2/3}\int d^{10}x\,\sqrt{g}\,\mathrm{Tr}\,F_i^2 + \cdots . \tag{28.51}$$

Note that the gauge coupling is simply proportional to the sixth power of the eleven-dimensional Planck length.

Support for this picture comes from a variety of sources. First, there is a subtle cancelation of gauge and gravitational anomalies. Second, the long-wavelength limit of this theory is ten-dimensional gravity plus Yang–Mills theory, with a relation between the gauge and gravitational couplings appropriate to the heterotic string (this is one way to determine the relations between the coupling constants). Further compactifications provide further checks.

28.7.1 Compactification of the strongly coupled heterotic string

One puzzle in the phenomenology of the weakly coupled heterotic string concerns the value of the gauge coupling and the unification scale. In the MSSM the unification scale is two orders of magnitude below the Planck scale. If we imagine that the unification scale corresponds to a scale of compactification then

$$\alpha_{\mathrm{gut}} \propto \frac{g_{\mathrm{s}}^2}{V}. \tag{28.52}$$

If we treat the left-hand side as fixed then as V becomes large so does g_{s}. Substituting in the observed values, we see that g_{s} is quite large. As we will now show, the situation in the strong coupling limit is quite different – and much more promising.

Now consider the compactification of the strongly coupled theory on a Calabi–Yau space. The full compact manifold, from the point of view of an eleven-dimensional observer, is the product of the interval times a Calabi–Yau space X. Such a configuration is an approximate solution of the lowest-order equations of motion. Even at the level of the classical equations of this theory, there are corrections arising from the coupling of bulk and boundary fields. These corrections can be constructed in a power series expansion. Terms in the expansion grow with R_{11}, owing to the one-dimensional geometry in the eleventh dimension. They are proportional to $\kappa^{2/3}$, from the bulk–brane coupling in Eq. (28.51). On dimensional grounds there is a factor R^{-4}, where R is the Calabi–Yau radius. The expansion parameter is thus

$$\epsilon = \kappa^{2/3}\frac{R_{11}}{R^4}. \tag{28.53}$$

We can readily obtain the relation between the four-dimensional and eleven-dimensional quantities. Using the string relations (here we need to be careful about factors of 2 and π)

$$G_N = \frac{e^{2\phi}(\alpha')^4}{64\pi V}, \quad \alpha_{\mathrm{gut}} = \frac{e^{2\phi}(\alpha')^3}{16\pi V}, \tag{28.54}$$

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where V is the volume of the compact space X, and the eleven-dimensional relations

$$G_N = \frac{\kappa^2}{16\pi^2 V R_{11}}, \quad \alpha_{\rm gut} = \frac{(4\pi\kappa^2)^{2/3}}{2V}, \tag{28.55}$$

we have

$$R_{11}^2 = \frac{\alpha_{\rm gut}^3 V}{512\pi^4 G_N^2}, \quad M_{11} = R^{-1}\left[2(4\pi)^{-2/3}\alpha_{\rm gut}\right]^{-1/6}. \tag{28.56}$$

where $R = V^{1/6}$. Substituting value of $\alpha_{\rm gut}$ obtaining from running the couplings as in the MSSM (Chapter 11) and the four-dimensional Planck mass gives:

$$R_{11}M_{11} = 18, \quad R = 2\ell_{11} = 3 \times 10^{16}\ \text{GeV}. \tag{28.57}$$

The regime of validity of the strongly coupled description is the regime where V and R_{11} are large compared with ℓ_{11}. We see that nature might well be in such a regime. When we evaluate the expansion parameter ϵ, we find $\epsilon \sim 1$. Adopting the viewpoint that the ground state of string theory which describes nature should be strongly coupled, this, again, seems promising: the parameters of grand unification correspond to the point where the eleven-dimensional expansion is just breaking down, $\epsilon \approx 1$. This is in contrast with the weak coupling picture, which seems far from its range of validity.

Apart from this phenomenological application of string theory ideas, there are two new possibilities which this analysis suggests. First, some compact dimensions might be large compared with the Planck scale (or any fundamental scale). Second, in a case with a one-dimensional geometry, this dimension can be significantly warped, i.e. the metric need not be a constant. These ideas underlie the large-extra-dimension and Randall–Sundrum models of compactification which we will encounter in the next chapter.

28.8 Non-perturbative formulations of string theory

We have seen that, at least in cases with a great deal of supersymmetry, there is a surprisingly large access to non-perturbative dynamics. But much of the evidence for the various phenomena we have described is circumstantial, matching actions and spectra in various regions of a given string moduli space. We lack a general non-perturbative formulation of the theory, analogous to, say, the lattice formulations of Yang–Mills theories which we encountered in Part 1. One might have hoped that there would be a *string field theory* that would be analogous to ordinary quantum field theories, but such a possibility is fraught with conceptual and technical difficulties. We have mentioned some of these. In this section we will describe situations where one can give a complete non-perturbative description. These descriptions are specific to particular backgrounds: flat space in higher dimensions and certain AdS spaces. In eleven dimensions, the flat-space supersymmetric theory can be described as an ordinary quantum mechanical system, while the theory compactified on an n-dimensional torus is described by a field theory in $n + 1$ space–time dimensions, up to $n = 3$. Quite generally, string theory (gravity) in AdS spaces is

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described by *conformal* field theories (CFTs); this is known as the AdS–CFT correspondence. Both formulations exhibit what is believed to be a fundamental feature of any quantum theory of gravity: holography. The holographic principle asserts that the number of degrees of freedom of a quantum theory of gravity grows, not as the volume of the system, but as its area.

28.8.1 Matrix theory

We have seen that the strong coupling limit of the IIA theory is an eleven-dimensional theory, whose low-energy limit is eleven-dimensional supergravity; $D0$-branes were crucial in making the correspondence. The Kaluza–Klein states of the eleven-dimensional theory are bound states of $D0$-branes; states with momentum N/R_{11} correspond to zero-energy ("threshold") bound states of N $D0$-branes. The world-line theory of N $D0$-branes is ten-dimensional $U(N)$ Yang–Mills theory reduced to zero dimensions. The action which describes this system is

$$S = \int dt \left[\frac{1}{g} \operatorname{Tr}(D_t X^i D_t X^i) + \frac{1}{2g} M^6 R_{11}^2 \operatorname{Tr}([X^i, X^j][X^i, X^j]) \right.$$
$$\left. + \frac{1}{g} \operatorname{Tr}(i\theta^T D_t \theta + M^3 R_{11} \theta^T \gamma^i [X^i, \theta]) \right], \tag{28.58}$$

where R_{11} is the eleven-dimensional radius, M is the eleven-dimensional Planck mass and $g = 2R_{11}$. The Xs are the bosonic variables $X_I, I = 1, \ldots, 9$; the θs are the fermionic coordinates. It is necessary to impose Gauss's law as a constraint on states.

Classically and quantum mechanically this system has a large moduli space, corresponding to configurations with commuting X^Is. For large X^I, the spectrum in these directions consists, in the language of quantum mechanics, of $9N$ free particles and a set of oscillators with frequencies of order $|\vec{X}|$. We can integrate out the fast degrees of freedom, obtaining an effective action for the low-energy degrees of freedom, the X^Is and their superpartners. The bosonic states are just momentum states for these particles. They are the states corresponding to the collective modes of the D-branes.

Banks, Fischler, Shenker and Susskind made the bold hypothesis of identifying these degrees of freedom and the Lagrangian of Eq. (28.58), as a complete description of the eleven-dimensional theory, in the limit that $N \to \infty$. They called this the *matrix model*. The Hamiltonian following from the action of (28.58) is identified with the light cone Hamiltonian, and N is identified with the light cone momentum, $P^+ = N/R$. In the large-N limit this becomes a continuous variable; it is necessary to take $R \to \infty$ at a suitable rate. The first step in this identification is to note that the spectrum of low-lying states of the matrix model is precisely that of the light cone supergravity theory. We have already noted that the states are labeled by a momentum nine-vector $\vec{p}$. In addition, there are 16 fermionic variables, the partners of the bosons. As in other contexts we can define eight fermionic creation operators and eight fermionic destruction operators. From these we can construct a Fock space with 256 states, of which half are space–time bosons (i.e. they have integer spin) and half are fermions. This is just the correct number to describe a graviton and an

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antisymmetric tensor in eleven dimensions and their superpartners. The states transform correctly under the little group.

A more convincing piece of evidence comes from studying the S-matrix of the matrix theory. Consider, for example, graviton–graviton scattering. Integrating out the massive states of the theory gives an action involving derivatives of x. We will not reproduce the detailed calculation here but the basic behavior is easy to understand. One can compute the action from Feynman graphs, just as in field theory. With four external Xs, simple power counting gives an action, in coordinate space, behaving as

$$\mathcal{L}_I \approx \dot{X}^4 \int \frac{dk}{(k^2+M^2)^4} \cong c\frac{v^4}{M^7}. \tag{28.59}$$

Here $M \propto |X| = R$, the separation of the gravitons. The four factors of v correspond to the four derivatives in the graviton–graviton amplitude; $1/R^7$ is precisely the form of the graviton propagator in coordinate space. With a little more work one can show that one obtains precisely the four-graviton amplitude in eleven dimensions, for suitable kinematics.

The M theory compactified on an n-torus is described by an $(n+1)$-dimensional field theory. We won't argue this through but will just note that in this case the power counting gives the right graviton–graviton scattering amplitude. If $n > 3$, however, the theory is non-renormalizable and the description does not make sense. An alternative description can be formulated for dimensions down to six. The matrix model has been subjected to a variety of other tests. It turns out that the large-N limit is not necessary; for fixed N one can describe a discretized version of the light cone theory (DLCQ). One can actually derive this result from with the assumed duality between IIA theory and eleven-dimensional supergravity.

All this is quite remarkable. Without even postulating the existence of ordinary space–time we have actually uncovered space–time and general relativity in a simple quantum mechanics model. One interesting feature of these constructions is the crucial role played by supersymmetry. Without it, quantum effects would lift the flat directions and one would not have space–time – though one would still have a sensible quantum system. One might speculate that what we think of as space–time is not fundamental but almost an accident associated with the dynamics of particular systems. Lacking, however, a formulation for a realistic non-supersymmetric system, this remains as speculation.

28.8.2 The AdS–CFT correspondence

An equally remarkable equivalence arises in the case of string theory on anti-de Sitter spaces. This connection was first conjectured by Maldacena and is referred to as the AdS–CFT correspondence. It asserts that gravitational theories in AdS spaces have a description in terms of conformal field theories on the boundary of the space.

28.8.2.1 A little more general relativity: AdS space

We could construct anti-de Sitter space by solving the Friedmann equation with a negative cosmological constant. Instead we will adopt a more geometrical viewpoint. Starting with a flat $(p+3)$-dimensional space, with metric

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$$ds^2 = -dx_0^2 - dx_{p+2}^2 + \sum_{i=1}^{p+1} dx_i^2, \tag{28.60}$$

we consider the hyperboloid

$$x_0^2 + x_{p+2}^2 - \sum_{i=1}^{p+1} x_i^2 = R^2. \tag{28.61}$$

These coordinates can be parameterized in various ways. For example, one can take

$$\begin{aligned} x_0 =& R\cosh\rho\cos\tau, \quad x_{p+2} = R\cosh\rho\sin\tau, \\ x_i =& R\sinh\rho\,\Omega_i, \quad i = 1, \ldots p+1, \quad \Omega_i^2 = 1. \end{aligned} \tag{28.62}$$

This automatically satisfies (28.61) and yields the metric

$$ds^2 = R^2(-\cosh^2\ \rho d\tau^2 + d\rho^2 + \sinh^2\ \rho\, d\Omega^2). \tag{28.63}$$

In making the AdS–CFT correspondence, another parameterization is helpful. This covers half the hyperboloid

$$\begin{aligned} x_0 &= \frac{1}{2u}[1 + u^2(R^2 + \vec{x}^2 - t^2)], \quad x_{p+2} = Rut, \\ x^i &= Rux^i, \quad i = 1, \ldots p, \\ x^{p+1} &= \frac{1}{2u}[1 - u^2(R^2 - \vec{x}^2 + t^2)]. \end{aligned} \tag{28.64}$$

The metric is then

$$ds^2 = R^2\left[\frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2)\right]. \tag{28.65}$$

Anti-de Sitter space has interesting features, which we will not fully explore here. There is a boundary at spatial infinity ($u = \infty$). Light can reach the boundary in finite time, but massive particles cannot do so. In a cosmological context, the negative cosmological constant leads not to an eternal AdS space but to a singularity. The last form of the metric will be useful in making the AdS–CFT correspondence in a moment. The metric has isometries (symmetries); the group of isometries can be seen from the form of the hyperboloid and the underlying metric of the $(p+3)$-dimensional space; it is $SO(2, p+1)$. This turns out to be the same symmetry as conformal symmetry in $p+1$ dimensions; this, again, is a crucial aspect of the AdS–CFT correspondence.

28.8.2.2 Maldacena's conjecture

Maldacena originally discovered this connection for the case of string theory on $AdS_5 \times S_5$. One suggestive argument starts by considering a set of N parallel $D3$-branes. We have discussed such configurations as open-string configurations but they can also be uncovered as solitonic solutions of the supergravity equations, here of the IIB theory. For these, the

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metric has the form

$$ds^2 = H(y)^{-1/2} dx^\mu dx_\mu + H(y)^{1/2}\big(dy^2 + y^2 d\Omega_5^2\big)$$
$$F_{\mu\nu\rho\sigma\tau} = \epsilon_{\mu\nu\rho\sigma\tau\alpha}\partial^\alpha H. \tag{28.66}$$

Here the x^μs are the coordinates tangent to the branes, while the ys (and their associated angles) are the transverse coordinates. The dilaton in this configuration is a constant; the other antisymmetric tensors vanish. The function H, for N parallel branes, is

$$H(\vec{y}) = 1 + \sum_{i=1}^{N} \frac{4\pi g_s(\alpha')^2}{|\vec{y} - \vec{y}_i|^4}. \tag{28.67}$$

This can be rewritten as

$$ds^2 = \left(1 + \frac{L^2}{y^4}\right)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{L^2}{y^4}\right)^{1/2} \big(dy^2 + y^2 d\Omega_5^2\big). \tag{28.68}$$

The parameter L is related to the string coupling g_s, the brane charge (the number of branes) N and the string tension α' by:

$$L^4 = 4\pi g_s N(\alpha')^2. \tag{28.69}$$

It is convenient to introduce a coordinate $u = L^2/y$ and to take a limit where N and g_s are fixed while $\alpha' \to 0$. The metric then becomes:

$$ds^2 = L^2\left(\frac{1}{u^2}\eta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2} + d\Omega_5^2\right). \tag{28.70}$$

We have seen the terms involving u and x previously; this is the geometry of AdS_5. The remaining terms describe a five-sphere of radius L.

Now, from a string point of view the low-energy limit of the system of N $D3$-branes is described by $N = 4$ Yang–Mills theory. So we might, with Maldacena, conjecture that there is just such an equivalence between the brane configuration of the string theory (a gravity theory in AdS space) and the field theory. Not surprisingly, demonstrating this equivalence is not simple. One needs to argue that on the string side the bulk modes (graviton, antisymmetric tensors and so on) decouple, as do the massive excitations of the open strings ending on the branes. One cannot argue this at weak coupling, and it would be surprising if one could since in that case one could calculate any quantity in the gravity theory in a weak coupling perturbation expansion in the Yang–Mills theory. This is similar to the situation in the matrix model. There are, however (as in the matrix model), many quantities which are protected by supersymmetry, and these permit quite detailed, consistency checks both in this case and for many other examples of the correspondence.

Suggested reading

Non-perturbative string dualities are discussed extensively in the second volume of Polchinski's (1998) book. This provides an excellent introduction to D-branes. They are treated at length in the text by Johnson (2003), as well. The reader may want to consult earlier papers on duality, especially Witten (1995). Matrix theory and the AdS–CFT correspondence are treated in several excellent pedagogical reviews (Bigatti and Susskind, 1997; Aharony *et al.*, 2000; D'Hoker and Freedman, 2002), but the original papers are very instructive; see, for example, Banks *et al.* (1997); Seiberg (1997), Maldacena (1997) and Witten (1998).

Exercises

(1) *D-branes* For a stack of N D-branes, write the open-string mode expansions. Show that, for small separations, the spectrum looks like that of a Higgs $U(N)$ field theory, with the Higgs in the adjoint representation. In the light cone gauge, check the counting of supersymmetries for open strings and D-branes.

(2) Verify the construction of the bosonic terms in the ten-dimensional action from the dimensional reduction of the eleven-dimensional action.

(3) Verify that the NS5-brane is a solution of the ten-dimensional supergravity equations.

(4) Take the long-wavelength limit of the Horava–Witten theory (see Section 28.7). Write down the Lagrangian in the ten-dimensional Einstein frame and verify that the gauge and gravitational couplings obey a relation appropriate to the heterotic string theory,

$$g_{\rm ym}^2 = 4\kappa^2\alpha'^{-1}. \tag{28.71}$$

(5) Calculate the effective action of the matrix model at one loop Eq. (28.58) in more detail. Verify that, treated in the Born approximation, this yields the correct graviton–graviton scattering matrix element for the eleven-dimensional theory. You may find the background-field method helpful for this computation.

(6) Check that the configuration of Eq. (28.66) solves the field equations of IIB supergravity in the case of a single brane. You may want to use some available programs for evaluating the curvature. Verify that in the Maldacena limit, the metric can be recast as in Eq. (28.30). If one requires that the curvature of the AdS space is small, it needs to be checked that the D-brane theory is strongly coupled. Discuss the problem of decoupling.

https://doi.org/10.1017/9781009290883.035 Published online by Cambridge University Press