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Non-local rings whose ideals are all quasi-injective

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A ring is a left Q-ring if all of its left ideals are quasi-injective. For an integer $m \geq 2$, a sfield D, and a null D-algebra V whose left and right D-dimensions are both equal to one, let $H(m, D, V)$ be the ring of all $m \times m$ matrices whose only non-zero entries are arbitrary elements of D along the diagonal and arbitrary elements of V at the places $(2, 1), \ldots, (m, m-1)$ and $(1, m)$. We show that the only indecomposable non-local left Q-rings are the simple artinian rings and the rings $H(m, D, V)$. An arbitrary left Q-ring is the direct sum of a finite number of indecomposable non-local left Q-rings and a Q-ring whose idempotents are all central.

A ring is a *(left) Q-ring* if all of its left ideals are quasi-injective. The study of Q-rings was initiated in [2]; in this note we determine the structure of indecomposable non-local Q-rings, and reduce the general problem to the investigation of Q-rings whose idempotents are all central. We show that there are only two types (both artinian) of indecomposable non-local Q-rings, and represent them as matrix rings. Our methods depend on the existence of non-central idempotents, a condition satisfied by all indecomposable non-local Q-rings, since the local Q-rings are precisely those with no idempotents other than their identities [1, Proposition 5.8].

Throughout this note all rings are unital and associative; all

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Q-rings are indecomposable and non-local; all modules are unital; and everything is done on the left (unless otherwise specified), so, for example, *ideal* means *left ideal*. The socle of a module M is denoted by $S(M)$.

We begin by showing that a Q-ring has non-zero socle: this guarantees that the ring has primitive idempotents. We then show that the ring cannot have an infinite number of ideals with the following two properties: the sum of the ideals is direct, and each ideal has a homomorphic image in the ring which intersects the ideal trivially. Using this we deduce that if at least one minimal ideal is injective then the ring is simple artinian. If the ring is not simple artinian then every indecomposable injective ideal has a unique proper submodule. It follows that the socle of the ring is essential. We then show that the ring is artinian and represent it as a full ring of matrices.

We will need the following result.

LEMMA 1 [2, Theorem 2.3]. *A ring is a Q-ring if, and only if, it is self-injective and all of its essential ideals are two-sided.*

LEMMA 2. *The socle of a Q-ring R is non-zero. If $e_1, e_2 \in R$ are orthogonal idempotents, then $e_2Re_1 \subseteq S(Re_1)$. If $Re_1 \cong Re_2$ then both Re_i are (finite) sums of minimal ideals.*

Proof. Let $e, f \in R$ be orthogonal idempotents whose sum is the identity. As R is indecomposable, either $eRf \neq 0$ or $fRe \neq 0$. Assume the latter and let L be an essential submodule of Re . By Lemma 1, the ideal $L \oplus Rf$ is two-sided: consequently $fRe = f^2Re$ is contained in L . Therefore the intersection L_0 of all essential submodules of Re is non-zero: it follows that L_0 is a sum of minimal ideals and so the socle of R is non-zero. By a similar argument we can show that $e_2Re_1 \subseteq S(Re_1)$. If $Re_1 \cong Re_2$ then $S(Re_1)$ contains e_2Re_1 which generates Re_1 : therefore $Re_1 = S(Re_1)$.

LEMMA 3. *Let R be a Q-ring and let $\{e_i \mid i \in I\}$ be an infinite set of mutually orthogonal idempotents. Then only finitely many e_i have the property that Re_i has a homomorphic image in $R(1-e_i)$.*

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Proof. Let I_0 be the set of all $i \in I$ such that Re_i has a homomorphic image in $R(1-e_i)$, and assume that I_0 is infinite. If any two ideals Re_i, Re_j, $i, j \in I$, have a common homomorphic image then, by the projectivity of Re_i, there is a homomorphism from Re_i to Re_j. It follows that there is an infinite subset $I_1 \subseteq I_0$ with the property that for each $i \in I_1$ there is a homomorphism $\varphi_i : Re_i \to R(1-e_i)$ satisfying the conditions that $\left(\bigoplus_{I_1} Re_i\right) \cap Re_i\varphi_i = 0$ and that all the $Re_i\varphi_i$ are distinct. Let Re be an injective hull of $\bigoplus_{I_1} Re_i$ and let $\varphi : \bigoplus_{I_1} Re_i \to R(1-e)$ be the sum of all the φ_i, $i \in I_1$. As $R(1-e)$ is injective, φ can be lifted to a homomorphism $\bar{\varphi} : Re \to R(1-e)$. But this is a contradiction since, by Lemma 2, $Re\bar{\varphi}$ is a finite sum of minimal ideals. Hence I_0 is finite.

THEOREM 1. *Let R be a Q-ring and $e \in R$ an idempotent. If Re is a minimal ideal then R is a simple artinian ring.*

Proof. By Lemma 3 the ring R has only a finite number of ideals isomorphic to Re. Let Rf be their sum, then $fR(1-f) = 0$. Since Re is projective it can be a homomorphic image only of modules which contain an isomorphic copy of itself. Therefore $(1-f)Rf = 0$. As R is indecomposable this means that it is a sum of mutually isomorphic minimal ideals. Therefore R is a simple artinian ring.

In view of Theorem 1 all Q-rings will, from now on, have the property that none of their minimal ideals is injective.

LEMMA 4. *Let e be a primitive idempotent in a Q-ring R. Then eRe is a sfield and the only submodule of Re is $S(Re) = (1-e)Re$.*

Proof. First we will show that $(1-e)Re \neq 0$. Assume that $(1-e)Re = 0$; then the ideals of eRe and the R-submodules of Re coincide, and as R is indecomposable, $eR(1-e) \neq 0$. By Proposition 5.8 of [1] the ring eRe has a unique maximal ideal, J say, which is not zero since R has no minimal ideals which are injective. Therefore there is an ideal $L \subseteq J$ which has a simple factor module. As eRe is a local ring and as every image in $R(1-e)$ of Re is a minimal ideal (Lemma 2), this means that there is a homomorphism from L to $R(1-e)$. By the injectivity of $R(1-e)$ this map can be lifted to Re : a contradiction, since by Lemma 2 every homomorphism from Re to $R(1-e)$ kills J and hence L. Therefore $(1-e)Re \neq 0$.

By Theorem 5.1 of [1] every element of J is annihilated on the left by an essential submodule of Re. Hence, by Lemma 2, $(1-e)ReJ = 0$ and so J is an ideal of R. But every non-zero submodule of Re must contain the minimal ideal $S(Re) \supseteq (1-e)Re$; therefore $J = 0$. That is, eRe is a sfield. Hence every non-zero element of eRe generates Re and, as $(1-e)Re$ generates $S(Re)$, the only submodule of Re is $S(Re) = (1-e)Re$.

LEMMA 5. *The socle of a Q-ring is essential.*

Proof. Let R be a Q-ring and let Re be an injective hull of $S(R)$. If $f_1, f_2 \in R(1-e)$ are orthogonal idempotents then, by Lemma 2, both $f_1Rf_2 = 0$ and $f_2Rf_1 = 0$: so as R is indecomposable, both products $f_iRe \neq 0$, $i = 1, 2$. As $R(1-e)$ does not contain any minimal ideals it does not contain any primitive idempotents (Lemma 4). Therefore $R(1-e)$ has an infinite set $\{e_i\}$ of mutually orthogonal idempotents such that each Re_i has a homomorphic image in Re : a contradiction to Lemma 3. Therefore $R(1-e) = 0$ and $S(R)$ is essential in R.

An idempotent is *finite* if it is a (finite) sum of primitive orthogonal idempotents; otherwise it is *infinite*.

THEOREM 2. *A Q-ring is artinian.*

Proof. Let R be a Q-ring and assume it is not artinian. Then, by Lemmas 4 and 5, the set $\{M_i \mid i \in I\}$ of minimal ideals of R is infinite. For each $i \in I$ let Re_i be an injective hull of M_i. It follows from Lemmas 2, 3 and 4 that only a finite number of the Re_i, say $Re_{t(1)}, \ldots, Re_{t(n)}$, have proper homomorphic images in R, and that these are minimal ideals, say $M_{s(1)}, \ldots, M_{s(n)}$. Let $I_0 = I\backslash\{t(1), \ldots, t(n)\}$; then $e_iM_j = 0$ for every pair

$(i, m) \in I_0 \times I$.

As all the M_i are mutually non-isomorphic minimal ideals it follows that for every pair $(i, j) \in I \times I$ there is an element $x \in l(M_j)$, the left annihilator of M_j , with the property that $xM_i \neq 0$. By Theorem 5.1 of [1], there is an element $y \in R$ such that yx is an idempotent modulo the Jacobson radical J of R with the property that $x + J = ryx + J$ for some $r \in R$. By Theorem 5.6 of [1] there is an idempotent $f \in R$ such that $f + J = yx + J$. Therefore $f \in l(M_j)$ and $fM_i = M_i$, that is, M_i is an image of Rf . This enables us to pick out an infinite number of orthogonal idempotents with the property that the ideal generated by each idempotent has an image outside itself, thus contradicting Lemma 3.

Let e', e'' be infinite orthogonal idempotents whose sum is the identity and having the property that all the ideals $Re_{t(1)}, \ldots, Re_{t(n)}$, $M_{s(1)}, \ldots, M_{s(n)}$ are contained in one and the same of the ideals Re', Re'' . Then there are an infinite number of minimal ideals which are images of one of the ideals Re', Re'' and are contained in the other. For assume that this is not true, and let Re_1' (respectively Re_1'') be an injective hull in Re'' (respectively Re') of all the images of Re' (respectively Re'') in Re'' (respectively Re'). As the idempotents e_1', e_1'' are both finite the ideals Re_1', Re_1'' have no images outside themselves in R : therefore the ideals $R(e'+e_1'-e_1'')$ and $R(e''+e_1''-e_1')$ annihilate each other. As R is indecomposable and $R = R(e'+e_1'-e_1'') \oplus R(e''+e_1''-e_1')$, this is impossible: therefore one of the ideals, Re' say, has an infinite number of images in the other, Re'' .

Let $I_1 = \{i \in I_0 \mid e'M_i = M_i \text{ and } M_i \subseteq Re''\}$. Let $k \in I_1$, and let $g \in l(M_k) \cap Re'$ be an idempotent which is a left identity for some M_i , $i \in I_1$. Then one of the idempotents g , $e' - g$ is a left identity for an infinite number of M_i , $i \in I_1$. Denote it by f_1' and let the other idempotent be f_1 . By a completely analagous argument we can show that f_1' is a sum of two orthogonal idempotents f_2, f_2' such that f_2' is a

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left identity for an infinite number of M_i , $i \in I_1$. As this procedure is clearly inductive, we can pick for each positive integer n an idempotent $f_n \in Re'$ such that all the f_n are mutually orthogonal and each Rf_n has an image in Re'' . But this is a contradiction to Lemma 3: therefore our original assumption is false. That is, R is artinian.

For any integer $m \geq 2$, any sfield D , and any null D-algebra V whose left and right D-dimensions are both equal to one, let $H(m, D, V)$ be the ring of all $m \times m$ matrices whose only non-zero entries are arbitrary elements of D along the diagonal, and arbitrary elements of V at the places $(2, 1), \ldots, (m, m-1)$ and $(1, m)$.

THEOREM 3. *Every Q-ring is isomorphic to one of the rings $H(m, D, V)$; conversely, every $H(m, D, V)$ is a Q-ring.*

Proof. Let R be a Q-ring; then, by Theorem 2 and Lemma 4, there is an integer $m \geq 2$ and a set $\{e_i \mid 1 \leq i \leq m\}$ of mutually orthogonal primitive idempotents such that $R = \bigoplus_1^m Re_i$. As the only minimal ideals of R are the socles of the Re_i , it follows from Lemma 4 that every minimal ideal is the image of an Re_i . If a minimal ideal M is the image of an Re_i , then as Re_i is projective it follows from Lemma 2 that M is not the image of any other Re_j , that is, $e_jM = 0$ for all $j \neq i$. Therefore each Re_i determines uniquely that Re_j whose socle is the image of Re_i and that Re_k which has $S(Re_i)$ as a factor module. As R is indecomposable a standard argument shows that we may assume the e_i to be indexed in such a way that $S(Re_i)$ is the image of Re_{i+1} if $1 \leq i \leq m-1$ and $S(Re_m)$ is the image of Re_1 .

We know that $R \cong \hom_R(R, R)$ which is isomorphic to H , say, the ring of all $m \times m$ matrices (φ_{ij}) where $\varphi_{ij} \in \hom_R(Re_i, Re_j) = H_{ij}$. It follows from the preceding paragraph that the only non-zero entries of H are along the diagonal and at the places $(2, 1), \ldots, (m, m-1)$ and $(1, m)$. As each Re_i is injective and e_iRe_i is a sfield, it follows that $H_{ii} \cong \hom_R\left(S(Re_i), S(Re_i)\right)$. If $S(Re_i)$ is the image of Re_j then $H_{jj} \cong \hom_R\left(S(Re_i), S(Re_i)\right)$, therefore all the sfields H_{ii} are mutually isomorphic; $H_{ii} \cong D$, say, for all i. Moreover, each non-zero H_{ij} is a one dimensional left vector space over H_{ii} and a one dimensional right vector space over H_{jj}. Hence all the non-zero H_{ij} are mutually isomorphic D-bivector spaces, all isomorphic to V, say. As $H_{ij}H_{jk} = 0$, the space V is a null algebra. Therefore $R \cong H(m, D, V)$.

We now prove the converse: we show that $H = H(m, D, V)$ is a Q-ring. First we show that H is injective. For any i let $e_i \in H$ be the matrix whose only non-zero entry is 1 at the place (i, i). If $L \subseteq H$ is an ideal then $L = L_1 \oplus L_2$ where $L_1 \subseteq He_i$ and $L_2 \cap He_i = 0$. Any homomorphism from L_1 to He_i can be lifted to an endomorphism of He_i. If there is a homomorphism φ from L_2 to He_i then L_2 contains a direct summand isomorphic to He_j ($j = i+1$ if $i \neq m$; $j = 1$ if $i = m$), and $\ker\varphi$ is a complement to this summand. Consequently every homomorphism from L to He_i can be lifted to H, that is, He_i is injective. Therefore, as H is a finite sum of injective ideals, it is itself injective. For any element $x \in H$ it is clear that $xH \subseteq S(H) + Hx$. Therefore every essential ideal of H is two-sided. Now we apply Lemma 1 to deduce that H is a Q-ring.

REMARK. It follows from Lemma 3 that an arbitrary Q-ring (without the restrictions imposed in this note) is the sum of a finite number of artinian non-local Q-rings (as described in Theorems 1 and 3) and a Q-ring whose idempotents are all central.

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