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High dimensional knot groups which are not two-knot groups

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This paper presents three arguments, one involving orientability, and the others Milnor duality and, respectively, the injectivity of cup product into H^2 for an abelian group and free finite group actions on homotopy 3-spheres to show that there are high dimensional knot groups which are not the groups of knotted 2-spheres in S^4, thus answering a question of Fox ("Some problems in knot theory", *Topology of 3-manifolds and related topics*", 168-176 (Proceedings of the University of Georgia Institute, 1961. Prentice-Hall, Englewood Cliffs, New Jersey, 1962).

In [6], Problem 29, Fox asked whether there was a knotted S^3 in S^5 whose group was not that of a knotted S^2 in S^4. Kervaire [10] showed that a group G was the group of a smooth knotted S^n in S^{n+2} (for $n \geq 3$) if and only if it was finitely presentable, of weight one, $H_1(G) = Z$ and $H_2(G) = 0$. There are given below three families of such groups which cannot occur as the groups of 2-knots (embeddings of S^2 in a homotopy 4-sphere).

The first type have Eilenberg-Mac Lane space a non-orientable $S^1 \times S^1 \times S^1$ bundle over S^1; the argument uses Wu's Theorem. The second type have abelian commutator subgroup; the argument uses Milnor

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duality and injectivity of cup-product into H^2 for an abelian group to show that for a 2-knot such a commutator subgroup must be Z^3 or finite. The third type have finite commutator subgroup (equivalently, have 2 ends [17]); the argument again uses Milnor duality, to show that the universal cover of the manifold obtained by surgery on such a 2-knot is a homotopy 3-sphere, and hence the finite groups which may occur must have cohomology of period dividing 4 .

The paper concludes with some remarks on the groups which may be realised by fibred 2-knots.

Non-orientable torus bundles

Let $A \in GL(3, Z)$ be such that $\det A = -1$, $|\det(A \pm I)| = 1$. Then $|\det(\Lambda^2 A - I)| = 1$ (since $\Lambda^2 A = (\det A).(A^{-1})^{tr}$). For example $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ is such a matrix. A determines an isotopy class of automorphisms φ of the 3-torus $S^1 \times S^1 \times S^1$ such that $H_1(\varphi) = A$. Let M_A be the mapping torus of such an automorphism (that is,

$$M_A = (S^1 \times S^1 \times S^1 \times [0, 1]) / (\langle s, 0\rangle \sim \langle \varphi(s), 1\rangle) ;$$

the homeomorphism type of M_A is well defined and depends only on the conjugacy class of A in $GL(3, Z)$. M_A is a non-orientable $K(\pi, 1)$-manifold, with fundamental group $\pi_A = \{Z^3, t \mid tzt^{-1} = Az \text{ for all } z \in Z^3\}$ (an HNN construction with base Z^3). π_A is finitely presentable, has weight one (that is, $\pi_A/\langle\langle t\rangle\rangle$ is trivial, where $\langle\langle S\rangle\rangle$ denotes the normal subgroup generated by a subset S of a group G), $H_1(\pi_A) = Z$ and $H_2(\pi_A) = 0$ (via the Wang sequence of the fibration $M_A \to S^1$). Therefore π_A is a high dimensional knot group.

Suppose that there were a knot $k : S^2 \to S^4$ with group

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$\pi_1(S^4-k(S^2)) \approx \pi_A$. Choose a framing for ν_k and let Y be the result of surgery on k with respect to this framing (that is,

$$Y = (S^4\text{-int } N(k)) \underset{S^1\times S^2}{\cup} S^1 \times D^3$$ for some choice of tubular neighbourhood

$N(k)$ and homeomorphism from $\partial N(k)$ to $S^1 \times S^2$). Y is an orientable 4-manifold with fundamental group π_A and homology $\mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}$. Since $M_A \approx K(\pi_A, 1)$, there is a map $f : Y \to M_A$ inducing an isomorphism of fundamental groups. $H_j(f, \mathbb{Z}/2\mathbb{Z})$ is clearly an isomorphism for $j = 0, 1, 2$; hence, by $\mathbb{Z}/2\mathbb{Z}$-Poincaré duality, for all j ; that is, f is a $\mathbb{Z}/2\mathbb{Z}$-(co)homology isomorphism. Therefore, by Wu's Theorem $W_1(Y) = f^*W_1(M_A) \neq 0$ which contradicts the orientability of Y .

REMARK. A similar construction has recently been used by Cappell and Shaneson to construct a PL-fake RP^4 , [2].

Two-knots with abelian commutator subgroup

In [1], Cappell considered fibred 2-knots with fibre a punctured $S^1 \times S^1 \times S^1$. For such knots the commutator subgroup of the knot group is isomorphic to $\mathbb{Z}^3$. In higher dimensions one may construct knots with commutator subgroup finitely generated free abelian of any rank not equal to 1 or 2 by using an HNN construction analogous to Cappell's and invoking Kervaire's characterization of high-dimensional knot groups. Indeed, if a finitely presented group K is the commutator subgroup of a knotted S^n in S^{n+2} then K admits an automorphism φ such that $H_1(\varphi) - 1$, $H_2(\varphi) - 1$ are automorphism of $H_1(K)$, $H_2(K)$ respectively. (Consider the Wang sequence of $K \to G \to \mathbb{Z}$.) Conversely, if a finitely presented group K has an automorphism φ satisfying these two conditions and such that $\langle\langle k^{-1}\varphi(k); k \in K\rangle\rangle = K$, then K is the commutator subgroup of a knotted S^n in S^{n+2} (for any $n \geq 3$), for the group $\{K, t \mid tkt^{-1} = \varphi(k) \text{ for all } k \in K\}$ satisfies the above criteria of Kervaire. It shall be shown that if a finitely generated abelian group is the commutator subgroup of a 2-knot group, then it is isomorphic to Z^3 or is finite.

First there is the following lemma, presumably well-known (*cf.* [18]):

LEMMA. *Let A be a finitely generated abelian group.*

(1) *If $F = Q$ or Z/pZ, p odd, then cup-product $\Lambda_2(H^1(A, F)) \to H^2(A, F)$ is injective.*

(2) *The kernel of cup-product $\mathrm{Sym}_2(H^1(A, Z/2Z)) \to H^2(A, Z/2Z)$ is isomorphic to the kernel of the Bockstein map*

$$Sq^1 : H^1(A, Z/2Z) \to H^2(A, Z/2Z) ;$$

that is the image of reduction mod 2 , $\rho : H^1(A, Z/4Z) \to H^1(A, Z/2Z)$.

The proof is elementary - more generally one can relate the kernel of cup-product into $H^2(G, F)$ to the subquotient G_2/G_3, for G any finitely generated group (*cf.* [18]).

THEOREM 1. *Let $k : S^2 \to \Sigma^4$ be a 2-knot with group G such that G' is abelian. Let F be a prime field (that is, Q, $Z/2Z$, or Z/pZ). Then $\beta_F = \mathrm{rk}_F(G' \otimes_Z F) \leq 3$.*

Proof. Let $X = \Sigma^4 - \mathrm{int}\, N(k)$ for some tubular neighbourhood $N(k)$ of $k(S^2)$, and let $Y = X \underset{S^1 \times S^2}{\cup} S^1 \times D^3$. Then $\pi_1(X) = \pi_1(Y) = G$. Let X', Y' be the maximal abelian covers of X, Y respectively (so $\pi_1(X') = \pi_1(Y') = G'$). Then $H_*(X', F)$ is finitely generated over F, [14], so $H_*(Y', F)$ is finitely generated over F, since $Y' \sim X' \underset{S^2}{\cup} D^3$. By the duality theorem of Milnor [13], Y' satisfies F-Poincaré duality with formal dimension 3 (that is $H^3(Y', F) \approx F$ and cup-product $H^i(Y , F) \otimes H^{3-i}(Y', F) \to H^3(Y', F)$ is a perfect pairing). Therefore $\mathrm{rk}_F(H^2(Y', F)) = \mathrm{rk}_F(H^1(Y', F)) = \beta_F$. Also by a theorem of Hopf [8, p.201]

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$H^2(G', F) \subseteq H^2(Y', F)$. Therefore, by the lemma, if $F = Q$ or Z/pZ , p odd, we must have $\frac{1}{2}\beta_F(\beta_F-1) = \text{rk}\left[\Lambda_2\left(H^1(G', F)\right)\right] \le \text{rk}_F\left(H^2(G', F)\right) \le \beta_F$; hence $\beta_F \le 3$. If $F = Z/2Z$, we must have

$$\tfrac{1}{2}\beta_F(\beta_F+1) = \text{rk}_F\left[\text{Sym}_2\left(H^1(G', F)\right)\right] \le \text{rk}_F\left(H^2(G', F)\right) + \text{rk}_F(\text{Im}\ \rho) \le 2\beta_F ;$$

hence again $\beta_F \le 3$. //

COROLLARY. *Suppose G' is finitely generated and infinite. Then $G' \approx Z^3$.*

Proof. By assumption, $\beta_Q > 0$. G' must admit an automorphism φ such that $\varphi - 1$ is also an automorphism (as above), so G' cannot be isomorphic to Z + torsion ; so $\beta_Q > 1$. If $\beta = 2$, then cup-product : $H^1(Y', Q) \times H^1(Y', Q) \to H^2(Y', Q)$ would have to be null (otherwise there would be an element of $H^1(Y', Q)$ nontrivially paired with the image of cup-product; hence a non-zero element of $\Lambda_3\left(H^1(Y', Q)\right)$). Hence $\beta_Q = 3$; that is, $G = Z^3 + T$, T a finite group. If there were a prime p that divided the order of T then $\beta_{Z/pZ} > 3$ which would contradict the theorem. Thus $G = Z^3$. //

REMARK. Conversely, following Cappell one may realize all possible 2-knot groups with commutator subgroup Z^3 by surgery on a cross-section of the mapping torus of an automorphism of $S^1 \times S^1 \times S^1$. The equivalence classes of such knots are determined by the conjugacy classes of matrices $M \in GL(3, Z)$ such that $\det M = 1$ and $\det(M-I) = \pm 1$. By a theorem of Latimer and MacDuffee [*14*] the conjugacy classes of matrices in $GL(n, Z)$ with given irreducible characteristic polynomial correspond to the ideal classes of the field generated over Q by a root of the polynomial. Hence such knots are determined (among all fibred 2-knots with fibre a punctured $S^1 \times S^1 \times S^1$) up to a finite ambiguity by their first Alexander polynomial.

In a similar vein, one can show the answer to Problem 28 in [*6*] is no.

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Let $k : S^1 \to S^3$ be a Neuwirth-Stallings knot of genus 1 (for example the trefoil knot or the figure eight knot) with group G. Then $\pi = G/G''$ is a finitely presentable quotient of a knot group with abelianization $\mathbb{Z}$, which is not the group of a knot in any dimension, since $H_2(\pi) = \mathbb{Z}$ (π has for Eilenberg-Mac Lane space an $S^1 \times S^1$-bundle over S^1).

Two-knots with finite commutator subgroup

As is well known, all classical knot groups are torsion-free [15]. This is not the case in higher dimensions [5]. Any finite group admitting an automorphism φ as above may occur as the commutator subgroup of a knotted S^n in S^{n+2} for $n \geq 3$. Stronger restrictions must be imposed on K for it to be the commutator subgroup of a 2-knot.

THEOREM 2. *Let $k : S^2 \to \Sigma^4$ be a 2-knot with group G such that G' is finite. Then G' has cohomological period dividing 4.*

Proof. Let X, Y, X', Y', F be as in Theorem 1. Let $\tilde{Y}$ be the universal cover of Y. Then $H_*(\tilde{Y}, F)$ is finitely generated over F, as $\tilde{Y}$ is a finite cover of Y'. But $\tilde{Y}$ is also the infinite cyclic cover of the closed 4-manifold Z, where Z is the irregular cover of Y associated with the image of a chosen splitting of the abelianization map $G \to \mathbb{Z}$. So by the duality theorem of Milnor $\tilde{Y}$ satisfies F-Poincaré duality with formal dimension 3. Let $A = H_2(\tilde{Y}, \mathbb{Z})$, $B = H_3(\tilde{Y}, \mathbb{Z})$. A, B are finitely generated Λ-modules (where Λ is the group ring of $\mathbb{Z}$, and is isomorphic to $\mathbb{Z}[t, t^{-1}]$) [13]. By the universal coefficient theorem, $\hom(A, F) = 0$. Since this is true for all fields F, A is torsion and p-divisible for all p. Let $a_1, \ldots, a_k$ generate A over Λ, and suppose $n_j a_j = 0$, $1 \leq j \leq K$. Then $N.A = 0$, where $N = \prod_{j=1}^{K} n_j$. Hence $A = 0$. Now by Milnor [13] the sequence

$$0 \to H_4(Z) \to H_3(\tilde{Y}) \xrightarrow{t-1} H_3(\tilde{Y}) \to H_3(Z) \to H_2(\tilde{Y})$$

(that is, $0 \to \mathbb{Z} \to B \xrightarrow{t-1} B \to \mathbb{Z} \to 0$) is exact. Therefore $B = \mathbb{Z} \oplus C$, where $C = \mathrm{Im}(t-1)$ is a finitely generated Λ-submodule. As before, by the universal coefficient theorem, $\hom(C, F) = 0$ for all fields F, and hence $C = 0$. Thus $\tilde{Y}$ is homotopy equivalent to S^3, and so G' has cohomological period dividing 4, [3]. //

REMARK. In particular, every abelian subgroup of G' must be cyclic [3].

COROLLARY 1.* *If G' is finite nilpotent then it is cyclic of odd order, or the direct product of such a cyclic group and a quaternion group.*

Proof. If every abelian subgroup of a finite p-group is cyclic, then the group is cyclic if p is odd, and contains a cyclic subgroup of index 2 if $p = 2$ ([9], Theorem III.7.6). It is not hard to check that the quarternion group is the only 2-group that admits an automorphism φ as above (see the discussion on pp. 456, 457 below); hence G', the product of its Sylow subgroups, is cyclic of odd order.

COROLLARY 2. *Let I^* denote the binary icosahedral group (a perfect group of order 120, with a presentation*

$$\{x, y \mid x^2 = (xy)^3 = y^5\}).$$

Then $I^* \times I^*$ is not the commutator subgroup of a 2-knot group, although it is the commutator subgroup of a high-dimensional knot group.

Proof. $I^* \times I^*$ clearly contains noncyclic abelian subgroups. On the other hand $H_1(I^*) = H_2(I^*) = 0$ (for example, since I^* is the fundamental group of the Poincaré homology 3-sphere) so $H_1(I^* \times I^*) = H_2(I^* \times I^*) = 0$ by the Künneth Theorem. To show that $I^* \times I^*$ is the commutator subgroup of a high dimensional knot group it will suffice to give an automrophism φ of $I^* \times I^*$ such that

$$\{I^* \times I^*, t \mid tjt^{-1} = \varphi(j) \text{ for all } j \in I^* \times I^*\}$$

is of weight one. One such automorphism is $\psi : \langle u, v\rangle \mapsto \langle xux^{-1}, yvy^{-1}\rangle$. (Since I^* has only 1 non-trivial normal subgroup, it is easy to verify that $\langle\langle j^{-1}\psi(j)\rangle\rangle = I^* \times I^*$.) //

REMARK. By a similar application of Milnor duality, one can easily prove Giffen's weak unknotting theorem [7] that if $\pi_1\left(S^4 - k\left(S^2\right)\right) = Z$, then

* [Amended in proof, 12 May 1977].

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simply-connected, and Y' is a homotopy 3-sphere, so X' is acyclic; hence contractible. Likewise one may weaken the assumption of Shaneson's *unknotting theorem for* S^3's *in* S^5, [16], to: *the complement has the same first and second homotopy groups as* S^1.

As above, the commutator subgroup K of a knot group must admit an automorphism φ with property

(a) $\langle\langle k^{-1}\varphi(k);\ k \in K\rangle\rangle = K$;

a fortiori, φ then has property

(b) $H_1(\varphi) - 1$ is an automorphism of $H_1(K)$.

After deleting from Milnor's list [12] of finite groups with cohomology of period 4 those which have abelianization cyclic of even order (hence which admit no automorphism with property (b)) there remain:

$$1 = \text{the trivial group},$$

$$Q(8n) = \{x,\ y \mid x^2 = (xy)^2 = y^{2n}\} ,$$

$$I^* = \{x,\ y \mid x^2 = (xy)^3 = y^5\} ,$$

$$T(k) = \{x,\ y,\ z \mid x^2 = (xy)^2 = y^2,\ zxz^{-1} = y,\ zyz^{-1} = xy,\ z^{3^k} = 1\} ,$$

$$Q(8n,\ k,\ l) = \{x,\ y,\ z \mid x^2 = (xy)^2 = y^{2n},\ z^{kl} = 1,\ xzx^{-1} = z^r,\ yzy^{-1} = z^{-1}\} ,$$

(where $8n$, k, l are pairwise relatively prime, $r \equiv -1 \bmod k$, $r \equiv +1 \bmod l$, if n is odd $n > k > l \geq 1$, and if n is even $n \geq 2$, $k > l \geq 1$), and direct products with cyclic groups of relatively prime, odd order.

If $G = H \times J$ with $(|H|,\ |J|) = 1$ then an automorphism φ of G corresponds to a pair of automorphisms φ_H, φ_J of H, J respectively and φ has property (a) (respectively (b)) if and only if φ_H and φ_J each have it. Clearly an automorphism $[s] : x \to x^s$ of the cyclic group $\{x \mid x^m = 1\}$ has property (a) (equivalently property (b)) if and only if $(s-1,\ m) = (s,\ m) = 1$; hence m must be odd.

If $n > 1$, the only elements of $Q(8n)$ of order $4n$ are powers of y and so any automorphism of $Q(8n)$ must map x to $y^a x$, y to y^b with $(b, 4n) = 1$. But such an automorphism clearly does not have property (b) so only the case $n = 1$, that is the quaternion group $Q = \{x, y \mid x^2 = (xy)^2 = y^2\}$, may occur. $\operatorname{aut}(Q)$ has just one conjugacy class of elements with property (a) (since Q is nilpotent, this is equivalent to having property (b)), represented by $\zeta : x \to y$, $y \to xy$. (Notice that the pair

$$\left\langle \text{group } G(\varphi) = \left\{G_1 t \mid tgt^{-1} = \varphi(g)\right\}, \text{ element } t \in G(\varphi) \text{ of weight one}\right\rangle$$

depends up to isomorphism only on the conjugacy class of φ in $\operatorname{aut}(G)$. $G(\varphi)$ itself depends only on the conjugacy class of φ in $\operatorname{aut}(G)$.

The group $Q(8n, k, l)$ maps onto $Q(8n)$ (with kernel the characteristic subgroup generated by z), so this case cannot occur (since $n > 1$).

The binary icosahedral group I^* was first considered in the context of 2-knots by Mazur [11]. $\operatorname{aut} I^*$ is isomorphic to S_5 , and has seven conjugacy classes, one for each partition of 5 . The first four classes, the identity, products of 2 disjoint 2-cycles, 3-cycles, 5-cycles (those which lie in A_5) contain inner automorphisms, and give rise to the HNN group $\mathbb{Z} \times I^*$. The other three classes (2-cycles, products of a 3-cycle and its complementary 2-cycle, and 4-cycles) give rise to the group

$$\{x, y, t \mid x^2 = (xy)^3 = y^5,\ txt^{-1} = x,\ tyt^{-1} = y^{-1}x^{-1}y^2xy\} .$$

All the automorphisms satisfy (b) (since I^* is perfect), and it is easily seen that all except the identity automorphism satisfy (a) (since I^* has only one proper normal subgroup, its centre $\langle x^2\rangle$).

There is a short exact sequence $1 - Q \to T(k) \xrightarrow{ab} \mathbb{Z}/3^k\mathbb{Z} \to 1$ where $Q = \{x, y \mid x^2 = (xy)^2 = y^2\}$ is the commutator subgroup of $T(k)$. An automorphism φ of $T(k)$ induces automorphisms $\bar{\varphi}$, $\overline{\varphi/Q}$ of $\mathbb{Z}/3^k\mathbb{Z}$, Q/Q' respectively; let α map $\operatorname{aut}(T(k))$ to $\operatorname{aut}(\mathbb{Z}/3^k\mathbb{Z}) \to \operatorname{aut}(Q/Q')$ by

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$\alpha : \varphi \to \langle\bar{\varphi}, \overline{\varphi/Q}\rangle$. Define automorphisms $\theta, \gamma, \psi, \rho$ of $T(k)$ by

	θ	γ	ψ	ρ
x	x^{-1}	x	y	y^{-1}
y	y	y^{-1}	xy	x^{-1}
z	$x^{-1}z$	$x^{-1}yz$	z	z^2 .

Then $\alpha(\rho) = \left\langle [2], \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\rangle$, so $pr_1 \circ \alpha : \mathrm{aut}(T(k)) \to \mathrm{aut}(\mathbb{Z}/3^k\mathbb{Z})$ is onto (since 2 generates $(\mathbb{Z}/3^k\mathbb{Z})^*$). $\ker\alpha$ is the four-group generated by θ and γ . $\ker(pr_1 \circ \alpha)$ has a presentation

$$\{\theta, \gamma, \psi \mid \theta^2 = \gamma^2 = (\theta\gamma)^2 = \psi^3 = 1, \ \psi\theta\psi^{-1} = \theta\gamma, \ \psi\gamma\psi^{-1} = \theta\} ,$$

which is equivalent to

$$\{\theta, \psi \mid \theta^2 = \psi^3 = 1, \ [\theta, \psi] = \psi^{-1}\theta\psi, \ [\theta, [\theta, \psi]] = 1\} .$$

$(\ker(pr_1 \circ \alpha) \approx A_4 \approx \mathrm{In}(T(k)))$. $\mathrm{aut}(T(k))$ then has the presentation

$$\{\theta, \psi, \rho \mid \theta^2 = \psi^3 = \rho^{2.3^{k-1}} = 1, \ [\theta, \psi] = \psi^{-1}\theta\psi, \ [\theta, [\theta, \psi]] = 1, \\ \rho\psi\rho^{-1} = \psi^2, \ \rho\theta\rho^{-1} = [\theta, \psi]\} .$$

The conjugacy classes in $\mathrm{aut}(T(k))$ have the following representatives: $\rho^{2l}, \rho^{2l}\theta, \rho^{2l}\psi, \rho^{2l+1}$ for $0 \le l < 3^{k-1}$. Since 3 divides $2^{2l} - 1$, the only automorphisms satisfying (b) are those conjugate to an odd power of ρ . These automorphisms do in fact also satisfy (a), since

$$\langle\langle x^{-1}\rho^{2l+1}(x), \ y^{-1}\rho^{2l+1}(y), \ z^{-1}\rho^{2l+1}(z)\rangle\rangle \\ = \langle\langle x^{-1}y^{-1}, \ y^{-1}x^{-1}, \ z^{2^{2l+1}-1}\rangle\rangle = T(k) .$$

All the groups of the form

$(1, Q(8n), I^*$ or $T(k)) \times$ (relatively prime odd cyclic group)

are 3-manifold groups [12], and so have trivial second homology; hence by earlier remarks all such groups and automorphisms with property (a) can be realised by high dimensional knots. We shall finally consider briefly which can be realised by fibred 2-knots. First some general remarks.

Let $\varphi : M \to M$ be an orientation preserving self-homeomorphism of a

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3-manifold M, with at least one fixed point P, and let $M(\varphi)$ denote the mapping torus of φ. The pair $(M(\varphi), S^1\times P)$ depends up to isomorphism only on the conjugacy class of φ in the homeotopy group of M. Let N be the manifold obtained by choosing a framing of the normal bundle of the cross-section $S^1 \times P \subset M(\varphi)$ and performing surgery (that is, $N = (M(\varphi)-\text{int } N(P\times S^1)) \underset{S^2\times S^1}{\cup} S^2\times D^2$).

$$\pi_1(M(\varphi)) = \{\pi_1(M), t \mid twt^{-1} = \varphi_*(w) \text{ for all } w \in \pi_1(M)\} ,$$

so

$$\pi_1(N) = \pi_1(M)/\langle\langle w^{-1}\varphi_*(w); w \in \pi_1(M)\rangle\rangle$$

(where φ_* is the induced map on $\pi_1(M)$).

If φ_* has property (b) then $H_1(M(\varphi)) = Z$; hence $H_2(M(\varphi)) = 0$ (since $\chi(M(\varphi); R) = \chi(M; R)\chi(S^1; R) = 0$ for any ring R), so $H_1(N) = 0$ and $H_2(N) = 0$ (since $\chi(N; R) = \chi(M(\varphi); R) + 2$); that is N is an homology 4-sphere. If also φ_* has property (a), then N is simply connected and so an homotopy 4-sphere. S^2 is embedded in N via $j : S^2 \times 0 \hookrightarrow S^2 \times D^2 \subset N$, and $N - j(S^2) \approx M(\varphi/M-P)$. Thus the 2-knot $j : S^2 \to N$ has commutator subgroup $\pi_1(M-P) \approx \pi_1(M)$ and associated automorphism φ_*.

For cyclic groups, the only automorphism with property (a) realisable by a self homeomorphism of a classical lens space is the involution $x \to x^{-1}$ (*cf.* [4]). The example of a 2-knot group with torsion given by Fox in [5] is of this form (with $G' = Z/3Z$); is the knot fibred? Notice also that, for example, $\{a, t \mid a^5 = 1, tat^{-1} = a^2\}$ is a high dimensional knot group; is it a 2-knot group (perhaps even for a fibred knot with fibre a punctured fake lens space)?

Q is isomorphic to the subgroup of S^3, the group of unit quaternions, generated by i (corresponding to x) and j (corresponding to y), and conjugation of S^3 by $z = -\frac{1}{2}(1+i+j+k)$ passes to a self-homeomorphism of S^3/Q inducing $\xi : x \to y$, $y \to xy$ on Q , its fundamental group (and which preserves orientation because the covering map of S^3 is isotopic to the identity).

I^* is isomorphic to a subgroup of S^3 ; for example that generated by i (corresponding to x) and $\left(\frac{1+\sqrt{5}}{4}\right) - \frac{1}{2}i + \left(\frac{1-\sqrt{5}}{4}\right)j$ (corresponding to y). $H = S^3/I^*$ is the Poincaré homology sphere. Conjugation of S^3 by an element of I^* induces an automorphism of H with at least one fixed point, orientation preserving (as above) and inducing conjugation by that element on the fundamental group. Thus all the inner automorphisms of I^* can be realised, and they all have mapping tori isomorphic to $H \times S^1$, but correspond to different cross sections of the projection $H \times S^1 \to S^1$. Can an outer automorphism be realised? (*Cf.* Zeeman [*19*], §8, Question 3.)

$T = T(1)$ is isomorphic to the subgroup of S^3 generated by i (corresponding to x), j (corresponding to y), and $-\frac{1}{2}(1+i+j+k)$ corresponding to z , and the automorphism ρ is realised by conjugation by $\frac{\sqrt{2}}{2}(i-j)$. What can one say about the other cases, $T(k)$ and $(Q, I^*, \text{or } T(k)) \times$ (relatively prime odd cyclic group) ?

Zeeman [*19*] showed that the homotopy 4-sphere constructed by Mazur [*11*] was standard; is this true for all the homotopy spheres constructed in the above manner? Finally one might ask, is every 2-knot with finite commutator subgroup fibred?

Note added in proof [12 May 1977]. After announcing the above results in the *Notices of the American Mathematical Society* (February 1977), the author received a preprint from M.A. Gutiérrez (Homology of knot groups, III: knots in S^4 , *Proc. London Math. Soc.*, to appear) in which it is proved that if the commutator subgroup of a 2-knot group is finite presentable, then it is a 3-manifold group. Theorems 1 and 2 of the present paper are immediate consequences of this result. Gutiérrez pointed out that M.S. Farber has also answered Fox's questions 28, 29, and 35; he constructed a "linking" pairing on the torsion of $H_1(X')$ (Linking coefficients and two-dimensional knots, *Soviet Math. Dokl.* 16 (1975), no. 3, 647-650). Using this pairing, one can show that for G' cyclic, only the involution may occur, and for $G' = T(k)$ only the map sending x, y, z to x^{-1}, y^{-1}, z^{-1} (respectively) may occur.

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