A THEOREM ON POLYNOMIAL LORENTZ STRUCTURES

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Let M be a differentiable manifold of dimension m. A tensor field f of type $(1,1)$ on M is called a polynomial structure on M if it satisfies the equation:

$$R(f)=f^n+a_1f^{n-1}+\ldots+a_{n-1}f+a_nI=0 \tag{1}$$

where $a_1, a_2, \ldots, a_n$ are real numbers and I denotes the identity tensor of type $(1,1)$.

We shall suppose that for any $x \in M$

$$R(\xi)=\xi^n+a_1\xi^{n-1}+\ldots+a_{n-1}\xi+a_n \tag{2}$$

is the minimal polynomial of the endomorphism $f_x: T_xM \to T_xM$.

We shall call the triple (M, f, g) a polynomial Lorentz structure if f is a polynomial structure on M, g is a symmetric and nondegenerate tensor field of type $(0,2)$ of signature

$$(-, \underbrace{+, +, \ldots, +}_{m-1 \text{ times}})$$

such that $g(fX, fY)=g(X, Y)$ for any vector fields X, Y tangent to M. The tensor field g is a (generalized) Lorentz metric.

In **[5]** B. Opozda gave a necessary and sufficient condition that the tensor field f be parallel with respect to the Riemannian connection induced by the metric tensor $\bar{g}$ such that $\bar{g}(f,f)=\bar{g}$. We are going to show that in general this is not true for polynomial Lorentz structures.

We prove that an analogous theorem is true for a certain class of these structures.

Let us decompose the polynomial $R(\xi)$ into prime factors:

$$R(\xi)=R_1'(\xi)\ldots R_r'(\xi)R_1''(\xi)\ldots R_s''(\xi),$$

where

$$R_i'(\xi)=(\xi-b_i)^{k_i}, \qquad k_i \geqslant 1, \qquad i=1,\ldots,r,$$
$$R_j''(\xi)=(\xi^2+2c_j\xi+d_j)^{l_j}, \qquad l_j \geqslant 1, \qquad c_j^2<d_j, \qquad j=1,\ldots,s.$$

The polynomials $\xi-b_i$, $i=1,\ldots,r$, as well as the polynomials $\xi^2+2c_j\xi+d_j$, $j=1,\ldots,s$ are pairwise distinct.

The main result of **[2]** is the following theorem:

THEOREM 1. *There exist exactly eleven types of polynomial Lorentz structures classified as follows by their minimal polynomials*:

(I) $$R(\xi)=(\xi-c)(\xi-c^{-1})(\xi-1)(\xi+1)Q(\xi),$$

(II) $$R(\xi)=(\xi-c)(\xi-c^{-1})(\xi-1)Q(\xi),$$

(III) $$R(\xi)=(\xi-c)(\xi-c^{-1})(\xi+1)Q(\xi),$$

(IV) $R(\xi) = (\xi - c)(\xi - c^{-1})Q(\xi),$

(V) $R(\xi) = (\xi - 1)(\xi + 1)Q(\xi),$

(VI) $R(\xi) = (\xi - 1)Q(\xi),$

(VII) $R(\xi) = (\xi + 1)Q(\xi),$

(VIII) $R(\xi) = (\xi - 1)^3(\xi + 1)Q(\xi),$

(IX) $R(\xi) = (\xi - 1)^3Q(\xi),$

(X) $R(\xi) = (\xi + 1)^3(\xi - 1)Q(\xi),$

(XI) $R(\xi) = (\xi + 1)^3Q(\xi),$

where $Q(\xi) = (\xi^2 + 2a_2\xi + 1)\ldots(\xi^2 + 2a_s\xi + 1)$; $a_j^2 < 1$, $a_i \neq a_j$ *for* $i \neq j$; $i, j = 2, \ldots, s$, $|c| \neq 1$, $c \neq 0$.

Now we denote:

$$\begin{aligned} D_0 &= \ker(f - cI) + \ker(f - c^{-1}I),\\ D_1 &= \ker(f - I)\\ \tilde{D}_1 &= \ker(f - I)^3,\\ D_{-1} &= \ker(f + I),\\ \tilde{D}_{-1} &= \ker(f + I)^3,\\ D_j &= \ker(f^2 + 2a_j + I) \quad \text{for} \quad j = 2, \ldots, s. \end{aligned} \tag{3}$$

Let $(T_1, \ldots, T_k)$ be the decomposition of T_xM by the distributions of type (3).

PROPOSITION 1. *The almost product structure* $(T_1, \ldots, T_k)$ *is orthogonal i.e.* D_i *is orthogonal to* D_j *if* $i \neq j$.

In view of Proposition 5 of [5] it is sufficient to prove, that:

$$\tilde{D}_1 \perp D_{-1},\ \tilde{D}_1 \perp D_j,\ \tilde{D}_{-1} \perp D_j,\ D_0 \perp D_1,\ D_0 \perp D_{-1},\ D_0 \perp D_j,\ \tilde{D}_{-1} \perp D_1 \quad \text{for} \quad j = 2, \ldots, s. \tag{4}$$

If $v \in \tilde{D}_1$and $w \in D_{-1}$ we have $(f - I)^3v = 0$, $fw = -w$. We compute:

$$\begin{aligned} 0 = g((f - I)^3v, w) &= g((f^3 - 3f^2 + 3f - I)v, w)\\ &= -g(f^3v, f^3w) - 3g(f^2v, f^2w) - 3g(fv, fw) - g(v, w)\\ &= -8g(v, w) \end{aligned}$$

and hence $\tilde{D}_1 \perp D_{-1}$.

Now, if $v \in \tilde{D}_1$, $w \in D_j$ for $j = 2, \ldots, s$ we have

$$\begin{aligned} 0 &= g((f - I)^3v, f^2w)\\ &= g(fv, w) - 3g(v, w) + g(3fv, -2a_ifw - w) - g(v, -2a_ifw - w)\\ &= g(fv, w) - 3g(v, w) - 6a_ig(v, w) - 3g(fv, w) + g(v, w) + 2a_ig(v, fw), \end{aligned}$$

hence

$$a_i g(v, fw) - (1 + 3a_i) g(v, w) - g(fv, w) = 0. \tag{5}$$

On the other hand

$$0 = g(fv, (f^2 + 2a_i f + I)w) = g(v, fw) + 2a_i g(v, w) + g(v, w),$$

hence

$$g(v, fw) + 2a_i g(v, w) + g(fv, w) = 0. \tag{6}$$

From conditions (5) and (6) we have

$$g(v, w) = g(v, fw) = g(v, f^2 w)$$

and consequently

$$\begin{aligned}(2 + 2a_i) g(v, w) &= g(v, w) + 2a_i g(v, w) + g(v, w)\\ &= g(v, f^2 w) + 2a_i g(v, fw) + g(v, w)\\ &= g(v, (f^2 + 2a_i f + I)w) = 0.\end{aligned}$$

The last condition implies that $\tilde{D}_1 \perp D_j$. Similarly we check that other conditions (4) are also true.

Let D be an l-dimensional distribution on M. A chart $(U, \varphi = (x^1, \ldots, x^m))$ is said to be flat with respect to a distribution D if the vector fields $\frac{\partial}{\partial x^\alpha}$ $(\alpha = 1, \ldots, l)$ form a basis for D in U.

A distribution on M is integrable if each point of M lies in the domain of a flat chart.

We say that a polynomial Lorentz structure (M, f, g) is integrable if for every point of M there exists a chart in which the matrix representation of f is constant.

We can prove that if (M, f, g) is integrable, so are the distributions (3).

From the theorem of E. Kobayashi [**3**, p. 967] we deduce immediately the following:

THEOREM 2. (i) *A polynomial Lorentz structure of type (I)–(VII) is integrable if the Nijenhuis tensor of f is equal to zero.*

(ii) *A polynomial Lorentz structure of type (VIII)–(XI) with* $\dim \tilde{D}_1 = 3$ *or* $\dim \tilde{D}_{-1} = 3$ *is integrable if the Nijenhuis tensor of f is equal to zero.*

In the following example $\dim \tilde{D}_1 > 3$, the Nijenhuis tensor of f is equal to zero, but the polynomial structure is not integrable. The example given by Kobayashi cannot be applied to our case, because f is not an isometry for g.

EXAMPLE 1. Let $M = R^5$ and $(x^1, \ldots, x^5)$ denote the canonical coordinate system in R^5. Let $X_i = \frac{\partial}{\partial x^i}$ for $i = 1, 2, 3, 4$ and $X_5 = \frac{\partial}{\partial x^5} + x^4 \frac{\partial}{\partial x^2}$. We define f by:

$$\begin{aligned} fX_1 &= X_1, \\ fX_2 &= X_1 + X_2, \\ fX_3 &= X_2 + X_3, \\ fX_4 &= X_4, \\ fX_5 &= X_5 \end{aligned}$$

and the Lorentz metric tensor g by the following components matrix in the basis $X_1, \ldots, X_5$:

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then f is a polynomial structure with minimal polynomial $R(\xi) = (\xi - 1)^3$, $g(f, f) = g$ and the Nijenhuis tensor of f is equal to zero. The polynomial Lorentz structure (M, f, g) is not integrable because the distribution $\ker(f - I)$ fails to be involutive:

$$[X_4, X_5] = \frac{\partial}{\partial x^2} = X_2$$

is not in $\ker(f - I)$.

J. Lehman-Lejeune [**4**] proved that a polynomial structure f is integrable if and only if there exists a symmetric linear connection ∇ such that $\nabla f = 0$.

A triple $(M, \bar{f}, \bar{g})$ is called a metric polynomial structure if $\bar{f}$ is a polynomial structure on M and $\bar{g}$ is a Riemannian metric such that $\bar{g}(\bar{f}, \bar{f}) = \bar{g}$.

B. Opozda has proved the following theorem [**5**]:

THEOREM 3. *Let $(M, \bar{f}, \bar{g})$ be a metric polynomial structure. Then the following conditions are equivalent*:

1^0 $\nabla\bar{f} = 0$,

2^0 $[\bar{f}, \bar{f}] = 0$, *the fundamental 2-form $\Phi(X, Y) = \bar{g}(X, \bar{f}Y) - \bar{g}(\bar{f}X, Y)$ is closed and the distributions* $\ker(\bar{f} - I)$, $\ker(\bar{f} + I)$ *are parallel with respect to* ∇,
where ∇ denotes the Riemannian connection on M induced by $\bar{g}$ and $[\bar{f}, \bar{f}]$ is the Nijenhuis tensor of $\bar{f}$.

Now we are going to prove that a certain class of polynomial Lorentz structures satisfies a theorem analogous to Theorem 3.

THEOREM 4. *Let (M, f, g) be a Lorentz polynomial structure of type (I)–(VII). Then the following conditions are equivalent*:

1^0 $\nabla f = 0$,

2^0 $[f, f] = 0$, *the fundamental* 2-*form* $\Psi(X, Y) = g(X, fY) - g(fX, Y)$ *is closed and the distributions* D_0, D_1, D_{-1} *are parallel with respect to* ∇.

Proof. 1^0 implies 2^0 because then f is integrable by the theorem of Lehman-Lejeune.

Assume 2^0. Now, there exists exactly one distribution (say D) of type D_0, D_1, D_{-1} such that the restriction of g to D has Lorentz signature. The distribution D is parallel and so is $D^{\perp}$. A parallel distribution is involutive and integrable [**2**]. For any $x \in M$ let N_x be the integral manifold of the distribution $D^{\perp}$. Let $\tilde{f}$, $\tilde{g}$ and $\tilde{\Psi}$ be restrictions of f, g and Ψ to N_x respectively.

$(N_x, \tilde{f}, \tilde{g})$ is a metric polynomial structure; so according to Theorem 3 $\tilde{\nabla}\tilde{f} = 0$, where $\tilde{\nabla}$ is the Riemannian connection for $\tilde{g}$. If $X, Y \in D^{\perp}$, then

$$0 = (\tilde{\nabla}_X \tilde{f})Y = \tilde{\nabla}_X \tilde{f}Y - \tilde{f}(\tilde{\nabla}_X Y) = \nabla_X fY - f\nabla_X Y = (\nabla_X f)Y.$$

If $X \in D$, $Y \in D^{\perp}$ or $X \in D^{\perp}$, $Y \in D$, then by Proposition 6 in [**5**] $(\nabla_X f)Y = 0$.

Now let $X, Y \in D$ ($D = D_1$ or $D = D_{-1}$ or $D = D_0$). In the first two cases it is obvious that $(\nabla_X f)Y = 0$ because

$$(\nabla_X f)Y = \nabla_X fY - f\nabla_X Y = \varepsilon\nabla_X Y - \varepsilon\nabla_X Y = 0 \quad (|\varepsilon| = 1).$$

In the case $D = D_0$ we are going to apply the following proposition:

PROPOSITION 2. *Let (M', f', g') be a 2-dimensional Lorentz structure of type (IV) i.e. with minimal polynomial $R(\xi) = (\xi - c)(\xi - c^{-1})$, $|c| \neq 1$, $c \neq 0$. Then $\nabla f' = 0$.*

Proof. Let $f'X = cX$, $f'Y = c^{-1}Y$. We show that the Nijenhuis tensor of f' is zero:

$$\begin{aligned} [f', f'](X, Y) &= f'^2[X, Y] + [f'X, f'Y] - f'[X, f'Y] - f'[f'X, Y] \\ &= (f'^2 - cf' - c^{-1}f' + I)[X, Y] \\ &= (f' - cI)(f' - c^{-1}I)[X, Y] = 0. \end{aligned}$$

From the integrability of f' there exists a chart $(U, \varphi = (x^1, x^2))$ in which $X = \dfrac{\partial}{\partial x^1}$, $Y = \dfrac{\partial}{\partial x^2}$. Now we have

$$\begin{aligned} &2g'(\nabla_X X, X) = 0, \quad \text{hence} \quad \nabla_X X \in \ker(f' - cI), \\ &2g'(\nabla_Y Y, Y) = 0, \quad \text{hence} \quad \nabla_Y Y \in \ker(f' - c^{-1}I), \\ &2g'(\nabla_X Y, X) = 0, \\ &2g'(\nabla_X Y, Y) = 0, \quad \text{hence} \quad \nabla_X Y = \nabla_Y X = 0. \end{aligned}$$

We compute:

$$\begin{aligned} &(\nabla_X f')X = \nabla_X f'X - f'\nabla_X X = c\nabla_X X - f'\nabla_X X = 0, \\ &(\nabla_X f')Y = \nabla_X f'Y - f'\nabla_X Y = 0. \end{aligned}$$

Similarly we check:

$$(\nabla_Y f')X = (\nabla_Y f')Y = 0.$$

Hence the condition 1^0 of Theorem 4 is satisfied.

Theorem 4 is not true for polynomial Lorentz structures of type (VIII)–(IX).

EXAMPLE 2. Let $M = R^4$. Define $X_1 = \exp(x^4)\dfrac{\partial}{\partial x^1}$ and $X_i = \dfrac{\partial}{\partial x^i}$ for $i = 2, 3, 4$. In the basis X_1, X_2, X_3, X_4 we assume

$$f = \begin{bmatrix} 1 & -2 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is a polynomial structure on M such that $(f - I)^3 = 0$. We define g by the coordinate matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is easy to see that $g(f, f) = g$. We show, that the fundamental 2-form Ψ defined by

$$\Psi(X, Y) = g(fX, Y) - g(X, fY)$$

is closed.

$$\Psi(X_2, X_3) = -\Psi(X_3, X_2) = 4$$

and

$$\Psi(X_i, X_j) = 0 \quad \text{for other} \quad X_i, X_j.$$

On applying the formula

$$\begin{aligned} 3d\Psi(X, Y, Z) &= X\Psi(Y, Z) + Y\Psi(Z, X) + Z\Psi(X, Y) \\ &\quad - \Psi([X, Y], Z) - \Psi([Y, Z], X) - \Psi([Z, X], Y) \end{aligned}$$

we get $d\Psi = 0$.

On the other hand

$$\begin{aligned} 2g((\nabla_{X_2} f)X_2, X_4) &= 2g(\nabla_{X_2}(-2X_1) + \nabla_{X_2} X_2 + \nabla_{X_2} 2X_3, X_4) - 2g(\nabla_{X_2} X_2, f^{-1}X_4) \\ &= 2g([X_4, X_1], X_2) = -2. \end{aligned}$$

Thus $\nabla f \neq 0$.

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