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Thermal field and linear response theory

Thermal field theory, or finite temperature quantum field theory, deals with quantum systems in equilibrium. This is likely a familiar subject to the readers. Given the huge literature on this subject (see e.g. [LaLiPi80b, LeB91, LeB96, Kap89, Par88, AbGoDz75, Mil69, LanWer87, Ber74, DolJac74, Wei74, KuToHa91]), we will not attempt to give a full treatment of it *per se,* but rather emphasize how the CTP approach actually unifies the study of equilibrium and nonequilibrium systems. We will discuss thermal perturbation theory from a CTP perspective, including a discussion of screening and damping in quantum fields at finite temperature. In Chapter 11 we shall consider quantum kinetic field theory, including a derivation of its centerpiece, the Kadanoff–Baym equations, and in Chapter 12 the issue of thermalization, namely the processes which bring about equilibrium to systems out-of-equilibrium.

10.1 The thermal generating functional

A thermal state is a mixed state described by the density matrix

$$\rho = e^{\beta F} e^{-\beta H} \tag{10.1}$$

where H is the Hamiltonian and F the *free energy,* defined in terms of the partition function

$$e^{-\beta F} \equiv Z = \mathrm{Tr}\, e^{-\beta H} \tag{10.2}$$

We shall assume the trace exists. Hence we are treating systems in equilibrium in a canonical ensemble; the generalization to grand canonical ensemble is straightforward. Observe that the equilibrium state is stationary but not Lorentz invariant, since the form (10.1) of the density matrix holds only in a preferred frame, the so-called *rest frame* of the field.

To investigate the matrix elements of the thermal density matrix, it is natural to proceed by analogy with the evolution operator $U = e^{-itH/\hbar}$ introduced in Chapter 4. In general, we may consider an evolution operator in complex time $U(z) = e^{-izH/\hbar}$. Given two field eigenvectors the matrix element is

$$\left\langle \psi \left| e^{-izH/\hbar} \right| \varphi \right\rangle = \sum_n e^{-izE_n/\hbar} \left\langle \psi \,|\, n \right\rangle \left\langle n \,|\, \varphi \right\rangle \tag{10.3}$$

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Since energies are positive, the sum will converge when $\mathrm{Im}\, z \leq 0$ (we are mostly concerned with z's of the form $-i\beta$, of course). Also we have the semigroup property

$$U(z+z') = U(z)\, U(z') \tag{10.4}$$

provided all terms exist, namely, that $\mathrm{Im}\, z, z' \leq 0$. In general, given a complex path $\gamma(u)$ connecting 0 to z, if the imaginary part of $\gamma(u)$ is nonincreasing throughout we may decompose $U(z)$ as a product of infinitesimal evolution operators, and obtain the path integral representation

$$\left\langle \psi \left| e^{-izH/\hbar} \right| \varphi \right\rangle = \int_{\varphi(0)=\varphi, \varphi(z)=\psi} D\varphi\, e^{(i/\hbar)S[\varphi]} \tag{10.5}$$

where the integration is carried over field configurations defined on the complex path. The fundamental property of the representation (10.5) is path independence: the path integral is independent of the choice of γ, as long as the difference between the endpoints is z, and the imaginary part is nonincreasing [Mil69, McL72a, McL72b].

In order to obtain the partition function, we set $\psi = \varphi$ and integrate over all choices. Thus Z is given by a path integral over *periodic* (for Bose–Einstein statistics) or antiperiodic (for Fermi–Dirac statistics) [Ber66, NegOrl98]) field configurations defined on a complex time path going from an arbitrary point z to $z - i\beta\hbar$ with nonincreasing imaginary parts.

In practice, different choices of time path lead to different formulations. If the main object is the computation of the partition function itself, then possibly the simplest choice is the so-called *Matsubara contour*, which goes straight down from 0 to $-i\beta\hbar$ [LaLiPi80b, KuToHa91]. If the goal is to compute real time correlations, then it is convenient to include (patches of) the real time axis into the contour. For example, if we choose a time path going along the real axis from $-T$ to T', then down to $T' - i\beta\hbar/2$, back on a reverse time line to $-T - i\beta\hbar/2$, and finally down again to the endpoint $-T - i\beta\hbar$, we get a functional representation of Umezawa's *thermo field dynamics* [UmMaTa82, NieSem84a, NieSem84b]. Although these different formulations are of course equivalent from the physical point of view, one or the other could be more adept to particular perturbative calculations.

From the point of view of making contact with the CTP approach to nonequilibrium dynamics, the natural choice of contour is from $-T$ to T', then immediately back to $-T$ and straight down to $-T - i\beta\hbar$ [CSHY85, LanWer87]. Comparing with the CTP generating functional, we see that this procedure is equivalent to replacing the density matrix by its path integral representation. The thermal CTP generating functional is given by

$$e^{(i/\hbar)W_\beta[J]} = e^{\beta F} \int D\varphi^A \exp\left\{(i/\hbar)\left[S\left[\varphi^A\right] + J_A \varphi^A\right]\right\} \tag{10.6}$$

with the branch index $a = 1, 2, 3$, with the new value 3 corresponding to the downward imaginary branch, and we adopt the convention $J^3 = 0$. Observe that the thermal generating functional shares with the vacuum one the property that the choice of quantum state has been encoded into the time path. The formal manipulations leading from the generating functional to the effective action, as well as the perturbative set-up for the evaluation of this latter object, are the same as in the zero-temperature theory.

10.2 Linear response theory

A profound property of the thermal state is that the dynamic response of a system in thermal equilibrium to small external perturbations can be described rigorously in terms of *equilibrium* expectation values, by means of the so-called *linear response theory* (LRT) [KuToHa91]. We shall describe very briefly the basics of LRT, and then show how it can be trivially derived from the CTP generating functional just introduced.

The basic set-up for LRT is a system which at time $t = 0$ is in equilibrium (namely, its density matrix is given by equation (10.1)) and is subsequently perturbed by an addition of a time-dependent term $-\sigma(t) P(t)$ to the Hamiltonian, where σ is a c-number external source, and P some Heisenberg operator acting on the system. We wish to follow the nonequilibrium, driven evolution of the field, through the time-dependent expectation value of some other observable $Q(t)$.

To this end, it is most efficient to adopt an interaction picture approach. The density matrix at time t then is given by

$$\rho(t) = T\left[e^{(i/\hbar)\int_0^t dt'\sigma P}\right]\rho(0)\,T\left[e^{-(i/\hbar)\int_0^t dt'\sigma P}\right] \tag{10.7}$$

The desired expectation value is $\langle Q\rangle_J(t) = \mathrm{Tr}\, Q(t)\,\rho(t)$. For small sources, we may linearize

$$\langle Q\rangle(t) = \langle Q\rangle_0(t) + \int_0^t dt'\,\mathcal{R}(t,t')\,\sigma(t') \tag{10.8}$$

where $\langle Q\rangle_0(t)$ is the expectation value for $Q(t)$ in equilibrium, and $\mathcal{R}$ is the *response function*

$$\mathcal{R}(t,t') = \left(\frac{i}{\hbar}\right)\langle[Q(t), P(t')]\rangle_0\,\theta(t-t') \tag{10.9}$$

The general relationship between the retarded and Jordan propagators introduced before is but a particular case of this general identity. Furthermore, certain transport coefficients may be written in terms of time integrals of response functions, by means of the Kubo formulae, to be discussed in Chapter 12. Then equation (10.9) may be used to link those transport coefficients to equilibrium expectation values of Heisenberg operators.

To obtain equations (10.8) and (10.9) in a CTP framework, we first introduce a generating functional for $\langle Q\rangle(t)$

$$e^{(i/\hbar)W_Q[J]} = e^{\beta F}\int D\varphi^A \exp\left\{(i/\hbar)\left[S_\sigma\left[\varphi^A\right] + J_A Q^A\right]\right\} \tag{10.10}$$

so that the desired expectation value is

$$\langle Q\rangle(t) = \left.\frac{\delta W_Q[J]}{\delta J^1(t)}\right|_{J=0} \tag{10.11}$$

(observe that we *doubled the degrees of freedom*). In this equation, the action S_σ contains the σ-dependent term $\int dt\, \sigma(t)\left(P^1 - P^2\right)(t)$; it is unnecessary to add CTP indices to σ, since this is a physical source, and in any case we would obtain $\sigma^1 = \sigma^2 = \sigma$. Since σ turns on for $t > 0$ only, the free energy is independent of it. We may therefore expand

$$W_Q[J] = \left.W_Q[J]\right|_{\sigma=0} + \frac{\int D\varphi^A\ e^{\left\{(i/\hbar)\left[S_0\left[\varphi^A\right]+J_A Q^A\right]\right\}}\int dt'\ \left(P^1 - P^2\right)\sigma(t')}{\int D\varphi^A\ e^{\left\{(i/\hbar)\left[S_0\left[\varphi^A\right]+J_A Q^A\right]\right\}}} \tag{10.12}$$

The path integral in the numerator of the second term vanishes at $J = 0$. Performing the J derivative in equation (10.11), we see that equations (10.8) and (10.9) are a simple consequence of the time ordering properties of the path integral.

10.3 The Kubo–Martin–Schwinger theorem

Since we have succeeded in incorporating the information about the state in the time path, we may adopt *verbatim* the perturbative approaches already discussed in Chapter 6. In particular, any expectation value may be developed as a sum of Feynman graphs, carrying in internal legs the *thermal propagators*

$$G^{11}_\beta(x, x') \equiv G_{\beta F}(x, x') = e^{\beta F}\mathrm{Tr}\ \left\{e^{-\beta H} T\left[\Phi(x)\Phi(x')\right]\right\} \tag{10.13}$$

etc. Since the thermal time path has three branches, it may seem that now we need nine different propagators to carry out perturbation theory. However, the path independence of the path integral may be invoked to push the third branch arbitrarily into the distant past, whereby it decouples from the other branches, and the usual set-up, based on four propagators, is sufficient. Nevertheless, the third branch is essential in enforcing the fundamental property of the thermal propagators, namely, that they can be analytically continued to complex time, and when so continued, they obey certain periodicity conditions, embodied in the so-called *Kubo–Martin–Schwinger* (KMS) *theorem* [Kub57, KuYoNa57, MarSch59]. The KMS theorem plays such a central role in thermal perturbation theory that it is often adopted as the *definition* of the thermal propagators.

To state the KMS theorem, let us consider the thermal positive frequency propagator

$$G^{21}_\beta(x,x') \equiv G^{+}_\beta(x,x') = e^{\beta F}\,\mathrm{Tr}\,\left\{e^{-\beta H}\Phi(x)\,\Phi(x')\right\} \tag{10.14}$$

and insert energy–momentum eigenstates

$$G^{+}_\beta(x,x') = e^{\beta F}\sum_{n,m} e^{i(\mathbf{P}_m-\mathbf{P}_n)\left(\mathbf{x}-\mathbf{x}'\right)/\hbar} e^{-\beta E_n} e^{-i(E_m-E_n)\left(t-t'\right)/\hbar}\left|\langle n\left|\Phi(0)\right|m\rangle\right|^2 \tag{10.15}$$

Given suitable conditions on the matrix elements of the field operators, we may regard the sum in equation (10.15) as converging when the integrand is exponentially suppressed, namely for $0 \geq \mathrm{Im}\ (t-t') \geq -\beta$. In this strip, equation (10.15) defines a complex variable function, which is the definition of the thermal propagator for complex time. G^{+}_β may be analytically continued beyond the strip, of course, but this constitutes its fundamental domain of definition.

If we apply the same reasoning to the "negative frequency" propagator we obtain

$$G^{-}_\beta(x,x') = e^{\beta F}\sum_{n,m} e^{i(\mathbf{P}_m-\mathbf{P}_n)\left(\mathbf{x}-\mathbf{x}'\right)/\hbar} e^{-\beta E_m} e^{-i(E_m-E_n)\left(t-t'\right)/\hbar}\left|\langle n\left|\Phi(0)\right|m\rangle\right|^2 \tag{10.16}$$

Comparing equations (10.15) and (10.16), it is immediate that

$$G^{+}_\beta\left((t,\mathbf{x}),x'\right) = G^{-}_\beta\left((t+i\hbar\beta,\mathbf{x}),x'\right) \tag{10.17}$$

This is the KMS theorem for Bose–Einstein fields. The KMS theorem for Fermi–Dirac fields will be discussed in a later section.

A shorter argument may serve as a mnemotechnic device, and also underscores the generality of this result. Let us define the complex time Heisenberg operators

$$\Phi(z) = e^{izH/\hbar}\Phi(0)\,e^{-izH/\hbar} \tag{10.18}$$

and observe the identity

$$e^{-\beta H}\Phi(t) = \Phi(t+i\hbar\beta)\,e^{-\beta H} \tag{10.19}$$

Then a simple cyclic permutation under the trace yields

$$\begin{aligned} G^{+}_\beta(t,t') &= e^{\beta F}\,\mathrm{Tr}\,\left\{e^{-\beta H}\Phi(t)\,\Phi(t')\right\} \\ &= e^{\beta F}\,\mathrm{Tr}\,\left\{e^{-\beta H}\Phi(t')\,\Phi(t+i\hbar\beta)\right\} = G^{-}_\beta(t+i\hbar\beta,t') \end{aligned} \tag{10.20}$$

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Given the importance of the KMS theorem for this subject, we shall show a third proof of it, now based on the properties of the path integral representation. The propagator $G^{+}_\beta(t,t') = G^{21}_\beta(t,t')$ may be computed by inserting the product $\varphi^2(t)\,\varphi^1(t')$ inside the path integral (10.6); namely, the field at t is put on the second branch, and at t' on the first branch. Given the path independence,

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we may choose a time contour beginning at t' and ending at $t' - i\beta\hbar$. Because field configurations are periodic, we obtain $\varphi^2(t)\,\varphi^1(t') = \varphi^3(t' - i\beta\hbar)\,\varphi^2(t)$. Since the path integral automatically sets field operators on the third branch to the left of field operators in the second branch, when we integrate we obtain the negative frequency propagator $G^-_\beta(t, t' - i\beta\hbar)$. Thermal propagators being translation invariant, this is again the KMS theorem.

The KMS theorem implies a new relationship among the Fourier transforms of the thermal propagators, namely

$$G^+_\beta(\omega) = e^{\beta\hbar\omega} G^-_\beta(\omega) \tag{10.21}$$

Together with the universal relationship $G^+_\beta - G^-_\beta = G_\beta$, the latter being the thermal Jordan propagator, we obtain

$$G^+_\beta(\omega) = \frac{e^{\beta\hbar\omega}}{e^{\beta\hbar\omega} - 1} G_\beta(\omega)\,; \quad G^-_\beta(\omega) = \frac{1}{e^{\beta\hbar\omega} - 1} G_\beta(\omega) \tag{10.22}$$

The two formulae can be combined into

$$G^\pm_\beta(\omega) = 2\pi\hbar\left[\theta(\pm\omega) + f_0(\omega)\right]\mathcal{D}(\omega) \tag{10.23}$$

where

$$2\pi\hbar\mathcal{D}(\omega) = \operatorname{sign}(\omega)\, G_\beta(\omega) = G_\beta(|\omega|) \tag{10.24}$$

and f_0 is the Bose–Einstein distribution

$$f_0(\omega) = \frac{1}{e^{\beta\hbar|\omega|} - 1} \tag{10.25}$$

These formulae generalize the vacuum Lehmann representation. As in the vacuum case, they allow us to find all the propagators, once the Jordan propagator is known. For example, the Hadamard propagator becomes

$$G_{\beta 1}(\omega) = 2\pi\hbar\left[1 + 2f_0(\omega)\right]\mathcal{D}(\omega) \tag{10.26}$$

For a free field the commutator of two field operators is a c-number, and therefore its expectation value is independent of the state. In this case the zero-temperature Jordan propagator remains valid at all temperatures.

The linear response equation (10.9) allows us to connect the Jordan and retarded propagators

$$G_{\beta\mathrm{ret}}(x, x') = \left(\frac{i}{\hbar}\right) G_\beta(x, x')\,\theta(t - t') \tag{10.27}$$

which is the same relationship as at zero temperature. Fourier transforming, we obtain as in Chapter 3

$$G_\beta(p) = 2\hbar \mathrm{Im} G_{\beta\mathrm{ret}}(p) \tag{10.28}$$

and therefore equation (10.26) may be regarded as a statement of the fluctuation–dissipation theorem.

10.4 Thermal self-energy: Screening

As remarked earlier, the CTP perturbation theory at finite temperature is formally identical to its zero-temperature counterpart, only now propagators must be consistent with the KMS theorem. In particular, the 2PIEA is formally the same, but it is now evaluated at a different set of propagators. We may even give a direct proof of this statement, by noting that thermal field theory is the usual field theory on a three-branched path. We may push the third branch to the remote past, and at the same time switch interactions off adiabatically, so that the action in the third branch corresponds to a free theory. Then the path integral over Euclidean configurations will be Gaussian, and the net effect will be to add a two-point source $K_{\beta AB}$ (concentrated at the initial time), as we did when we formally constructed the 2PIEA. The construction will then go as in Chapter 6, only that the final equation for the thermal propagators becomes

$$\frac{\delta\Gamma_2}{\delta G_\beta^{AB}} = -\frac{1}{2}K_{\beta AB} \tag{10.29}$$

Because the right-hand side turns on only at the initial time, in practice we may ignore it; its only role is to enforce the KMS initial conditions. We may therefore write down the thermal 2PI Schwinger–Dyson equation: it is just the same as the vacuum equation we discussed in Chapter 6.

Because Lorentz invariance is lost for thermal fields, we cannot define the mass of the field from the location of the pole of the propagators. Luckily, this problem only appears at the three-loop level in the 2PIEA, so we may still make progress by analyzing the one-loop gap equation.

The analysis is in fact the same as in Chapter 6, only now the tadpole is computed with a thermal Feynman propagator or, what is the same at the coincidence limit, a thermal Hadamard propagator. Since the free Jordan propagator is independent of temperature, this means that there is a new term, corresponding to the integration over the thermal correction demanded by the KMS theorem. So the gap equation becomes

$$M^2 = m_b^2 + m_V^2 + \frac{\lambda_b \hbar}{2} M_T^2 \tag{10.30}$$

where

$$M_T^2 = \int \frac{d^4p}{(2\pi)^3}\delta\left(\Omega_0\right) f_0\left(p\right) = \frac{1}{\pi^2}\int_M^\infty d\omega\ \frac{\sqrt{\omega^2 - M^2}}{e^{\beta\omega} - 1} \tag{10.31}$$

and $\Omega_0 = p^2 + M^2$. The second term m_V^2 is a vacuum tadpole.

Although strictly speaking the gap equation must be renormalized before it makes sense, we may learn some of its implications from the following simple arguments. First, observe that there exists a critical temperature $T_{\rm c}$ (maybe imaginary) such as to make M^2 vanish. If we use dimensional regularization, the massless tadpole vanishes as well, and the massless thermal contribution gives

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the usual value of $M^2_{T_c} \sim (1/6)\,(k_B T_c/\hbar)^2$, so we get $m_b^2 = (-\lambda_b/12\hbar)\,(k_B T_c)^2$. If we adopt T_c rather than m_b as the fundamental object, then $M^2 \sim O\left(\lambda T^2\right) \ll T^2$, which justifies neglecting the tadpole term and computing the thermal mass as for a massless theory. We then get the simple gap equation

$$M^2 = \frac{\lambda k_B^2}{12\hbar}\left[T^2 - T_c^2\right] \tag{10.32}$$

The first and obvious consequence of the thermal gap equation is that a field theory which is massless at some temperature ($T = T_c = 0$, say) will not be massless at other temperatures. If we associate a massless field (such as the photon) with a long-range interaction, then at finite temperature the same interaction will be short range (of course, Maxwell's theory is a gauge theory, and we must be careful [LeB96]).This phenomenon is called *screening*, and because it fixes the screening length M^{-1}, M^2 is sometimes called the *Debye mass* M_D^2.

If T_c is real, the gap equation admits negative solutions at low temperature (meaning that the symmetric point $\phi = 0$ corresponds to an unstable configuration of the field, and we should not be doing perturbation theory around it) and regular solutions above T_c. This is the phenomenon of *symmetry restoration*, and T_c is the critical temperature which marks the destabilization of the symmetric point [DolJac74, Wei74].

In theories with multiple fields the thermal mass matrix may not be positive definite. In this case, there may be inverse symmetry restoration (namely, a symmetry is broken at high temperature) or else symmetry nonrestoration (a symmetry is never restored) [Wei74, PinRam06].

The generation of the Debye mass and the corresponding screening length means that the behavior of thermal propagators is totally different from their zero temperature counterparts, at least for *soft momenta* $p \le M_D$. In this range, to assume that one can develop a meaningful perturbation theory without a careful consideration of screening is simply wrong. At the very least, one should use the physical mass throughout. Since a mass shift in the inverse propagator is equivalent to resumming an infinite number of graphs in the perturbative expansion of the propagator itself, and the shift may be seen as coming mostly from the high momentum sector where all masses can be neglected, the techniques necessary to implement a consistent perturbation theory are generally known as *hard thermal loops resummation*.

10.5 Landau damping

In addition to screening, at finite temperature a collective excitation will be damped by scattering off quanta in the heat bath. Therefore, there are decay channels unavailable at zero temperature. The most important of these decay processes is the so-called Landau damping, originally discussed by Landau in the context of collisionless plasma theory [LifPit81].

In this section, we shall discuss Landau damping through a concrete example, namely, the damping of a Maxwell field interacting with a Dirac quantum field at finite temperature. To stress the physics involved, we shall begin with a brief review of Landau damping in its original plasma physics context.

10.5.1 Landau damping in a relativistic collisionless plasma

Before we consider the issue of damping in the equation for a Maxwell background field coupled to quantum fluctuations in a Dirac spinor field, let us discuss the same issue in the simpler context of a classical relativistic collisionless plasma.

Under the collisionless approximation, particles evolve independently under the electromagnetic field, which in turn is sourced by the average charge density and current. The dynamics of each particle is determined by the Hamiltonian (we set the speed of light $c = 1$)

$$p^0 = \left[(\mathbf{p} - e\mathbf{A})^2 + m^2 \right]^{1/2} + eA^0 \tag{10.33}$$

Therefore we have the velocity

$$\mathbf{v} = \nabla_{\mathbf{p}} p^0 = \frac{(\mathbf{p} - e\mathbf{A})}{(p^0 - eA^0)} \tag{10.34}$$

and the Lorentz force (in a somewhat unusual notation)

$$[p_i]^{\cdot} = -\partial_i p^0 = e\left[v^j A_{j,i} - \partial_i A^0 \right] \tag{10.35}$$

The charge and current densities are given by

$$j^0 = e \int \frac{d^3\mathbf{p}}{(2\pi)^3} f(\mathbf{x}, \mathbf{p}, t) \tag{10.36}$$

$$\mathbf{j} = e \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathbf{v}\, f(\mathbf{x}, \mathbf{p}, t) \tag{10.37}$$

Charge is conserved as a consequence of Hamilton's equations, provided f satisfies the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \dot{\mathbf{p}} \cdot \nabla_{\mathbf{p}} f = 0 \tag{10.38}$$

Explicitly

$$(p - eA)^\mu \frac{\partial f}{\partial x^\mu} + e\left[(p - eA)^\nu A_{\nu,j} \right] \frac{\partial f}{\partial p_j} = 0 \tag{10.39}$$

We are interested in linearized Maxwell fields, so we may expand $f = f^0 + f^1 + \ldots$ in powers of A. Assume $f^0 = f^0(\mathbf{p})$. The first-order terms read

$$p^\mu \frac{\partial f^1}{\partial x^\mu} + e\left[p^\nu A_{\nu,j} \right] \frac{\partial f^0}{\partial p_j} = 0 \tag{10.40}$$

If the Maxwell background corresponds to a plane wave

$$A_\mu = A_{k\mu} e^{ikx} \tag{10.41}$$

we may seek a solution

$$f^1 = f^1_k e^{ikx} \tag{10.42}$$

where, adopting Landau's causal boundary conditions,

$$f^1_k = e\left[\frac{p^\nu A_{k\nu}}{-p^\lambda\left(k+i\varepsilon\right)_\lambda}\right] k_j \frac{\partial f^0}{\partial p_j} \tag{10.43}$$

We may now write the charge density (10.36). Using our perturbative solution and discarding the zeroth-order term this becomes

$$j^0 = j^0_k e^{ikx} \tag{10.44}$$

$$j^0_k = e^2 \int \frac{d^3\mathbf{p}}{(2\pi)^3}\left[\frac{p^\nu A_{k\nu}}{-p^\lambda\left(k+i\varepsilon\right)_\lambda}\right] k^i \frac{\partial f^0}{\partial p^i} \tag{10.45}$$

Observe that the induced current is gauge invariant. Essentially, what we have done is to compute the conductivity of the plasma. The important point is that equation (10.45) develops an imaginary part when there are particles whose momenta satisfy $p^\lambda k_\lambda = 0$. The imaginary part reads

$$\mathrm{Im}\left[j^0_k\right] = \frac{-\pi e^2}{\left(k^0\right)^2}\left[k^0 A^j_k - k^j A_{k0}\right] k^i \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\,\delta\left[p^0 - \frac{\mathbf{k}.\mathbf{p}}{k^0}\right] p_j \frac{\partial f^0}{\partial p^i} \tag{10.46}$$

If we use this as a source for the Maxwell equations, the imaginary part in the charge density will engender the so-called Landau damping of the background plane wave. As can be seen from equation (10.46), Landau damping occurs when there are charged particles moving at the phase speed of the wave. These particles see the wave as a time-independent field, and may extract energy from it. Actually, the expression for $\mathrm{Im}\left[j^0_k\right]$ does not have a definite sign; however, damping obtains generally for isotropic distributions [LifPit81].

10.5.2 A nonequilibrium problem with fermions: The case of QED

As a simple example of fermionic nonequilibrium field theory, we wish to consider the linearized equations of motion for a Maxwell background field coupled to a Dirac spinor (representing the electron field). The action is given by

$$S = S_{\mathrm{M}} + S_{\mathrm{D}} + S_{\mathrm{int}} \tag{10.47}$$

where S_{M} is the free Maxwell action

$$S_{\mathrm{M}} = \left(\frac{-1}{4}\right)\int d^4x\, F^{\mu\nu}F_{\mu\nu} \tag{10.48}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{10.49}$$

is the Maxwell field tensor, and A_μ the photon field. $S_{\rm D}$ is the free Dirac action

$$S_{\rm D} = \int d^4x\, \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi \tag{10.50}$$

where ψ is a four-component Dirac spinor, the γ^μ are the Dirac matrices obeying

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu} \tag{10.51}$$

and $\overline{\psi}$ is the Dirac adjoint spinor

$$\overline{\psi} = \psi^\dagger\gamma^0 \tag{10.52}$$

The interaction term is

$$S_{\rm int} = \int d^4x\, eA_\mu\overline{\psi}\gamma^\mu\psi \tag{10.53}$$

We wish to compute the lowest order $\left(O\left(e^2\right)\right)$ linearized equation of motion for the Maxwell background field. To this order, photon quantum fluctuations are decoupled, so we need not consider them further. The CTPEA for the Maxwell field reads

$$\Gamma\left[A^a_\mu\right] = S_{\rm M}\left[A^a_\mu\right] + \Gamma_1\left[A^a_\mu\right] \tag{10.54}$$

$$S_{\rm M}\left[A^a_\mu\right] = S_{\rm M}\left[A^1_\mu\right] - S_{\rm M}\left[A^2_\mu\right] \tag{10.55}$$

$$\Gamma_1\left[A^a_\mu\right] = -i\hbar\,\ln\int D\psi^a D\overline{\psi}^b\, e^{(i/\hbar)\{S_{\rm D}+S_{\rm int}\}} \tag{10.56}$$

where the path integral is over Grassmann fields [Ber66, NegOrl98] defined on the closed time path, and the actions in the integrand also are CTP actions.

To quadratic order in the external field,

$$\begin{aligned}\Gamma_1\left[A^a_\mu\right] = \frac{ie^2}{2\hbar}\, c_{acd}c_{bef}\int d^4x d^4x'\; A^a_\mu(x)\, A^b_\nu(x')\\ \left\langle\left[\overline{\psi}^c(x)\,\gamma^\mu\psi^d(x)\right]\left[\overline{\psi}^e(x')\,\gamma^\nu\psi^f(x')\right]\right\rangle_0\end{aligned} \tag{10.57}$$

where

$$\langle O\rangle_0 = \int D\psi^a D\overline{\psi}^b\, e^{iS_D}\, O \tag{10.58}$$

and we have used the fact that

$$\left\langle\left[\overline{\psi}(x)\,\gamma^\mu\psi(x)\right]\right\rangle_0 = 0 \tag{10.59}$$

(see below). Since the integration measure is Gaussian, Wick's theorem holds and

$$\left\langle\left[\overline{\psi}^c(x)\,\gamma^\mu\psi^d(x)\right]\left[\overline{\psi}^e(x')\,\gamma^\nu\psi^f(x')\right]\right\rangle_0 = -\gamma^\mu G^{de}(x,x')\,\gamma^\nu G^{fc}(x',x) \tag{10.60}$$

where

$$G^{de}(x,x') = \left\langle \psi^d(x)\,\overline{\psi}^e(x')\right\rangle_0 \tag{10.61}$$

and we have used the fact that in this problem

$$\left\langle \psi^d(x)\,\psi^e(x')\right\rangle_0 = 0 \tag{10.62}$$

which follows directly from the Gaussian integration.

The G^{de} are the CTP free Dirac propagators. From the ordering properties of the CTP path integral

$$G^{21}(x,x') = \left\langle \hat{\psi}(x)\,\hat{\overline{\psi}}(x')\right\rangle \tag{10.63}$$

$$G^{12}(x,x') = -\left\langle \hat{\overline{\psi}}(x')\,\hat{\psi}(x)\right\rangle \tag{10.64}$$

where $\hat{\psi}$ is the Heisenberg picture field operator, and the brackets denote vacuum expectation value. We also have

$$G^{11}(x,x') = G^{21}(x,x')\,\theta(t-t') + G^{12}(x,x')\,\theta(t'-t) \tag{10.65}$$

$$G^{22}(x,x') = G^{21}(x,x')\,\theta(t'-t) + G^{12}(x,x')\,\theta(t-t') \tag{10.66}$$

The Heisenberg equation for the free field operator is the Dirac equation, whereby

$$\left(i\gamma^\mu\partial_\mu - m\right)G^{21}(x,x') = 0 \tag{10.67}$$

and similarly for $G^{12}(x,x')$, while

$$\left(i\gamma^\mu\partial_\mu - m\right)G^{11}(x,x') = i\hbar\delta(x-x') \tag{10.68}$$

$$\left(i\gamma^\mu\partial_\mu - m\right)G^{22}(x,x') = -i\hbar\delta(x-x') \tag{10.69}$$

The solution to these equations with the proper CTP boundary conditions is

$$G^{ab}(x,x') = \left(i\gamma^\mu\partial_\mu + m\right)\Delta^{ab}(x,x') \tag{10.70}$$

where Δ^{ab} are the Klein–Gordon CTP propagators. With the representation (10.70) it is immediate to obtain equation (10.59).

The new term in the CTPEA induces a new term in the wave equation for the photon field

$$\frac{e^2}{\hbar}\int d^4x'\,\Pi^{\mu\nu}(x,x')\,A_\nu(x') \tag{10.71}$$

$$\Pi^{\mu\nu}(x,x') = (-i)\,\gamma^\mu\left\{G^{11}(x,x')\,\gamma^\nu G^{11}(x',x) - G^{12}(x,x')\,\gamma^\nu G^{21}(x',x)\right\} \tag{10.72}$$

As in the case of the scalar field, the first term is what we would have found from the “in-out” EA, and the second term enforces reality and causality.

Using the representation (10.70) the calculation of this term is a standard exercise in quantum field theory [Ram80, PesSch95] and we will not repeat it. The photon retarded propagator acquires an imaginary part for off-shell momenta $-p^2 \geq 4m^2$. This represents damping of a photon wave from pair creation. We have seen in Chapter 4 that damping also goes on below threshold, but it is exponentially suppressed.

10.5.3 KMS and thermal Fermi propagators

The proof of the KMS theorem works as well for Fermi fields, but now we must take into account the proper relation between the Fermi propagators and their expression as averages of Heisenberg fields. If we introduce the thermal positive and negative frequency propagators $G_\beta^+ \equiv G_\beta^{21}$ and $G_\beta^- \equiv G_\beta^{12}$ the KMS condition becomes

$$G_\beta^+(\omega) = -e^{\beta\hbar\omega} G_\beta^-(\omega) \tag{10.73}$$

We may introduce a Fermi Jordan propagator

$$G_\beta = G^+ - G^- = \left\langle \left\{ \psi, \overline{\psi} \right\} \right\rangle_\beta \tag{10.74}$$

(for a free field, G_β is independent of the temperature) and a density of states

$$\mathcal{D}_F(\omega) = \frac{1}{2\pi\hbar} \operatorname{sign}(\omega)\, G_\beta(\omega) \tag{10.75}$$

Then the KMS condition becomes

$$G_\beta^\pm(\omega) = 2\pi\hbar \left\{ \theta(\pm\omega) - f_{\mathrm{FD}}(\omega) \right\} \mathcal{D}_F(\omega) \tag{10.76}$$

where f_{FD} is the Fermi–Dirac distribution

$$f_{\mathrm{FD}}(\omega) = \left[e^{\beta\hbar|\omega|} + 1 \right]^{-1} \tag{10.77}$$

All other propagators may be built from these two.

10.5.4 Induced charge density from a finite temperature Dirac quantum field

We now return to the quantum field problem. A nontrivial Maxwell background induces a current (cf. equation (10.71))

$$j^\mu(x) = \frac{\delta\Gamma_1}{\delta A_\mu(x)} = \frac{e^2}{\hbar} \int d^4x'\, \Pi^{\mu\nu}(x, x')\, A_\nu(x') \tag{10.78}$$

where $\Pi^{\mu\nu}(x, x')$ is defined in equation (10.72). As in our simple plasma example, we shall look into the induced charge density only. If the background is a single plane wave as in equation (10.41), then

$$j_k^0 = \frac{e^2}{\hbar} \Pi_k^{0\nu} A_{k\nu} \tag{10.79}$$

$$\Pi_k^{0\nu} = (-i)\,\gamma^0 \int \frac{d^4p}{(2\pi)^4} \left\{ G^{11}(p)\, \gamma^\nu G^{11}(p-k) - G^{12}(p)\, \gamma^\nu G^{21}(p-k) \right\} \tag{10.80}$$

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To continue, let us decompose all propagators into a zero-temperature and a statistical component

$$G^{ab}(p) = G_0^{ab}(p) - G_{\text{stat}}(p) \tag{10.81}$$

where $G_0^{ab}(p)$ has the form of the zero-temperature propagators, but maybe with temperature-dependent coefficients, and

$$G_{\text{stat}}(p) = 2\pi\hbar\, f_{\text{FD}}\left(p^0\right)\left[-\gamma^\mu p_\mu + m\right]\delta\left(-p^2 - m^2\right) \tag{10.82}$$

is the same for all basic propagators. Clearly

$$\Pi_k^{0\nu} = \Pi_{k0}^{0\nu} + \Pi_{k\text{stat}}^{0\nu} \tag{10.83}$$

where the first term is the zero-temperature contribution. As we know, this term describes, among other things, damping from pair creation out of the vacuum. Our interest here is the other term

$$\Pi_{k\text{stat}}^{0\nu} = \hbar\gamma^0 \int \frac{d^4p}{(2\pi)^4} J^\nu \tag{10.84}$$

$$J^\nu = G_{\text{ret}}\left(p + \frac{k}{2}\right)\gamma^\nu G_{\text{stat}}\left(p - \frac{k}{2}\right) + G_{\text{stat}}\left(p + \frac{k}{2}\right)\gamma^\nu G_{\text{adv}}\left(p - \frac{k}{2}\right) \tag{10.85}$$

The exact expression for $\Pi_{k\text{stat}}^{0\nu}$ is involved and shall not be discussed further. It becomes simpler at high temperature, where we may argue that the leading contribution to the integral comes from momenta $p \approx T \gg k, m$. The leading contribution in this limit is

$$4\hbar^2 \int \frac{d^4p}{(2\pi)^3} \frac{p^0 p^\nu}{p\,(k + i\varepsilon)} J \tag{10.86}$$

$$J = f_{\text{FD}}\left(p - \frac{k}{2}\right)\delta\left(\left(p - \frac{k}{2}\right)^2 + m^2\right) - f_{\text{FD}}\left(p + \frac{k}{2}\right)\delta\left(\left(p + \frac{k}{2}\right)^2 + m^2\right) \tag{10.87}$$

If $k \ll p$, we may expand inside the brackets. The leading contribution to the imaginary part comes from a term

$$4\hbar^2 \int \frac{d^4p}{(2\pi)^3} \frac{2p^0\delta\left(-p^2 - m^2\right)\theta\left(p^0\right)}{\left[-p\,(k + i\varepsilon)\right]} p^\nu k^0 \frac{\partial f_{\text{FD}}}{\partial p^0} \tag{10.88}$$

The momenta which contribute to the imaginary part satisfy $p \cdot k = 0$, and so

$$k^0 \frac{\partial f_{\text{FD}}}{\partial p^0} = \frac{k^0 p^0}{p^0} \frac{\partial f_{\text{FD}}}{\partial p^0} = \frac{\mathbf{k}\cdot\mathbf{p}}{p^0} \frac{\partial f_{\text{FD}}}{\partial p^0} = \mathbf{k}\nabla_{\mathbf{p}} f_{\text{FD}} \tag{10.89}$$

We see a quantum field theory version of the Vlasov equation, only now we have a factor of 4 reflecting the presence of electrons and positrons with two spin states each, and $\hbar f_{\text{FD}}$ instead of the classical one-particle distribution function.

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10.6 Hard thermal loops

In this section we show how the above machinery can be applied to one set of important problems – the dynamics of long-wavelength or "soft" modes of a nonlinear quantum field, as affected by the short-wavelength or "hard" modes towering over them. Historically, the techniques of this subject were developed in an attempt to understand the physics of soft modes in non-abelian gauge theories, for application to the quark–gluon plasma in relativistic heavy ion collisions (RHICs), and to topology change in electroweak theory. This specific context imposed a number of constraints, such as the need to deal with gauge invariance, derivative couplings, ghost fields, etc. We wish to isolate the basic physical ideas relating to nonequilibrium and statistical behavior from the technical devices specific to a given application, and therefore we shall not follow the historical path, but rather present a fictitious toy model which retains sufficient fundamental physics.

The important lesson to be gleaned from this is that formal questions, such as which is the best perturbative scheme or to which order should it be pursued, cannot be separated from physical questions. As illustrated in this example, we get different answers depending on whether we wish to discuss soft or ultrasoft modes. In either case, simply counting powers of coupling constants gives the wrong result. It is necessary to analyze the contents of the theory to make sure that what looks small is indeed small, and to realize that different scales pertain to different physics.

In this section we shall use some tools of quantum kinetic field theory which will be discussed in detail in Chapter 11. The reader unfamiliar with these techniques may return to this section after getting acquainted with them.

10.6.1 The model

The essential elements we need to keep from the physics of non-abelian gauge fields are the presence of massless fields and a derivative cubic coupling. For massless fields radiative corrections are infrared sensitive, and will require special care to evaluate them. There is a term in the action containing three gauge fields and one derivative. In momentum space, the derivative becomes one momentum component. At finite temperature T, typical momenta are of the order of T, and so the effective coupling strength increases with temperature.

Therefore we postulate as our toy model a massless scalar field theory with cubic interaction

$$S = \int d^4x \, \left\{ -\frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{gT}{6}\Phi^3 + h\Phi \right\} \tag{10.90}$$

with the constitutive relation $h = gT^3/12$. We assume $g \ll 1$. The linear term is necessary to cancel a tadpole term later on; non-abelian gauge theories are protected against such terms by gauge invariance. Of course, this model is not

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by itself a viable theory, since it has no stable ground state; we shall use it only as an ersatz for a fully consistent, but necessarily more involved, non-abelian gauge theory.

The basic problem, as already stated, is to find the dynamics of the "soft" modes as modified by the virtual "hard" particles, the so-called "hard thermal loops" (HTLs). Here a "soft" mode corresponds to wavenumbers $k \sim gT$, while a "hard" mode has a much larger wavenumber $k \sim T$. Taking the open systems approach as introduced in Chapter 5, we partition the scalar field $\Phi = \phi + \varphi$, where ϕ is the soft field, and φ is the hard field. The soft and hard field equations are, respectively

$$\partial^2 \phi - \frac{gT}{2}\phi^2 - \frac{gT}{2}\,\varphi^2\big|_{\text{soft}} + h = 0 \tag{10.91}$$

$$\partial^2 \varphi - gT\phi\varphi - \frac{gT}{2}\,\varphi^2\big|_{\text{hard}} = 0 \tag{10.92}$$

In a naive perturbation expansion, we would argue that since the hard modes appear in equation (10.91) in a $O(g)$ term, we only need to solve equation (10.92) up to $O(1)$ accuracy. We would neglect the second and third terms in equation (10.92), proceeding to treat φ as a massless Klein–Gordon field. The treatment simplifies even further if we actually think of the soft modes as a classical field (which may be justified on the grounds of the large occupation numbers prevalent in the soft sector of the theory). Then we may replace $\varphi^2\big|_{\text{soft}}$ in equation (10.91) by the thermal expectation value appropriate for a massless field $\left\langle \varphi^2 \right\rangle \sim T^2/6$. In this approximation the $\left\langle \varphi^2 \right\rangle$ term is canceled by the h term, by design, and we find that to leading order in g, the hard modes have no effect on the soft modes.

10.6.2 Hard thermal loops

Braaten and Pisarski [BraPis90a, BraPis90b, BraPis92] and Frenkel and Taylor [FreTay90] were the first to point out that this argument is not only naive, but actually wrong. The reason is resonance. Assume for simplicity that the soft modes undergo a homogeneous oscillation $\phi = \phi_0 \cos \omega t$. Assume also a perturbative expansion $\varphi = \varphi_0 + \varphi_1 + \dots$ for the hard modes, where $\varphi_n \propto g^n$, and neglect interactions between hard modes. Then $\partial^2 \varphi_0 = 0$, and we may expand (in this section, we use natural units)

$$\varphi_0 = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{e^{i\mathbf{k}\mathbf{x}}}{\sqrt{2k}}\varphi_{0\mathbf{k}}; \qquad \varphi_{0\mathbf{k}} = a_{0\mathbf{k}}e^{-ikt} + a^{\dagger}_{0-\mathbf{k}}e^{ikt} \tag{10.93}$$

with

$$\left\langle a^{\dagger}_{0\mathbf{k}} a_{0\mathbf{p}} \right\rangle = n_k \delta(\mathbf{k} - \mathbf{p}) \tag{10.94}$$

where n_k is the Bose–Einstein distribution

$$n_k = \left[e^{k/T} - 1\right]^{-1} \tag{10.95}$$

For not-so-hard modes, $n_k \sim T/k$.

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Now consider the first-order correction. Assuming a Fourier decomposition

$$\varphi_1 = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{e^{i\mathbf{kx}}}{\sqrt{2k}} \varphi_{1\mathbf{k}} \tag{10.96}$$

we have

$$\frac{\partial^2}{\partial t^2}\varphi_{1\mathbf{k}} + k^2 \varphi_{1\mathbf{k}} = -\left[gT\phi_0 \cos\omega t\right]\varphi_{0\mathbf{k}} \tag{10.97}$$

Neglecting the homogeneous solution, we get, in the limit of soft ω,

$$\varphi_{1\mathbf{k}} = \frac{gT\phi_0}{2k^2\omega} \sin\omega t \, \frac{\partial}{\partial t}\varphi_{0\mathbf{k}} \tag{10.98}$$

Glossing over the details of actually computing the expectation values, we see that a priori we expect a correction to the soft equation of motion of order of magnitude $gT\langle\varphi_0\varphi_1\rangle \sim g^2 (T/\omega) T^2\phi_0$. For $\omega \sim gT$, this is larger than the classical term $\omega^2\phi_0$ by a factor of g^{-1}, and it cannot possibly be neglected.

In diagrammatic terms, we may represent $\left\langle\varphi^2\right\rangle$ as a tadpole graph (cf. Fig. 6.1 in Chapter 6). By considering a soft field insertion, it turns into a fish graph (Fig. 6.7 in Chapter 6). Explicitly

$$\left\langle\varphi^2\right\rangle = \Delta(x,x) + gT\int d^4y\, \Delta^2(x,y)\,\phi(y) + \ldots \tag{10.99}$$

where Δ represents a massless scalar propagator. We postpone the question on exactly which propagator is involved, and work for now with the Feynman propagators (as would be the case in the “in-out” formulation). Fourier transforming,

$$\left\langle\varphi^2\right\rangle = \int \frac{d^4p}{(2\pi)^4}\,\Delta(p) + gT\int \frac{d^4k}{(2\pi)^4}\int \frac{d^4p}{(2\pi)^4}\,\Delta(p)\,\Delta(p-k)\,\phi(k) \tag{10.100}$$

We are interested in the contribution from the second term. Consider the contribution from the thermal part in $\Delta(p)$. Then p is on-shell. Since $p-k$ cannot be on-shell, $\Delta(p-k) \sim (p-k)^{-2} \sim 1/2p^0\omega$. The presence of an inverse power of ω in the integral invalidates the naive perturbation theory when ω is parametrically small. If in particular $\omega \sim gT$, the “correction” is a priori as large as the leading term.

Moreover, the problem appears with *every* soft insertion. Adding a soft insertion to a pre-existing graph adds a power of $gT\phi$ but also a power of $\left(p^0\omega\right)^{-1}$. If we are considering large field amplitudes $\phi \sim T$, hard momenta $p \sim T$, and soft frequencies $\omega \sim gT$, then $gT\phi \sim p^0\omega \sim gT^2$. The overall amplitude of the graph with the insertion is not smaller than without it, and we must sum over all soft field insertions at once. The result is called a HTL resummed perturbation theory.

10.6.3 Hard thermal loops from the 2PI CTP effective action

To derive the HTL resummed theory, we shall use the 2PI CTP formalism. Since graphs with more than one external field insertion cannot be two-particle irreducible, they do not appear explicitly in the 2PI effective action (that is, the graph with a single insertion represents them all). The CTP technique warrants causality of the resulting equations of motion.

To simplify things, we shall make the semiclassical approximation for the soft field (that is, we shall not include soft field propagators) and set the hard background field to zero (that is, the hard field will be represented only by its propagators). The 2PI CTP effective action Γ_2 is thus a functional of a CTP soft background field ϕ^A (as before, the index A comprises a branch index $a = 1, 2$ and a spacetime location x) and hard propagators G^{AB}, and takes the form

$$\Gamma_2 = S\left[\phi^A\right] + \frac{1}{2} S_{,AB}\left[\phi^A\right] G^{AB} - \frac{i}{2}\mathrm{Tr}\,\ln G + \Gamma_Q \tag{10.101}$$

where $S\left[\phi^A\right] = S\left[\phi^1\right] - S\left[\phi^2\right]$, and

$$S_{,AB}\left[\phi^A\right] = \left\{c_{ab}\partial_x^2 - c_{abc} g T \phi^c(x)\right\}\delta(x-y) \tag{10.102}$$

(the tensors $c_{abc\ldots}$ take the value 1 when all their indices are 1, -1 when all the indices are 2, and vanish otherwise). Γ_Q is the sum of all 2PI vacuum bubbles with a cubic vertex and G propagators. It represents the hard field's self-interactions, and we shall disregard it for the time being.

Variation of Γ_2 yields the 2PI Schwinger–Dyson equation for G

$$\left\{c_{ab}\partial_x^2 - c_{abc} g T \phi^c(x)\right\} G^{bd}(x,y) = i\delta_a^d \delta(x-y) \tag{10.103}$$

(where $\phi^1(x) = \phi^2(x) = \phi(x)$) and the field equation for the soft modes

$$\partial^2\phi(x) - \frac{gT}{2}\phi^2(x) - \frac{gT}{2} G^{11}(x,x) + h = 0 \tag{10.104}$$

In principle, one is to solve equation (10.103) for G and plug into equation (10.104) for ϕ. In practice, solving equation (10.103) is nontrivial, because of the spacetime dependence in $\phi(x)$. One possibility is to take advantage of the slow variation of the soft field to write the hard propagator in terms of a Wigner function. Since there are no hard self-interactions, the result is a Vlasov equation for the hard-field Wigner function.

10.6.4 The Vlasov equation for hard modes

Observe that the soft field only couples to the hard Hadamard propagator, which, neglecting hard self-interactions, obeys the simple equation

$$\left(\partial_x^2 - gT\phi(x)\right) G_1(x,y) = 0 \tag{10.105}$$

To avoid formal problems particular to the cubic interaction, we shall assume $\phi(x) > 0$ throughout.

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Following the usual quantum kinetic theory approach, to be discussed further in Chapter 11, we decompose

$$G_1(x,y) = \int \frac{d^4p}{(2\pi)^4}\, e^{ipu}\, G_1(X,p) \tag{10.106}$$

$$u = x - y; \qquad X = \frac{1}{2}(x+y) \tag{10.107}$$

We also expand

$$\phi(x) \sim \phi(X) + \frac{u}{2}\partial\phi(X) + \dots \tag{10.108}$$

Keeping the first derivatives only (formally, $\partial\phi \sim gT\phi$) we obtain the mass shell condition

$$\left[p^2 + gT\phi(X)\right] G_1(X,p) = 0 \tag{10.109}$$

therefore

$$G_1(X,p) = 2\pi\delta\left(p^2 + gT\phi(X)\right)\left[1 + 2f(X,p)\right] \tag{10.110}$$

where the distribution function f obeys the Vlasov equation

$$\left(p\frac{\partial}{\partial X} - \frac{1}{2}gT\frac{\partial\phi}{\partial X}\frac{\partial}{\partial p}\right) f = 0 \tag{10.111}$$

We solve this equation perturbatively

$$f = f_0 + f_1 + \dots \tag{10.112}$$

off the thermal distribution

$$f_0 = n_{p^0} \tag{10.113}$$

Then

$$p\frac{\partial}{\partial X} f_1 = \frac{1}{2}g\frac{\partial\phi}{\partial X^0} n_{p^0}\left(1 + n_{p^0}\right) \tag{10.114}$$

which admits the particular solution

$$f_1 = \frac{(-i)}{2} g n_{p^0}\left(1 + n_{p^0}\right)\int d^4Y \int \frac{d^4Q}{(2\pi)^4}\,\frac{e^{iQ(X-Y)}}{p\,(Q + i\varepsilon)}\frac{\partial\phi}{\partial Y^0} \tag{10.115}$$

To find the soft equation of motion we must compute the coincidence limit $G_1(x,x)$

$$G_1(x,x) = G_1^0(x,x) + G_1^1(x,x) + \dots \tag{10.116}$$

$$G_1^0(x,x) = \int \frac{d^4p}{(2\pi)^3}\,\delta\left(p^2 + gT\phi(x)\right)\left[1 + 2n_{p^0}\right] \sim \frac{T^2}{6} + O(g) \tag{10.117}$$

$$\begin{aligned} G_1^1(x,x) = (-i)\, g \int d^4Y \int \frac{d^4Q}{(2\pi)^4}\, e^{iQ(X-Y)}\frac{\partial\phi}{\partial Y^0} \\ \times \int \frac{d^4p}{(2\pi)^3}\,\delta\left(p^2 + gT\phi(x)\right)\frac{n_{p^0}\left(1 + n_{p^0}\right)}{p\,(Q + i\varepsilon)} \end{aligned} \tag{10.118}$$

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Assume for simplicity that ϕ is spatially homogeneous, then the integral over the space components $\mathbf{Y}$ is immediate, and we obtain

$$G_1^0(x,x) = ig \int dY \int \frac{dQ\, e^{-iQ\left(X^0-Y\right)}}{(2\pi)(Q+i\varepsilon)} \frac{\partial \phi}{\partial Y} \int \frac{d^4p}{(2\pi)^3} \times \delta\left(p^2 + gT\phi(x)\right) \frac{n_{p^0}\left(1+n_{p^0}\right)}{p^0} \tag{10.119}$$

Now

$$\int \frac{d^4p}{(2\pi)^3}\, \delta\left(p^2 + gT\phi(x)\right) \frac{n_{p^0}\left(1+n_{p^0}\right)}{p^0} \sim \alpha\sqrt{\frac{T^3}{g\phi}} \tag{10.120}$$

where α is some numerical constant. In the other integral, we integrate by parts

$$\int dY \int \frac{dQ}{(2\pi)(Q+i\varepsilon)}\, e^{-iQ\left(X^0-Y\right)} \frac{\partial \phi}{\partial Y} \sim -i\phi(X) \tag{10.121}$$

Finally, we retrieve the equation for the soft modes

$$\partial^2\phi(x) - \frac{gT}{2}\phi^2(x) - \frac{\alpha g T^2}{4}\sqrt{gT\phi} = 0 \tag{10.122}$$

The correction is actually larger than the classical potential term when $(\phi/T) < g^{1/3}$; in any case, the corrections behave like $g^{3/2}$ rather than the expected g^2. The infrared sensitivity of the theory invalidates naive perturbation theory.

10.6.5 Ultrasoft modes and Boltzmann equation

We now return to address a possible concern about the consistency of neglecting interactions among hard modes. A moment's reflection shows that for soft modes $\omega \sim gT$ Feynman graphs containing hard cubic vertices are indeed of higher order. For example, if we compare the graph in Fig. 10.1 to the fish graph, we see that both lead to one power of ω^{-1}, but Fig. 10.1 has four powers of gT against 2 in the fish. Even the graph in Fig. 10.2 is safe, because although it scales as ω^{-2}, it also has four powers of gT in the denominator.

Figure 10.2 becomes unsafe, however, if we push the theory to deal with *ultra-soft* modes $\omega \sim g^2T$. To include it into the model, we must reconsider the role of Γ_Q. In particular, we obtain Fig. 10.2 if we approximate Γ_Q by the setting-sun

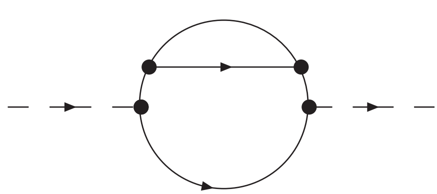

Figure 10.1

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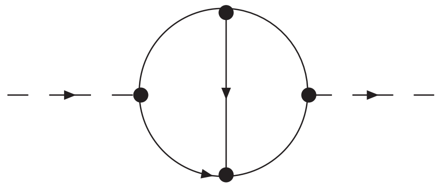

Figure 10.2

graph (Fig. 6.4 in Chapter 6). The three loops graph in Fig. 6.10 of Chapter 6 will lead to graphs of higher order even in the ultrasoft regime.

Adding a nontrivial Γ_Q has the important effect that the equation for the hard propagators becomes nonlinear. We may still make use of Wigner function techniques, but now the transport equation acquires a collision term – for the cubic self-interaction, the corresponding kinetic equation has been worked out by Danielewicz in the early days of NEqQFT [Dan84a, Dan84b], and it is not quite Boltzmann's. Of course, if we are only interested in small oscillations, we may linearize the collision term around the equilibrium solution for zero soft background. Even after linearization, the presence of the collision term affects the physics in important ways.

To get an idea of the changes brought by the collision term, we may adopt the simple "collision time approximation" [Lib98], and write the full kinetic equation as

$$\left(p\frac{\partial}{\partial X} - \frac{1}{2}gT\frac{\partial\phi}{\partial X}\frac{\partial}{\partial p}\right) f = -\frac{T}{\tau}\left(f - f_0\right) \tag{10.123}$$

where τ is the relaxation time. From naive power counting and dimensional analysis, we see $\tau \sim \left(g^2T\right)^{-1}$. The unperturbed solution is still a Bose–Einstein distribution, but now the first correction is

$$\left(p\frac{\partial}{\partial X} + \frac{T}{\tau}\right) f_1 = \frac{1}{2}g\frac{\partial\phi}{\partial X^0}n_{p^0}\left(1 + n_{p^0}\right) \tag{10.124}$$

and we may approximate

$$f_1 = \frac{g\tau}{2T}\frac{\partial\phi}{\partial X^0}n_{p^0}\left(1 + n_{p^0}\right) \tag{10.125}$$

This introduces a dissipative term in the equation for ultrasoft modes

$$\partial^2\phi\left(x\right) - \frac{gT}{2}\phi^2\left(x\right) - 2\gamma\frac{\partial\phi}{\partial X^0} = 0 \tag{10.126}$$

$$\gamma = \frac{g^2\tau}{8}\int\frac{d^4p}{\left(2\pi\right)^3}\,\delta\left(p^2 + gT\phi\left(x\right)\right)n_{p^0}\left(1 + n_{p^0}\right) \tag{10.127}$$

An explicit calculation yields

$$\gamma \sim \alpha' g^2T^2\tau\ \ln\left[\frac{T}{g\phi}\right] \tag{10.128}$$

where α' is (another) numerical constant.

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Once again, the contribution from HTLs is of order $g^2T^2\phi$, much larger than the classical term $\partial^2\phi/\partial X^{02} \sim g^4T^2\phi$.

10.6.6 Langevin dynamics of ultrasoft modes

We have seen in the previous section that the dynamics of ultrasoft modes is dissipative. From the discussion in Chapter 8, we know that it will be noisy as well. We shall use this as a model problem, by deriving the fluctuations in three different ways.

Method 1: fluctuation–dissipation theorem for ultrasoft modes

The simplest approach is to apply to the ultrasoft equation of motion (10.126) the fluctuation-dissipation theorem as discussed in Chapter 8. We modify equation (10.126) to read

$$\partial^2\phi\,(x) - \frac{gT}{2}\phi^2\,(x) - 2\gamma\frac{\partial\phi}{\partial X^0} = \xi\,(x) \tag{10.129}$$

$$\langle\xi\,(x)\,\xi\,(x')\rangle = \sigma^2\delta\,(x-x') \tag{10.130}$$

The fluctuation–dissipation theorem yields

$$\sigma^2 = 4\gamma T \tag{10.131}$$

Method 2: fluctuations from the CTPEA

Our second approach to the derivation of noise in the dynamics of ultrasoft modes will be based on the derivation of equation (10.129) from the CTPEA, a problem we already confronted in Chapter 8. The equation of motion, as derived from the CTPEA, reads

$$\partial^2\phi\,(x) - \frac{gT}{2}\phi^2\,(x) - \frac{gT}{2}\left[\left\langle\varphi^2\right\rangle_\phi - \left\langle\varphi^2\right\rangle_0\right](x) = \frac{gT}{2}\zeta\,(x) \tag{10.132}$$

We have shown in Chapter 8 that

$$\langle\zeta\,(x)\,\zeta\,(x')\rangle = \frac{1}{2}\left[\left\langle\left\{\varphi^2\,(x)\,,\varphi^2\,(x')\right\}\right\rangle_\phi - 2\left\langle\varphi^2\right\rangle_\phi(x)\left\langle\varphi^2\right\rangle_\phi(x')\right] \tag{10.133}$$

To compare these expressions to the explicit derivation above, we recall the linear response theory result (discussed earlier in this chapter)

$$\left\langle\varphi^2\right\rangle_{\phi+\delta\phi}(x) = \left\langle\varphi^2\right\rangle_\phi(x) - \frac{igT}{2}\int d^4x'\;\left\langle\left[\varphi^2\,(x)\,,\varphi^2\,(x')\right]\right\rangle_\phi\theta\left(x^0-x'^0\right)\delta\phi(x') \tag{10.134}$$

Comparison with equation (10.129) yields

$$\left\langle\left[\varphi^2\,(x)\,,\varphi^2\,(x')\right]\right\rangle_\phi = \frac{16i\gamma}{g^2T^2}\frac{\partial}{\partial x^0}\delta\,(x-x') \tag{10.135}$$

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On the other hand, commutator and anticommutator are related through the KMS theorem (also discussed earlier in this chapter). Introducing the Fourier decomposition

$$\left\langle \left[\varphi^2(x), \varphi^2(x')\right]\right\rangle_\phi = \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-x')} R(p) \tag{10.136}$$

$$\left\langle \left\{\varphi^2(x), \varphi^2(x')\right\}\right\rangle_\phi = \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-x')} R_1(p) \tag{10.137}$$

then

$$R(p) = \frac{16\gamma}{g^2T^2} p^0 \tag{10.138}$$

and the fluctuation–dissipation theorem yields

$$R_1(p) = \frac{16\gamma}{g^2T^2}\left(1 + 2n_{p^0}\right)\left|p^0\right| \tag{10.139}$$

In the high-temperature limit, this leads to the classical result

$$\langle \zeta(x)\, \zeta(x')\rangle = \frac{16\gamma}{g^2T}\delta(x - x') \tag{10.140}$$

which agrees with the results from Method 1 after identifying $\xi = gT\zeta/2$.

Method 3: fluctuations in the Boltzmann equation

Yet another method to derive the fluctuations in the ultrasoft modes is to keep to the derivation of the ultrasoft equation of motion in the previous section, but now using for the hard modes the full Boltzmann equation which, as discussed in Chapter 2, must contain stochastic terms over and above the usual collision term, thus becoming a Boltzmann–Langevin equation. This means we replace equation (10.123) by

$$\left(p\frac{\partial}{\partial X} - \frac{1}{2}gT\frac{\partial \phi}{\partial X}\frac{\partial}{\partial p}\right) f = -\frac{T}{\tau}(f - f_0) + J(X, \mathbf{p}) \tag{10.141}$$

where it is understood that p^0 is given as a function of the spatial components $\mathbf{p}$ through the mass shell condition. The noise self-correlation has been derived in Chapter 2. Under the collision time approximation (10.123) for the collision integral, we get

$$\langle J(X, \mathbf{p})\, J(Y, \mathbf{q})\rangle = 2(2\pi)^3\, \delta(X - Y)\, \delta(\mathbf{p} - \mathbf{q})\, \frac{T}{\tau} p^0\; n_{p^0}\left(1 + n_{p^0}\right) \tag{10.142}$$

For the ultrasoft components of the distribution function, we obtain

$$f_1 = f_{1\mathrm{det}} + \frac{\tau}{T}J \tag{10.143}$$

where $f_{1\mathrm{det}}$ is the deterministic solution given in equation (10.125). The equation of motion for ultrasoft modes (10.126) is transformed into equation (10.129), where now we have a model for the noise

$$\xi(x) = \frac{gT}{2} \int \frac{d^4p}{(2\pi)^3}\, \delta\left(p^2 + gT\phi(x)\right) \left[\frac{\tau}{T} J(x, \mathbf{p})\right] \tag{10.144}$$

leading to the self-correlation

$$\langle \xi(x)\, \xi(x') \rangle = \frac{g^2 T \tau}{2} \delta(x - x') \int \frac{d^4p}{(2\pi)^3}\, \delta\left(p^2 + gT\phi(x)\right) n_{p^0} \left(1 + n_{p^0}\right) \tag{10.145}$$

Comparing this to equation (10.127), we recover equation (10.130)

10.6.7 A note on the literature

As already mentioned, Braaten, Pisarsky, Frenkel and Taylor were the first to point out the need to restructure perturbation theory to account for the physics at different scales. Their work was motivated by the need to derive a reliable estimate of the decay constants of various fields in a hot non-abelian plasma. In this context, there are a number of issues associated with the gauge nature of the fundamental theory (such as whether the right decay constants are automatically gauge invariant) which have no analog in our toy model. We have only attempted to give a flavor of the physical ideas behind the formalism.

The subsequent literature on hard thermal loops is voluminous. Le Bellac's book [LeB96] has a nice chapter on this subject. Our presentation here is mostly a retelling of work by Bödeker [Bod98, Bod99] and by Arnold, Moore, Son and Yaffe [Son97, ArSoYa99a, ArSoYa99b, ArMoYa00] (of course, any flaw incurred in our attempt to "simplify" their discussion is our own). Important contributions from Blaizot and Iancu are summarized in the *Physics Reports* review article by these authors [BlaIan02]. The Boltzmann–Langevin equation for Φ^4 field was investigated by the authors in [CalHu00] while for non-abelian plasmas by Litim and Manuel. We recommend their review article as a good entry point to the literature [LitMan02].

We shall return to some of these issues in later chapters.