AXIOMATIC PROOF OF J. LAMBEK'S HOMOLOGICAL THEOREM

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DEFINITION 1: A category $\mathcal{C}$ with zero-maps is called "quasi-exact" in the sense of D. Puppe (see [4], page 8, 2.4), if it satisfies the following axioms:

(Q_1) : Every may f is a product $f = \mu\varepsilon$ of an epimorphism ε followed by a monomorphism μ.

(Q_2) : a) Every epimorphism ε has a kernel $\kappa = \text{Ker}\ \varepsilon$.

b) Every monomorphism μ has a cokernel $\gamma = \text{Coker}\ \mu$, where Ker and Coker are characterized by the familiar universality properties (see [3], page 252, (1.10) and (1.11)).

(Q_3) : a) For epic ε, $\varepsilon = \text{Coker}(\text{Ker}\ \varepsilon)$.

b) For monic μ, $\mu = \text{Ker}(\text{Coker}\ \mu)$.

These axioms alone suffice to develop essential parts of the elementary homological algebra (see [4], page 8, 2.4 and also [2]). This was shown essentially by P. J. Hilton and W. Ledermann in their theory of ringoids (see [1]), where basic theorems are proved without really using the additivity assumptions.

PROPOSITION 2: Suppose α and β are cofinal monomorphisms such that $\beta = \alpha\omega$ for some (unique and monic) ω. Then for $\alpha' = \text{Coker}\ \alpha$, $\beta' = \text{Coker}\ \beta$ there exists (a unique and epic) ω' such that $\alpha' = \omega'\beta'$ and an object Q which is simultaneously the range of Coker ω (denoted by α/β) and the domain of Ker ω' (denoted by $\beta'\backslash\alpha'$).

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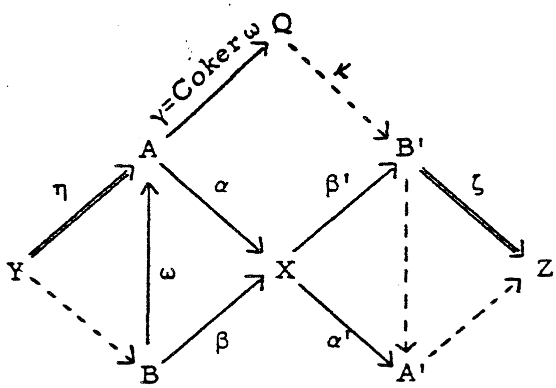
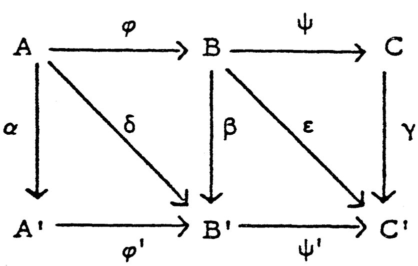


Proof: ω' is given by $\alpha'\beta = \alpha'\alpha\omega = 0$ and $\beta' = \text{Coker}\,\beta$. Put $\gamma = \text{Coker}\,\omega : A \to Q$; then by $(\beta'\alpha)\omega = \beta'\beta = 0$ one obtains κ such that $\beta'\alpha = \kappa\gamma$. Moreover $\omega'\kappa = 0$, since $\omega'\kappa\gamma = \alpha'\alpha = 0$ and γ is epic. Suppose $(\kappa\gamma)\eta = 0$ for some η; then $\beta'(\alpha\eta) = 0$, hence $\alpha\eta$ factorizes over $\beta = \text{Ker}\,\beta' = \alpha\omega$ and thus also η over ω, α being monic. Hence $\gamma\eta = 0$, which shows $\text{Ker}(\kappa\gamma) = \text{Ker}\,\gamma$. Since γ is epic, it follows from axioms (Q_1) and (Q_3) a) that κ is a monomorphism. Suppose $\zeta\kappa = 0$ for some ζ, then $\zeta\kappa\gamma = (\zeta\beta')\alpha = 0$, hence $\zeta\beta'$ factorizes over $\alpha' = \text{Coker}\,\alpha = \omega'\beta'$, and thus also ζ over ω', β' being epic. This proves $\omega' = \text{Coker}\,\kappa$, hence $\kappa = \text{Ker}\,\omega'$ by (Q_3) b).

THEOREM 3: Suppose both rows of the commutative diagram are exact. Then there exists an object Q which represents simultaneously $Q = \text{Ker}\,\varepsilon / (\text{Ker}\,\beta \cup \text{Ker}\,\psi)$ $= (\text{Im}\,\beta \cap \text{Im}\,\varphi')/\text{Im}\,\delta$.



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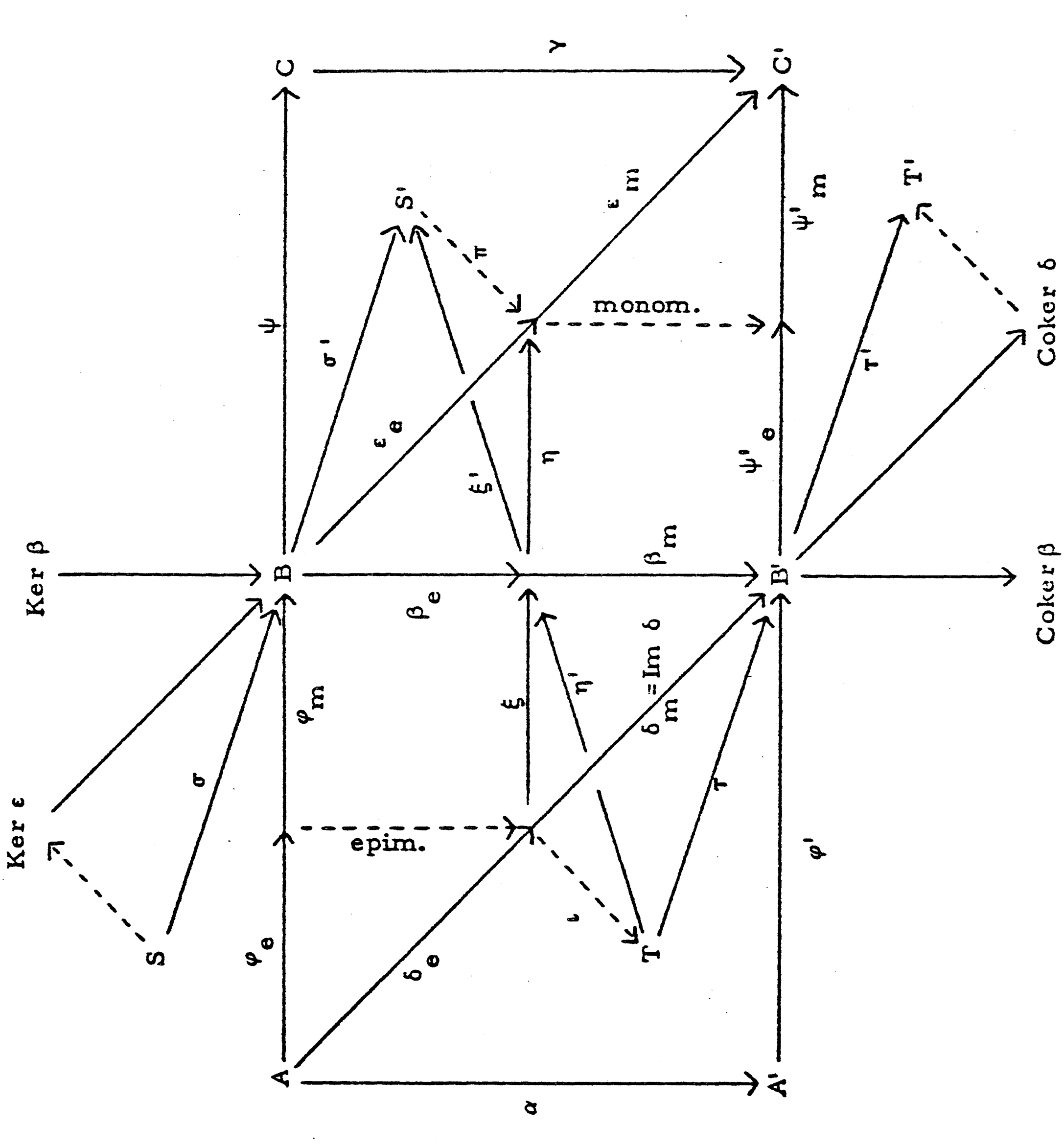

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Proof: The canonical factorization of a map $f : A \to B$ given by (Q_1) is essentially unique and will be described by $f = f_m f_e : A \twoheadrightarrow B$, so that $f_e = \text{Coim } f$, $f_m = \text{Im } f$. Factorize in this way φ, δ, ε, ψ', then $\varphi_m = \text{Ker } \psi$ and $\psi'_e = \text{Coker } \varphi'$ $= \text{Coker } (\varphi'_m)$. By uniqueness one obtains the monomorphism $\xi = (\beta_e \varphi_m)_m$ and the epimorphism $\eta = (\psi'_e \beta_m)_e$, so that $\xi' = \text{Coker } (\beta_e \varphi_m) = \text{Coker } \xi$ and $\eta' = \text{Ker } (\psi'_e \beta_m) = \text{Ker } \eta$. To construct the sum $\sigma = \text{Ker } \beta \cup \text{Ker } \psi$ form $\sigma' = \xi' \beta_e$, then $\sigma = \text{Ker } \sigma'$. Dually the intersection $\tau = \text{Im } \beta \cap \text{Im } \varphi' = \beta_m \eta'$ (see [1], pp. 2, 3, Props. 2.2 and 2.6). Since $\varepsilon_m \eta \xi \delta_e = \varepsilon \varphi$ $= \gamma \psi \varphi = 0$, one has $\eta \xi = 0$, hence there exists π and ι such that $\eta = \pi \xi'$ and $\xi = \eta' \iota$, thus also $\varepsilon_e = \eta \beta_e = \pi \xi' \beta_e = \pi \sigma'$ and $\delta_m = \beta_m \xi = \beta_m \eta' \iota = \tau \iota$. Since $\varepsilon_e \sigma = \pi \sigma' \sigma = 0$, σ factorizes over $\text{Ker } \varepsilon$ (similarly $\tau' = \text{Coker } \tau$ factorizes over $\text{Coker } \delta$). Repeated application of Prop. 2) gives an object $Q = \text{Ker } \varepsilon / \sigma = \sigma' \backslash \varepsilon_e = \xi' \backslash \eta = \eta' / \xi = \tau / \delta_m$ $(= \text{Coker } \delta \backslash \tau')$.

Remark: In the category of groups (or rings) axiom (Q_3) b) no longer holds, and thus a distinction must be made between "normal" monomorphisms $\mu = \text{Ker}(\text{Coker } \mu)$ and non-normal ones. Still valid however is axiom

(G) : If μ is a normal monomorphism and ε an epimorphism such that $\varepsilon \mu$ is defined, then also $(\varepsilon \mu)_m$ (the image of μ under ε) is normal.

The latter suffices to prove Prop. 2 for normal α and β, because then κ is also normal. Thm. 3 remains valid without restriction: Its premises imply the normality of ξ, so the same proof applies.

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