Can. J. Math., Vol. XXII, No. 4, 1970, pp. 878–880

ON BAER INVOLUTIONS OF FINITE PROJECTIVE PLANES

JUDITA COFMAN

1. An *involution* of a projective plane π is a collineation λ of π such that $\lambda^2 = 1$. Involutions play an important rôle in the theory of finite projective planes. According to Baer [**2**], an involution λ of a finite projective plane of order n is either a perspectivity, or it fixes a subplane of π of order $\sqrt{n}$; in the last case, λ is called a *Baer involution.*

While there are many facts known about collineation groups of finite projective planes containing perspectivities (see for instance [**4**; **5**]), the investigation of Baer involutions seems rather difficult. The few results obtained about planes admitting Baer involutions are restricted only to special cases.

Our aim in the present paper is to investigate finite projective planes admitting a large number of Baer involutions. It is known (see for instance [**3**, p. 401]) that in a finite Desarguesian projective plane of square order, the vertices of every quadrangle† are fixed by exactly one Baer involution. In this paper it will be shown that the converse is also true; we shall prove the following result.

THEOREM. *If in a finite projective plane π the vertices of every quadrangle are fixed by exactly one Baer involution, then π is Desarguesian and the collineation group generated by all Baer involutions of π contains the little projective group of π.*

2. Proof of the Theorem. Let π be a finite projective plane of order n such that the vertices of every quadrangle in π are fixed by exactly one Baer involution. Denote by Δ the collineation group generated by all Baer involutions of π. It follows immediately that:

(i) *Δ contains an even number of Baer involutions.*

For, let α be a Baer involution and denote its fixed subplane by π_0; then the vertices of all quadrangles in π_0 are fixed by α. Thus, if N denotes the number of quadrangles in π and N_0 the number of quadrangles in a square-root subplane of π, then the number of all Baer involutions contained in Δ is N/N_0. It is easy to see that $N = (n^2 + n + 1)(n^2 + n)n^2(n - 1)^2/4!$ and

$$N_0 = (n + \sqrt{n} + 1)(n + \sqrt{n})n(\sqrt{n} - 1)^2/4!;$$

hence $N/N_0 = (n\sqrt{n} + 1)(n + 1)n\sqrt{n}$, which is even.

Since the number of involutions in a finite group is odd, (i) implies:

Received October 6, 1969.

†A quadrangle is a set of four points no three of which are collinear.

https://doi.org/10.4153/CJM-1970-100-6 Published online by Cambridge University Press

(ii) Δ *contains involutions which are perspectivities.*

The next step is to show:

(iii) Δ *contains non-trivial elations.*

In view of (ii), the group Δ contains involutorial perspectivities. If n is even, according to Baer [**2**] every involutorial perspectivity of Δ is an elation. It remains to consider the case when n is odd. Let α be an involutorial perspectivity of Δ. By [**2**], the perspectivity α is a homology. Denote by a the axis and by A the centre of α, respectively. Take any two distinct points B and C on a and a point D not on any of the lines AB, AC, and BC. According to the assumption of our Theorem there exists a unique Baer involution σ fixing the points A, B, C, and D. Let Σ be the subplane of π fixed pointwise by σ. Take a point A_0 on the line AD but not in Σ. Then the unique Baer involution σ_0 fixing A_0, B, C, D cannot fix A (otherwise the vertices of the quadrangle A, B, C, D would be fixed by two Baer involutions). Thus $\sigma_0^{-1}\alpha\sigma_0$ is a homology with axis a and centre $A\sigma_0 \neq A$. The group generated by α and $\sigma_0^{-1}\alpha\sigma_0$ contains, according to André [**1**], an elation with axis a. This proves (iii).

(iv) *For every point A of π which is the centre of a non-trivial elation in Δ the group Δ contains elations with centre A and with distinct axes through A.*

Let α be a non-trivial elation of Δ with centre A and axis a. Take a point B on a, distinct from A and two points C and D not on a such that A, B, C, D form a quadrangle. Let σ be the Baer involution fixing A, B, C, D. Take a point D_0 on BD, not in the subplane Σ of π fixed pointwise by σ. Denote the involution fixing A, C, D, D_0 by σ_0. Obviously, σ_0 cannot fix a. Otherwise, σ_0 would fix the intersection of a with DD_0, that is the point B, contradicting our assumption. Thus $\sigma_0^{-1}\alpha\sigma_0$ is an elation with centre A and axis $a\sigma_0 \neq a$.

(v) *For every line a of π which is the axis of a non-trivial elation in Δ, the group Δ contains elations with axis a and with distinct centres on a.*

The proof of (v) is dual to the proof of (iv).

(vi) *No point of π is fixed under Δ.*

For, let X be any point of π. Consider a quadrangle containing X as a vertex. Denote its other vertices by Y, Z, U, respectively. Let σ be the Baer involution fixing X, Y, Z, U with the fixed square-root subplane Σ. Take any point X_0 not in Σ, forming a quadrangle with Y, Z, and U. Then the Baer involution Σ_0 fixing X_0, Y, Z, U cannot fix the point X.

(vii) *No line of π is fixed under Δ.*

This can be proved by arguments dual to those in the proof of (vi).

(viii) *For any incident point-line pair A, a denote by $\Delta(A, a)$ the subgroup of Δ consisting of all elations with centre A and axis a. If $\Delta(A, a)$ contains*

https://doi.org/10.4153/CJM-1970-100-6 Published online by Cambridge University Press

a non-trivial elation, then there are at least two non-trivial elations in $\Delta(A, a)$.

Let α be a non-trivial elation with centre A and axis a. Take a point B not on a and denote by B' its image under α. Let C and D be two distinct points of a, different from A. The group Δ contains a unique Baer involution fixing C, D, B, and B'. Choose a point B_0 on BB' which is not fixed under σ. Then the involution σ_0 fixing C, D, B, and B_0 must be different from σ. The collineation $\sigma_0^{-1}\alpha\sigma_0$ is obviously an elation α_0 with centre A and axis a, mapping B onto $B\sigma_0^{-1}\alpha\sigma_0 = B'\sigma_0$ which is different from B'; hence $\sigma_0^{-1}\alpha\sigma_0$ is different from α.

From (iv)–(viii) it follows, according to Piper [**4**], that:

(ix) *The centres and the axes of all elations in* Δ *are the points, respectively the lines, of a Desarguesian subplane* π_0 *of* π *and* Δ *contains the little projective group of* π_0.

It remains to show:

(x) π_0 *coincides with* π.

Suppose that π_0 is a proper subplane of π. Take three non-collinear points A, B, C in π_0. Through each of them there are at least two lines of π not in π_0. This enables us to choose three non-concurrent lines of π not in π_0, say a, b, c, such that $A \in a$, $B \in b$, and $C \in c$. Denote $a \cap b$ by C_1, $a \cap c$ by B_1, and $b \cap c$ by A_1. Take a point D on CC_1 distinct from C and C_1. The points A_1, B_1, C_1, and D are the vertices of a quadrangle Q; let σ be the involution fixing the vertices of Q with fixed square-root subplane Σ. Obviously the line CC_1 contains at least one point D_0 of π not in Σ. The points A_1, B_1, C_1, and D_0 are the vertices of a quadrangle Q_0 fixed by an involution σ_0, distinct from σ. Both involutions σ and σ_0 fix the lines a, b, c; since each of these lines contains only one point of π_0, namely A, B, and C, respectively, it follows that both involutions σ and σ_0 fix the vertices of the quadrangle A, B, C, C_1. This contradiction shows that π_0 cannot be a proper subplane of π.

Our Theorem is proved.

References

1. J. André, *Über Perspektivitäten in endlichen projektiven Ebenen*, Arch. Math. *6* (1954), 29–32.
2. R. Baer, *Projectivities of finite projective planes*, Amer. J. Math. *69* (1947), 653–684.
3. M. Hall, Jr., *The theory of groups* (Macmillan, New York, 1959).
4. F. C. Piper, *Collineation groups containing elations*. I, Math. Z. *89* (1965), 181–191.
5. A. Wagner, *On perspectivities of finite projective planes*, Math. Z. *71* (1959), 113–123.

Imperial College of Science and Technology,
London, England