BOUNDED GENERATORS IN LINEAR TOPOLOGICAL SPACES

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1. Ito and Seidman in [5] define a BG space as a locally convex space in which there exists a bounded set with a dense span. In this note we extend the idea to a class of not necessarily locally convex linear topological spaces (l.t.s.). We note the link between the idea of a BG space and Weston's characterization in [7] of separable Banach spaces. Finally we examine σ-BG spaces; here the bounded set in the definition of a BG space is replaced by the union of a sequence of bounded sets.

2. Let A be a subset of a linear space E. If $k \geqq 2$ and $A+A \subseteq kA$, then A is called a *k-convex set*. An l.t.s. which has a base of neighbourhoods of the origin consisting of balanced k-convex sets (for some fixed k) is called a *k-convex* l.t.s.

Every locally convex space is a k-convex l.t.s. for any $k \geqq 2$. Also, a locally bounded space (i.e. an l.t.s. which has a bounded neighbourhood) is a k-convex l.t.s. for some $k \geqq 2$. Thus a k-convex l.t.s. need not be a locally convex space. If for some fixed k, E_α is a k-convex l.t.s. for each α in an index set Ψ, then the product space $\mathbf{X}(E_\alpha : \alpha \in \Psi)$ is a k-convex l.t.s. However the product of a sequence of complete Hausdorff locally bounded spaces need not be a k-convex l.t.s. for any k (see for example, p. 179 of [6]).

In [6, p. 170] Simons defines the notion of a λ-pseudometric. It is clear from [6, Theorem 4] that if (E, u) is a k-convex l.t.s., then for some λ, there is a family (p_α) of u-continuous λ-pseudometrics which give the topology u.

Let E be a k-convex l.t.s. As in [5], let $\Phi(E)$ denote the set of all families $\varphi = (\varphi_\gamma)$ of continuous λ-pseudometrics (for some fixed λ) which give the topology of E, and if $\varphi = (\varphi_\gamma) \in \Phi(E)$, let

$$S_\varphi = (x : x \in E, \sup \varphi_\gamma(x) < \infty).$$

The proof of the equivalence of (A) and (B) in Theorem 1 of [5] goes through for a k-convex l.t.s. if we replace "seminorm" by "λ-pseudometric" throughout. We use the fact [6, Theorem 6] that if $(\varphi_\gamma) \in \Phi(E)$, then a subset A of E is bounded if and only if $\varphi_\gamma(A)$ is bounded for each γ.

If we call an l.t.s. which contains a bounded set with dense span, a BG *space*, we immediately have the following generalization of the corollary of Theorem 1 of [5].

· Theorem 1. *With the notation above, a Hausdorff k-convex l.t.s. E is a* BG *space if and only if there is* φ *in* $\Phi(E)$ *such that* S_φ *is dense in E.*

Lemma 1. *If A is a balanced k-convex bounded set in an l.t.s.* (E, u), *then the family* $(k^{-n}A : n = 1, 2, \ldots)$ *of sets is a base of neighbourhoods for a locally bounded k-convex topology* v_A *on the linear span* E_A *of A which is finer than the u-induced topology. The space* (E_A, v_A) *is Hausdorff and complete if* (E, u) *is Hausdorff sequentially complete and A is u-closed.*

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For the situation of a locally convex space (E, u), Lemma 1 is well known (see for example the proof of Chapter III, Section 3, No. 4, Lemma 1 of **[1]**).

It is easy to see that in a k-convex l.t.s., any bounded set is contained in a balanced k-convex bounded set.

THEOREM 2. *A (sequentially complete) Hausdorff k-convex l.t.s.* (E, u) *is a* BG *space if and only if there is a one-to-one continuous linear map t from a (complete) Hausdorff locally bounded k-convex l.t.s. F into* (E, u) *such that* $t(F)$ *is u-dense.*

Proof. If (E, u) is a BG space, let A be a balanced k-convex bounded set which has a dense span. With (E_A, v_A) as in Lemma 1, let t be the identity map from (E_A, v_A) into (E, u).

Given a subset A of an l.t.s. E, and $k \geqq 2$ a fixed real number, the intersection C of the non-empty set of all (closed), balanced k-convex subsets of E containing A is (closed) balanced and k-convex. The set C is called the *(closed) balanced k-convex envelope* of A.

LEMMA 2. *In a Hausdorff k-convex l.t.s. E, the balanced k-convex envelope C of a precompact set A is precompact.*

Proof. Let U be an open balanced k-convex neighbourhood in E. Since A is precompact, there is a finite subset B of E such that $A \subseteq B+U$, and therefore $C \subseteq B'+U$, where B is a compact set, being the closed absolutely convex envelope of the finite set B. As B' is compact and U is open, there is a finite subset D of B' such that $C \subseteq D+U$.

Let us call a linear map from one l.t.s. G into another H a *precompact* (*compact*) map if there is a neighbourhood which is mapped into a precompact (compact) set in H.

Weston in [7] proves that a Banach space (E, u) is separable if and only if there is a one-to-one compact map t say, from a Banach F into (E, u) such that $t(F)$ is u-dense. It is shown in **[2]** that this result is still valid if " Banach space " is replaced by " complete Hausdorff locally bounded space ".

THEOREM 3. *A (complete metrizable) metrizable k-convex l.t.s.* (E, u) *is separable if and only if there is a one-to-one (compact) precompact linear map t say, from a (complete) Hausdorff locally bounded k-convex l.t.s. F into* (E, u) *such that* $t(F)$ *is u-dense.*

Proof. Let (E, u) be a separable metrizable k-convex l.t.s. and let (U_n) be a shrinking base of u-neighbourhoods. If $(x_n : n = 1, 2, \ldots)$ is a countable u-dense subset of E, then for each n, there is a non-zero real number α_n such that $\alpha_n x_n \in U_n$. As (U_n) is shrinking, the sequence $(\alpha_n x_n)$ thus converges to zero in (E, u). By Lemma 2, the balanced k-convex envelope A of $(\alpha_n x_n : n = 1, 2, \ldots)$ is precompact; its closure is compact if (E, u) is complete. We now apply Lemma 1. With $F = (E_A, v_A)$, the identity map t from F into (E, u) is precompact, being compact if (E, u) is complete.

COROLLARY. *A separable infinite dimensional Fréchet space contains a dense subspace on which there is a finer Fréchet space topology.*

THEOREM 4. *A complete metrizable k-convex l.t.s.* (E, u) *is finite dimensional if and only if* $t(F)$ *is closed in* (E, u) *whenever t is a continuous linear map from a complete metrizable k-convex l.t.s. F into* (E, u).

Proof. Suppose first that (E, u) is separable. Then by Theorem 3, there is a one-to-one compact (and therefore continuous) linear map t say, from a complete metrizable k-convex l.t.s. F into (E, u) such that $t(F)$ is u-dense. If t has a closed range, $t(F) = E$ and t is a topological isomorphism by Banach's inversion theorem. Therefore (E, u) has a compact neighbourhood and is thus finite dimensional.

If (E, u) is not necessarily separable, let E_0 be a subspace of E of countable dimension. Let E_1 be the closure of E_0 in E and let u_1 be the u-induced topology on E_1. Then (E_1, u_1) is a separable complete metrizable k-convex l.t.s. If $t(F)$ is closed in (E, u) whenever t is a continuous linear map from a complete metrizable k-convex l.t.s. F into (E, u), then by the argument above, the dimension of E_1 is necessarily finite. The dimension of E must then be finite, otherwise, we could choose E_0 as above to have countably infinite dimension.

COROLLARY. *If E is a Fréchet space and every continuous linear map from any Fréchet space into E has a closed range then E is finite dimensional.*

Ito and Seidman in [5, p. 287] call a Hausdorff locally convex space E a HBG *space* if every closed linear subspace of E is a BG space. Let (E, u) be a normed linear space of infinite dimension. If v is the weak topology associated with u, then it follows from [5, Theorem 2(D)] that (E, v) is a BG space. In fact (E, v) is a HBG space. As (E, v) is not quasibarrelled, (E, v) is not the quotient of a product of normed linear spaces.

Cf. [5, p. 287, questions 2 and 3].

3. Let E be an l.t.s. We call E a σ-BG *space* if there is a sequence of bounded sets, the union of which spans a dense subspace of E.

Every BG space is a σ-BG space. Also, every separable l.t.s. is a σ-BG space.

If E is a linear space of countably infinite dimension, then under its finest locally convex topology $\tau(E, E^*)$, E is separable (and complete) and therefore the space $(E, \tau(E, E^*))$ is a σ-BG space. As each $\tau(E, E^*)$-bounded set is contained in some finite dimensional linear subspace of E, $(E, \tau(E, E^*))$ is not a BG space.

It follows from [1, Ch. III, section 2, exercise 5] that a metrizable k-convex l.t.s. is a BG space if and only if it is a σ-BG space. The example of Ito and Seidman [5, p. 286] then shows that a Fréchet space need not be a σ-BG space. However as in Theorem 2 of [5], a product of BG (σ-BG) spaces is a BG (σ-BG) space, and the image under a continuous linear map of a BG (σBG) space is of the same sort.

For a fixed $k \geqq 2$ and each positive integer n, let (E_n, u_n) be a k-convex l.t.s. such that $E_n \subset E_{n+1}$. If $E = \bigcup_n(E_n)$, then there is a finest linear topology u say, on E such that each identity map $(E_n, u_n) \to E$ is continuous [4, Definition 2.1]. By an application of Proposition 2.2 of [4], we see that (E, u) is a k-convex l.t.s., and that if each (E_n, u_n) is locally convex, so is (E, u). The space (E, u) is called the *generalized strict k-convex inductive limit* of (E_n, u_n). If in addition, each u_n coincides with the topology induced on E_n by u_{n+1}, then (E, u) is called the *strict k-convex inductive limit* of (E_n, u_n).

If (E, u) is the strict k-convex inductive limit of (E_n, u_n), then the topology u coincides with u_n on each E_n, (E, u) is Hausdorff if each (E_n, u_n) is [4, Proposition 2.7, Cor. 1], and in this case if each (E_n, u_n) is complete, (E, u) is also complete [4, Proposition 2.8, Cor.], but is not metrizable [4, Proposition 2.9, Cor.]. We shall prove:

THEOREM 5. *The strict k-convex inductive limit of a sequence of complete Hausdorff k-convex σ-BG spaces is also a σ-BG space.*

This theorem will follow immediately from the following result.

LEMMA 3. *If (E, u) is the strict k-convex inductive limit of (E_n, u_n) where each (E_n, u_n) is complete, k-convex and Hausdorff, then a subset of E is u-bounded if and only if it is contained in some E_n and is u_n-bounded.*

Proof. Suppose that A is a u-closed balanced k-convex u-bounded set which is not contained in any E_n. Then there is a subsequence $(n(i))$ of (n) such that for each i, some point of $A \cap E_{n(i+1)}$ is not in $E_{n(i)}$ and (E, u) is the strict k-convex inductive limit of $(E_{n(i)}, u_{n(i)})$. Observe that (E, u) is complete and Hausdorff and that each $E_{n(i)}$ is u-closed.

As in Lemma 1, let E_A be the linear span of A and v_A the linear topology on E_A with the family $(k^{-m}A: m = 1, 2, \ldots)$ of sets as a base of neighbourhoods. Similarly, let $F_{n(i)}$ be the linear span of $A \cap E_{n(i)}$ and $v_{n(i)}$ the linear topology on $F_{n(i)}$ with the family $(k^{-m}(A \cap E_{n(i)}): m = 1, 2, \ldots)$ of sets as a base of neighbourhoods. The spaces (E_A, v_A), $(F_{n(i)}, v_{n(i)})$ are complete Hausdorff locally bounded k-convex spaces, $F_{n(1)} \subset F_{n(2)} \subset F_{n(3)} \subset \ldots$, $E_A = \bigcup F_{n(i)}$, and $v_{n(i)}$ coincides with the $v_{n(i+1)}$-induced topology on $F_{n(i)}$. If (E_A, w) is the strict k-convex inductive limit of $(F_{n(i)}, v_{n(i)})$, w is finer than the u-induced topology on E_A, and it follows that the identity map from (E_A, w) onto (E_A, v_A) has a closed graph. By Theorem 4.2 of [3], we see that $v_A = w$, implying that (E_A, w) is metrizable. As this is not possible, the set A must be contained in some E_n, and is u_n-bounded because A is u-bounded and u induces the topology u_n on each E_n.

Thus any strict inductive limit of a sequence of Banach or separable Fréchet spaces is a σ-BG space. Also, if E is the sequence space $l^p (0 < p < 1)$ and F is the algebraic direct sum of countably many copies of E, then under the finest linear topology for which the injection maps $E \to F$ are continuous, F is a σ-BG space.

There is a parallel to Theorem 2.

THEOREM 6. *If a (sequentially complete) Hausdorff k-convex l.t.s. (E, u) is a σ-BG space but not a BG space, then there is a one-to-one continuous linear map, t say, from F into (E, u) such that $t(F)$ is u-dense, where F is the generalized strict k-convex inductive limit of a sequence of (complete) Hausdorff locally bounded spaces.*

Proof. Let a Hausdorff k-convex l.t.s. (E, u) be a σ-BG space but not a BG space. Let (A_n) be a sequence of u-closed balanced k-convex u-bounded sets, the union of which spans a dense linear subspace F of (E, u). We may assume that $A_1 \subset A_2 \subset A_3 \subset \ldots$; and since no A_n spans F, we may further assume that for each n, $A_{n+1} \not\subseteq E_{A_n}$. If v_{A_n} is the topology on E_{A_n} defined as in Lemma 1, let (F, v) be the generalized strict k-convex inductive limit of (E_{A_n}, v_{A_n}) and let the map $t: F \to F$ be the identity map.

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