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TENSOR PRODUCTS AND SINGULARLY CONTINUOUS SPECTRUM

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ABSTRACT. An example of a bounded self adjoint operator A is constructed so that $A\otimes I+\alpha(I\otimes A)$ is purely singularly continuous but $A\otimes 1+\beta(I\otimes A)$ is purely absolutely continuous, for some real α and β. In fact $\alpha-\beta$ can be chosen arbitrarily small.

1. **Introduction.** Consider two self adjoint operators A_1 and A_2 on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Then $A_1\otimes I$ and $I\otimes A_2$ are commuting self adjoint operators on the tensor product space $\mathcal{H}_1\otimes\mathcal{H}_2$, ($I$ is the identity operator). Therefore it is possible, given a reasonable Borel measurable function $F:\mathbb{R}^2\to\mathbb{R}$ to define the self adjoint operator $F(A_1\otimes I, I\otimes A_2)\equiv F(A_1, A_2)$ on $\mathcal{H}_1\otimes\mathcal{H}_2$ using spectral calculus. What can be said about the structure of the spectrum of $F(A_1, A_2)$ in terms of those of A_1 and A_2? In particular, what are the discontinuous, singularly continuous and absolutely continuous spectra of $F(A_1, A_2)$? Provided A_1 and A_2 do not both have singularly continuous spectrum then this question can easily be answered using spectral calculus. (The requisite results may be found in [1] see also [2]: for more generality see [3, 4].) However, if both A_1 and A_2 have singularly continuous spectrum then there is no simple answer as we shall show in this note by constructing an example.

In describing this example, we adopt the notation of [8, 9]. Recall that, if A is a self adjoint operator on a Hilbert space $\mathcal{H}$ with spectral measure E, then its *subspaces of absolute continuity* (denoted $\mathcal{H}_{ac}(A)$), of *discontinuity* $\mathcal{H}_{pp}(A)$) and of *singular continuity* ($\mathcal{H}_{sing}(A)$) are defined by

$$\mathcal{H}_{ac}(A)=\bigcap\{E(\Delta)\mathcal{H}\colon \Delta \text{ is a Borel set of Lebesgue measure } 0\}$$

$$H_{pp}(A)=\bigoplus_{\lambda\in\mathbb{R}} E(\{\lambda\})\mathcal{H};$$

$$\mathcal{H}_{sing}(A)=(\mathcal{H}_{ac}(A)\oplus\mathcal{H}_{pp}(A))^{\perp}$$

(See [9, p. 230] for more details.) We shall construct the following example.

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EXAMPLE. There exists a bounded self adjoint operator A_1, on a separable Hilbert space $\mathscr{H}_1$, for which $\mathscr{H}_{1,\text{sing}}(A_1)=\mathscr{H}_1$, such that: if, $\alpha>0$, $A^{(\alpha)}$ is the self adjoint operator on $\mathscr{H}=\mathscr{H}_1\otimes\mathscr{H}_1$,

$$A^{(\alpha)}=A_1\otimes I+\alpha I\otimes A_1$$

then

(i) $\mathscr{H}_{\text{sing}}(A^{(\alpha_j)})=\mathscr{H}$, for a sequence $\{\alpha_j\}$ convergent to 2

(ii) $\mathscr{H}_{\text{sing}}(A)^{(2)})=\{0\}=\mathscr{H}_{ac}(A^{(2)})^{\perp}$.

REMARK 1. The operator $A^{(\alpha)}$ is $A^{(\alpha)}=F(A_1,\alpha A_1)$ where $F(\lambda_1,\lambda_2)=\lambda_1+\lambda_2$. There are other choices of F that would work just as well in the above example but this choice corresponds to the operators of primary interest in multi-particle scattering theory, where it is desirable to show that $\mathscr{H}_{\text{sing}}(A_1\otimes I+I\otimes A_2)=\{0\}$, (See [5, 6]).

2. We explicitly determine the spectrum $\sigma(A_1)$ of A_1 and hence also of $A^{(\alpha)}:\sigma(A^{(\alpha)})=\sigma(A_1)+\alpha\sigma(A_1)$.

3. The anomaly illustrated by the example is peculiar to spectral calculus of several variables for suppose F were a function of only one variable then $\mathscr{H}_{1,\text{sing}}(F(A_1))=\mathscr{H}_{1,\text{sing}}(A_1)$ if, say $F\in C^1(\mathbb{R})$ and $\{x:F'(x)=0\}$ is a countable set.

We now construct the example. Our method is an adaption to the present situation of a construction of Jessen and Wintner's [5].

2. **Construction of the Example.** Define, for each $k\in\mathbb{N}$, ν_k to be the probability measure on the Borel subsets Σ of $\mathbb{R}$, such that

$$\nu_k(\{-4^{-k}\})=\tfrac{1}{2}=\nu_k(\{4^{-k}\})$$

For each $n\in\mathbb{N}$, let ϕ_n be the distribution function of the probability measure $\nu_1*\nu_2*\cdots*\nu_n$ ($*$ denotes convolution). Then $\phi_n(-\frac{1}{3})=0$, $\phi_n(\frac{1}{3})=1$ and ϕ_n has 2^n jumps of height 2^{-n} at the points

$$(2.1)\quad (-1)^{j_1}4^{-1}+(-1)^{j_2}4^{-2}+\cdots+(-1)^{j_n}4^{-n},\quad \text{where}\quad j_k\in\{0,1\},\qquad 1\le k\le n.$$

The sequence $\{\phi_n\}_{n\in\mathbb{N}}$ converges uniformly to some function ϕ. It follows that $\{\nu_1*\cdots*\nu_n\}_n$ converges weakly (i.e. in the weak $*$ topology on $C([-\frac{1}{3},\frac{1}{3}])$) to a probability measure μ_1 with distribution function ϕ. It is not difficult to check that ϕ is continuous. Therefore, the self adjoint operator A_1 defined on $\mathscr{H}_1=L^2([-\frac{1}{3},\frac{1}{3}],\Sigma,\mu_1)$ by

$$A_1f(\lambda)=\lambda f(\lambda)\quad\text{for all}\quad f\in\mathscr{H}_1$$

is purely continuous i.e. $\mathscr{H}_{1,pp}(A_1)=\{0\}$. Compute now the spectrum $\sigma(A_1)$ of A_1 which is the support of μ_1. If, for some $n\in\mathbb{N}$, x_k is one of the points in (2.1) for which $j_n=1$ then

$$x_k+\left(\sum_{l=n+1}^{\infty}4^{-l},2\cdot4^{-n}-\sum_{l=n+1}^{\infty}4^{-l}\right)$$

is an interval of constancy for ϕ_m, $m \geq n$ and hence for ϕ. The union of all these intervals over all 2^{n-1} values of k and all $n \geq 1$ has full Lebesgue measure in $[-\frac{1}{3}, \frac{1}{3}]$. its complement is $\sigma(A_1)$. Since $\sigma(A_1)$ has zero Lebesgue measure, $\mathcal{H}_{1,\text{sing}}(A_1) = \mathcal{H}_1$.

For each $\alpha > 0$, define $A^{(\alpha)} = A_1 \otimes I + I \otimes \alpha A_1$ on

$$\mathcal{H} \equiv \mathcal{H}_1 \otimes \mathcal{H}_1 \cong L^2([-\tfrac{1}{3}, \tfrac{1}{3}] \times [-\tfrac{1}{3}, \tfrac{1}{3}], \Sigma \otimes \Sigma, \mu_1 \otimes \mu_1).$$

If $E^{(\alpha)}$ denote the spectral measure of $A^{(\alpha)}$ then $E^{(\alpha)}(\Delta)$ is multiplication by $\chi_\Delta(\lambda_1 + \alpha\lambda_2)$ where χ_Δ is the indicator function for $\Delta \in \Sigma$. Therefore $\mathcal{H}_{pp}(A^{(\alpha)}) = \{0\}$. Now let μ_α be the probability measure, $\mu_\alpha(\Delta) = \mu_1(\alpha^{-1}\Delta)$ for $\Delta \in \Sigma$. We shall see that $\mu_1 * \mu_\alpha$ is either singular or absolutely continuous with respect to Lebesgue measure, depending on the value of α. (Recall that a probability measure μ is a *singular* with respect to Lebesgue measure if $\mu(\Delta) = 1$ for some $\Delta \in \Sigma$ of Lebesgue measure 0.) If $\mu_1 * \mu_\alpha$ singular then one easily sees that $\mathcal{H}_{\text{sing}}(A^{(\alpha)}) = \mathcal{H}$ whereas, if $\mu_1 * \mu_\alpha$ is absolutely continuous then $\mathcal{H}_{ac}(A^{(\alpha)}) = \mathcal{H}$.

Thus we need only determine for what values of $\alpha > 0$, $\mu_1 * \mu_\alpha$ is singular or absolutely continuous with respect to Lebesgue measure. To do so, we require the following result. Recall that a probability measure μ on Σ is said to be *purely discontinuous* if $\sum_{\lambda \in \mathbb{R}} \mu(\{\lambda\}) = 1$.

THEOREM (Jessen and Wintner). *If $\{\nu_k'\}_{k \in \mathbb{N}}$ is a sequence of purely discontinuous probability measure on $(\mathbb{R}, \Sigma)$ such that $\{\nu_1' * \cdots * \nu_n'\}_{n \in \mathbb{N}}$ converges weakly to μ then μ is either purely discontinuous or is singular or absolutely continuous with respect to Lebesgue measure.*

For a proof of this result see [5]; it is a consequence of Kolmogorov's zero one law.

To determine which of the three alternatives of the theorem applies to $\mu_1 * \mu_{\alpha'}$ compute its characteristic function $(\mu_1 * \mu_\alpha)\hat{}$. Because,

$$\hat{\mu}_1(p) \equiv \int_{\mathbb{R}} e^{-ipx}\, d\mu_1(x) = \lim_{n \in \mathbb{N}} \int_{\mathbb{R}} r^{-ipx}\, d\nu_1 * \cdots * \nu_n(x) = \prod_{n=1}^{\infty} \cos(4^{-n}p)$$

we have

$$(\mu_1 * \mu_\alpha)\hat{}(p) = \prod_{n=1}^{\infty} \cos(4^{-n}p)\cos(4^{-n}\alpha p).$$

Now suppose that α belongs to the set

$$\alpha \in \{(2k+1)4^{-n} : k \geq 0, n \geq 0\}.$$

Then $\mu_1 * \mu_\alpha(p)$ does not converge to 0 as $|p| \to \infty$, for consider $\{p_m\}_{m \in \mathbb{N}} = \{4^m\}_{m \in \mathbb{N}}$, and so $\mu_1 * \mu_\alpha$ is not absolutely continuous by the Riemann Lebesgue lemma. Therefore, by the theorem $\mu_1 * \mu_\alpha$ is singular. (It is clearly not purely discontinuous). This proves (i) of the example. To prove (ii), set $\alpha = 2$.

Then

$$(\mu_1 * \mu_2)\hat{}(p) = \prod_{n=1}^{\infty} \cos 2^{-n}p = \frac{\sin p}{p}$$

so that $\mu_1 * \mu_2$ is $\frac{1}{2}$ times Lebesgue measure on $[-1, 1]$, by the uniqueness of the characteristic function.

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