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FRACTIONAL DERIVATIVES AS INVERSES

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1. Introduction. We write formally

$$\underset{(C,p)}{I^k} g(x) = (\Gamma(k))^{-1} \int_{\substack{x\\(C,p)}}^{\infty} (t - x)^{k-1} g(t)dt \quad (k > 0, p > -1),$$

(C, p) indicating that the integral is summable (C, p), i.e., is

$$\lim_{X \to \infty} (\Gamma(k)\Gamma(p + 1))^{-1} \int_x^X (1 - t/X)^p (t - x)^{k-1} g(t)dt$$

if this limit exists. We note here that all integrals over a finite range are taken in the Lebesgue sense, and all inversions of such iterated integrals are justifiable by Fubini's Theorem.

In [**3**, Theorem 1] Bosanquet showed that if

$$f(u) = \underset{(C,0)}{I^k} g(u) \quad \text{a.e. for } u > \text{some } x,$$

and if g is continuous at x, then $g(x)$ is given by $g(x) = D^k f(x)$, the kth fractional derivative of f, where

$$(1) \quad D^k f(x) = (-d/dx)^{[k]} \lim_{w \to \infty} (\Gamma(1 - k'))^{-1} \times (-\partial/\partial x) \int_x^w (t - x)^{-k'} f(t)dt.$$

Here and below $k = [k] + k'$, where $[k]$ is the integral part of k.

In attempting to generalize this to (C, p) summability we replaced $(C, 0)$ in the hypothesis by (C, p), for some $p > 0$. Perhaps surprisingly, the conclusion continued to hold if p was $<k$:

THEOREM A. *If* $f(u) = \underset{(C,p)}{I^k} g(u)$ (k *fractional* >0) a.e. *for* $u >$ *some* x, *where* $0 < p < k$, *and if* g *is continuous on the right at* x, *then* $g(x) = D^k f(x)$. *The result is false if* $p = k$, *in the sense that* $D^k f(x)$ *need not exist in this case.*

From this we shall obtain Theorem B (below) which gives the integration of a fractional derivative, and Theorem C which provides a long-outstanding converse to a well-known result (Lemma 1 below).

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Bosanquet's proof depended on the fact that if $I^k g(u)$ is summable $(C, 0)$ then

$$\int_{\substack{w \\ (C,0)}}^{\infty} (t - u)^{k-1} g(t) dt = o(1) \quad \text{as } w \to \infty.$$

This is unfortunately not true if $(C, 0)$ is replaced by (C, p) for $p > 0$, as is illustrated by taking, say, $p = 1$, $k = 2$, $g(t) = t^{-1}e^{it}$. We find (Lemma 4 below) that a true version of this arises if we insert the factor $(1 - w/t)^r$ in the second integral, as long as $r \geqq p$ ($r > p$ if p fractional), a result of some interest in summability theory.

2. Lemmas. We have (see [**5**, Theorem 1] and for b integral [**2**, Theorem 3]):

LEMMA 1. *If $a, b > 0$ and if $I^{a+b}g(x)$ is summable (C, p) for some $p > b - 1$ then*

$$\underset{(C,p)}{I^{a+b}} g(x) = \underset{(C,q)}{I^{a}} \ \underset{(C,p)}{I^{b}} g(x)$$

where $q = p - b$ for b integral, and $q > p - b$ (not $= p - b$) for b fractional.

In the following lemma, if the $G^{[p]+2}$ condition is replaced by the lesser requirement

$$u^{[p]+1} G^{[p]+2}(u) \in L[w, \infty),$$

we obtain a well-known 'summability-factor' result; see [**4**, Theorem 1, p. 56] for p fractional >0:

LEMMA 2. *Suppose $\int_w^\infty h(u)du$ is summable (C, p) $(p > -1)$. Then so is $\int_w^\infty h(u)G(u)du$ if, for p fractional, $G^{([p]+1)}(u)$ is absolutely continuous on $w \leqq u \leqq W$ for all $W > w$ and*

$$G^{(r)}(u) = O(u^{-r}), \quad r = 0, 1, \dots, [p] + 1,$$

$$G^{([p]+2)}(u) = O(u^{-[p]-2-\epsilon}) \quad (\epsilon > 0)$$

as $u \to \infty$. (For p integral the conditions are as above, with $[p] + 1$ replaced by p.) If $G(u) = G(u, y)$ and the above orders hold uniformly for $0 \leqq y \leqq x$, for some $x < w$, then the summability of the second integral is uniform in $0 \leqq y \leqq x$.

LEMMA 3. *Let $k > 0$, $p > -1$, and let $I^k g(u)$ be summable (C, p)* a.e. *for $u >$ some x. Then writing*

$$H(x, w) = \int_{\substack{w \\ (C,p)}}^{\infty} g(t) dt \int_x^w (t - u)^{k-1} (u - x)^{-k'} du \quad (w > x) \tag{2}$$

we have, for $C = \Gamma(k)/\Gamma(k')$,

$$(3)\qquad (\partial/\partial x)^{[k]+1}H(x, w) = C(w-x)^{-k}\int_{w\,(C,p)}^{\infty}(t-w)^k(t-x)^{-1}g(t)dt.$$

Proof. We observe that the case k integral is immediate, the left and right sides of (3) being 0. For the case k fractional, now, it is clearly sufficient to show that H equals

$$(4)\qquad P(x) - \frac{C}{[k]!}\int_0^x (x-y)^{[k]}dy \times \int_{w\,(C,p)}^{\infty}(w-y)^{-k}(t-w)^k(t-y)^{-1}g(t)dt$$

where $P(x)$ is a polynomial in x of degree $[k]$. Now in (2) the inner integral, with $[x, w] = [x, t] - [w, t]$, becomes

$$c(t-x)^{[k]} - \int_w^t (t-u)^{k-1}u^{-k'}du - k'\int_w^t (t-u)^{k-1}du\int_0^x (u-y)^{-k'-1}dy$$

for some constant c. If the range of the second term is now split into $[0, t] - [0, w]$ we see that the first two terms form a polynomial of degree $[k]$ in x with $t^{[k]-1}$ and $(t-u)^{k-1}$ the highest powers involving t in the polynomial. The last term, after $[k]$ integrations by parts, becomes a polynomial in x of degree $[k]$ plus

$$(kC/[k]!)\int_w^t (t-u)^{k-1}du\int_0^x (x-y)^{[k]}(u-y)^{-k-1}dy.$$

Inverting this pair and then applying

$$\int_{w\,(C,p)}^{\infty} g(t)dt$$

to all the terms obtained gives, by Lemma 2, the polynomial $P(x)$ in (4) plus

$$(kC/[k]!)\int_{w\,(C,p)}^{\infty} g(t)dt\int_0^x (x-y)^{[k]}dy \times \int_w^t (t-u)^{k-1}(u-y)^{-k-1}du.$$

We now apply the rather curious identity

$$\int_w^t (t-u)^{k-1}(u-y)^{-k-1}du$$

$$= k^{-1}(t - w)^k(t - y)^{-1}(w - y)^{-k},$$

obtained by writing $(u - y)^{-k-1}$ as

$$(t - y)^{-k-1}\left(1 - \frac{t - u}{t - y}\right)^{-k-1}$$

and then expanding in a binomial series. Then the term above becomes

$$(C/[k]!) \int_{\substack{w \\ (C,p)}}^{\infty} g(t)dt \int_0^x (x - y)^{[k]}(t - w)^k(t - y)^{-1}(w - y)^{-k}dt$$

and inverting this gives the pair of integrals in (4); the inversion is justified because the inner integral of (4) is, by Lemma 2, uniformly summable on $0 \leqq y \leqq x$.

LEMMA 4. *Suppose $\int_0^\infty h(u)du$ is summable (C, p) for some fractional $p > 0$. Then*

$$\int_{\substack{w \\ (C,p)}}^{\infty} (1 - w/u)^k h(u)du = o(1) \quad \text{as } w \to \infty \text{ for any } k > p.$$

The result is false for $k = p$, and in fact if $k = p$ we can make the integral $> cw^{[p]+1}$ (say) as $w \to \infty$ through some sequence T_n. For p an integer the result holds with $k = p$.

Proof for p fractional. We may assume the given integral has value 0, i.e.,

$$h_{p+1}(u) = o(u^p) \quad \text{as } u \to \infty.$$

Also, in the integral of the conclusion the (C, p) existence follows from Lemma 2, and we shall replace it by $(C, [p] + 1)$ for convenience in the proof, this being permitted since '(C, p)' implies '(C, q)' $(q > p)$ to the same value. Now we write for $k > p$,

$$Q(X, w) = \int_w^X \{(1 - u/X)^{[p]+1}(1 - w/u)^k\}h(u)du$$
$$= (-1)^{[k]+1} \int_w^X h_{[p]+1}(u)(\partial/\partial u)^{[p]+1}\{\ldots\}du.$$

But

$$(5) \qquad (\partial/\partial u)^r(1 - w/u)^k = \sum_{j=0}^{r} c_j(u - w)^{k-j}u^{-k-r}w^j \quad (c_0 = 0 \text{ for } r \geqq 1)$$

as is easily verified by induction on r. Thus

$$Q(X, w) = \sum_{r=0}^{[p]+1} \sum_{j=0}^{r} c_{rj} \int_w^X L(u, w, X)h_{[p]+1}(u)du$$

where

$$L(u, w, X) = (1 - u/X)^r (u - w)^{k-j} u^{-k-r} X^{r-[p]-1} w^j$$

and $j \geqq 1$ for $r \geqq 1$. Now, suppressing multiplicative constants (as we shall do in similar contexts below) we have

$$\int_w^X = \int_w^X Ldu \int_0^u (u - t)^{-p'} h_p(t)dt = \int_w^X Ldu\left(\int_0^w + \int_w^u\right)$$
$$= A + B, \text{ say.}$$

But A equals

$$\int_w^X Ldu\left(h_{p+1}(w)(u - w)^{-p'} - \int_0^w (u - t)^{-p'-1} h_{p+1}(t)dt\right)$$
$$= A_1 - A_2, \text{ say.}$$

Since $h_{p+1}(w) = o(w^p)$ we see on separating the cases $1 \leqq r \leqq [p]$, $r = [p] + 1$, $r = 0$, that

$$\overline{\lim_{X\to\infty}}\, A_1 = o(1) \quad \text{as } w \to \infty;$$

in doing this we use

$$(6) \qquad \int_w^X (u - w)^{s-1} u^q (1 - u/X)^r du = O(1)(X^{s+q} \text{ or } \log X \text{ or } w^{s+q})$$

for large X, where $s > 0$, $r > -1$, and $s + q$ is >0, $=0$, or <0 respectively. Again, A_2 is essentially A_1. For B we get

$$B = -h_{p+1}(w) \int_w^X (u - w)^{-p'} Ldu$$
$$- \int_w^X h_{p+1}(t)dt(\partial/\partial t) \int_t^X (u - t)^{-p'} Ldu$$
$$= B_1 + B_2, \text{ say.}$$

Now $B_1 = -A_1$ and for B_2 we integrate the inner integral by parts and then differentiate. For $r \geqq 1$ this gives

$$B_2 = X^{r-[p]-1} w^j \int_w^X h_{p+1}(t)dt$$
$$\times \int_t^X (u - t)^{-p'} (\partial/\partial u) G(u, w, X)du,$$

where

$$G(u, w, X) = (u - w)^{k-j} u^{-k-r} (1 - u/X)^r.$$

Each of the three terms of the derivative is $\leqq (u - w)^{k-j-1} u^{-k-r}$, and if this expression is inserted in B_2 and the integrals inverted, (6) shows that

$$\overline{\lim_{X\to\infty}}\ B_2 = o(1) \quad \text{as } w \to \infty.$$

The case $r = 0$ is similarly treated.

For the negative part of Lemma 4 we let $k = p$ and choose a sequence T_n such that $1 < T_n < T_{n+1}/2$. Now let $h(t)$ be such that $h_{p+1}(t)$ is zero everywhere except for t between T_n and $T_n + 1$; here it is the constant $T_n^p/\log T_n$ except in two small intervals at the endpoints where it is smoothed down monotonically to 0 so as to remain $([p] + 2)$-fold differentiable. The two intervals have width $z_n = T_n^{-Q}$ where

$$Q = T_n^{[p]+1}.$$

We go through the proof above with $w = T_n$, $k = p$, and $h(t)$ as shown. All the terms work as before except for the contribution to B_2 arising from the one term in $(\partial/\partial u)G$ which is

$$(u - w)^{k-j-1}u^{-k-r}(1 - u/X)^r$$

$$\text{with } r = j = [p] + 1 \text{ (and } k = p).$$

This gives (with $w = T_n = T$, $z_n = z$ for short),

$$\begin{aligned}
&\underline{\lim_{X\to\infty}}\ B_2^* \text{ (say)} \\
&> c_1T^{-p} \int_{T+z}^{T+1-z} (u - T)^{p'-2}du \int_{T+z}^{u} (u - t)^{-p'}h_{p+1}(t)dt \\
&= c_2(\log T)^{-1} \int_{T+z}^{T+1-z} (u - t)^{p'-2}(u - T - z)^{1-p'}du \\
&> c_3(\log T)^{-1} \log(z^{-1}) > cT^{[p]+1} = cT_n^{[p]+1}
\end{aligned}$$

for all large n, by an integration by parts of the last integral followed by the substitution $u - T = y$, then expansion of $(1 - z/y)^{-p'}$ in a power series. This completes the proof.

3. Proof of theorem A. It is sufficient to prove the positive part for the case $0 < k < 1$ since if $k > 1$ we have

$$f(u) = \underset{(C,p)}{I^k}\ g(u) = \underset{(C,p-[k])}{I^{k'}}\ \underset{(C,p)}{I^{[k]}}\ g(u) \quad \text{a.e. for } u > x$$

by Lemma 1; hence by the case $0 < k < 1$ (assumed proved),

$$\underset{(C,p)}{I^{[k]}}\ g(x) = D^{k'}f(x) \quad \text{if } p - [k] < k', \text{ i.e., } p < k,$$

and hence $g(x)$ is given by

$$(-d/dx)^{[k]}D^{k'}f(x)$$

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by routine arguments, since g is continuous on the right at x. Since the last expression is just $D^k f(x)$ the result is proved.

The argument below proves the case $0 < k < 1$ and also gives a slightly weaker version of the (proved) case $k > 1$, with $p < k$, namely that $g(x)$ equals

$$(7) \quad \lim_{w\to\infty} (-\partial/\partial x)^{[k]+1}(\Gamma(1-k'))^{-1} \int_x^w (u-x)^{-k'} f(u)du;$$

we have supplied the latter version since we shall use it in constructing the counterexample for the case $p = k$.

We may take $p < k$, $[p] = [k]$. We write

$$\begin{aligned} J &= \int_x^w (u-x)^{-k'} f(u)du \\ &= \int_x^w (u-x)^{-k'} du \int_{u\,(C,p)}^{\infty} (t-u)^{k-1} g(t)dt \end{aligned}$$

by hypothesis. (We again suppress multiplicative constants.) Splitting the inner integral into

$$\int_u^w + \int_{w\,(C,p)}^{\infty}$$

we get $J_1 + J_2$, say. By inverting the integrals in J_1 we get

$$(-\partial/\partial x)^{[p]+1} J_1 = g(x)$$

by the right-continuity of g at x. It remains to show that

$$(-\partial/\partial x)^{[p]+1} J_2 = o(1) \quad \text{as } w \to \infty.$$

Splitting the inner integral of J_2 into

$$\int_w^{2w} + \int_{2w}^{\infty}$$

we can show by Lemma 2 and (5) that the second integral is uniformly summable in $x \leqq u \leqq w$. Hence we may invert the integrals in J_2 and get $J_2 = H(x, w)$ ((2) above). By Lemma 3, since $[p] = [k]$, $(-\partial/\partial x)^{[p]+1} J_2$ is given by (3). If we write $u = t - x$, $u^{k-1} g(u + x) = h(u)$ in (3) this gives

$$(8) \quad (-\partial/\partial x)^{[p]+1} J_2 = CW^{-k} \int_{W\,(C,p)}^{\infty} (1 - W/u)^k h(u)du$$

where $W = w - x$. By hypothesis $\int_0^\infty h(u)du$ is summable (C, p) and hence by Lemma 4 the expression in (8) is $o(1)$ as $W \to \infty$, i.e., as $w \to \infty$.

For the negative part we choose and fix x and write

$$g(u) = (u - x)^{1-p}h(u - x)(u > x), = 0 \quad (0 \leqq u \leqq x)$$

where $p = k$ and h is the function used in the proof of the negative part of Lemma 4. Then $I^k g(x) = f(x)$ is summable (C, p) $(=(C, k))$, and it is not difficult to show that g is continuous on the right at x. As in the proof of the positive part we again have

$$-(\partial/\partial x)^{[p]+1}J_1 = g(x)$$

and $(-\partial/\partial x)^{[p]+1}J_2$ is given by (8). By the negative part of Lemma 4 this $\to\infty$ as $W \to \infty$ through a suitable sequence. Thus (7) does not exist. By splitting the range of the integral there into $[x, c] + [c, w]$ we see that the existence of $D^k f(x)$ in (1) implies that of (7). Hence $D^k f(x)$ does not exist either for $p = k$.

4. Further results. We first define a more general concept of the kth derivative:

(9) $$\underset{(C,q)^*}{D^k} f(x) = (-d/dx)^{[k]} \lim_{X\to\infty} (-\partial/\partial x) c \int_x^X (1 - t/X)^q (t - x)^{-k'} f(t)dt$$

if this exists, where

$$c = (\Gamma(q + 1)\Gamma(1 - k'))^{-1}.$$

For $q = 0$ we obtain the definition (1). We now state:

THEOREM B. *Let $a > 0$, b fractional > 0, $q > -1$, and let x satisfy*

$$E(u, w) = (-\partial/\partial u)^{[b]} \int_u^w (t - u)^{-b'} f(t)dt$$

absolutely continuous on $x \leqq u \leqq w^*$, where $w^* < w$, for all $w > x$.

Suppose $I^a f(x)$ is summable (C, q). Then $D^b f(y)$ exists $(C, q)^*$ a.e. *for $y > x$. If, further, $E(u, w) = O(w - u)^{1-b}$ as $u \to w-$ when $b > 1$, and if f is continuous on the right at x, then*

$$\underset{(C,p)}{I^{a+b}} \underset{(C,q)^*}{D^b} f(x)$$

exists for $p > q + b$, and

(10) $$\underset{(C,p)}{I^{a+b}} \underset{(C,q)^*}{D^b} f(x) = \underset{(C,q)}{I^a} f(x) \quad (p > q + b).$$

The result is false for $p = q + b$. We observe that since an arbitrary constant may be added to $f(x)$ on the left side, the result does not necessarily hold under the assumption that the left side exists.

Proof of Theorem B. We may assume

$$\underset{(C,q)}{I^a f(0)} = 0,$$

so that with $p(t) = t^{a-1}f(t)$ we have

$$p_{q+1}(w) = o(w^q) \quad \text{as } w \to \infty.$$

The $(C, q)^*$ existence of $D^{b'}f(u)$ for $u > 0$ then follows by writing the integral in (9) as integrals with ranges $[x, c] + [c, X]$ $(x < c < X)$. In the proof we shall (as we may) use $(C, r)^*$ instead of $(C, q)^*$ where $r > [q] + 2$. We must show that

$$P = w^{-p} \int_0^w (w - u)^p u^{a+b-1} \underset{(C,r)^*}{D^b f(u)} du$$

satisfies $P = o(1)$ as $w \to \infty$ if $E(u, w) = O(w - u)^{1-b} (b > 1)$ as $u \to w-$ and $f(u)$ is continuous on the right at $u = 0$.

We split the integral of range $[u, X]$ in $D^b f(u)$ into integrals of ranges $[u, w] + [w, X]$ $(X > w)$, obtaining $P = Q + R$, say. In Q we integrate by parts $[b]$ times, then write

$$(t - u)^{-b'} f(t) = ((t - u)^{-b'} - t^{-b'})t^{1-a}p(t) + t^{1-a-b'}p(t),$$

giving $Q = Q_1 + Q_2$, say. It is easy to show that $Q_2 = o(1)$ as $w \to \infty$. For Q_1 we integrate once more by parts, obtaining

$$Q_1 = w^{-p} \int_0^w (\partial/\partial u)^{[b]+1}((w - u)^p u^{a+b-1})du$$
$$\times \int_u^w ((t - u)^{-b'} - t^{-b'})t^{1-a}p(t)dt.$$

We now invert the pair of integrals, put $u = ts$ and then integrate by parts $[q] + 1$ times:

$$\begin{aligned} Q_1 &= \int_0^1 ((1 - s)^{-b'} - 1)ds \sum_{r=0}^{[q]} (-1)^r w^{-p} p_{r+1}(w) \\ &\quad \times [(\partial/\partial t)^r (\partial/\partial s)^{[b]+1} M(w, t, s)]_{t=w} \\ &\quad + (-1)^{[q]+1} \int_0^1 ((1 - s)^{-b'} - 1)ds \\ &\quad \times w^{-p} \int_0^w p_{[q]+1}(t)(\partial/\partial t)^{[q]+1}(\partial/\partial s)^{[b]+1} M dt \\ &= Q_1' + Q_1'', \end{aligned}$$

say, where

$$M = (w - ts)^p s^{a+b-1}.$$

In Q_1' we differentiate only with respect to t, and then make $t \to w-$; then integrating by parts $[b] + 1$ times we obtain

$$Q_1' = (-1)^{[b]+1}(b')_{[b]+1} \sum_{r=0}^{[q]} (-1)^r(-p)_r w^{-r} p_{r+1}(w)$$
$$\times \int_0^1 (1 - s)^{p-r-b-1} s^{a+b-1+r} ds$$

where

$$(x)_k = x(x + 1) \ldots (x + k - 1) \quad (k \geqq 1), \ = 1(k = 0).$$

In Q_1'' we differentiate only with respect to t and then write

$$(11) \quad p_{[q]+1}(t) = c \int_0^t (t - y)^{-q'} p_q(y) dy.$$

Inversion of a pair of integrals plus an integration by parts of Q_1'' gives

$$Q_1'' = c \int_0^1 ((1 - s)^{-b'} - 1) ds \, w^{-p} \int_0^w p_{q+1}(y) dy$$
$$\times (\partial/\partial y) \int_y^w (t - y)^{-q'} (\partial/\partial s)^{[b]+1} N(w, t, s) dt$$

where

$$N = (w - ts)^{p-[q]-1} s^{a+b+[q]}.$$

Performing the last differentiation and then replacing the factor t^k by $(w - (w - t))^k$ and expanding in powers of $(w - t)$, then writing $t = y + (w - y)r$ and finally differentiating with respect to y, we get $Q_1'' = Q_1^* + Q_1^{**}$, where Q_1^* is given by

$$Q_1^* = \sum_{k=0}^{[b]+1} \sum_{m=0}^{k} c_{mk} \int_0^1 ((1 - s)^{-b'} - 1) s^{a+b'+[q]+k} ds$$
$$\times \int_0^1 r^{-q'} (1 - r)^{m+1} T(w, s, r),$$

where

$$T = w^{k-m-p} \int_0^w p_{q+1}(y)(w - y)^{m-q'+1}$$
$$\times (w - s(y + (w - y)r))^{p-[q]-2-k} dy;$$

and Q_1^{**} is the same double sum with m, k replaced by $m - 1$, $k - 1$ in each term.

We write

$$(w - s(y + (w - y)r))^{p-[q]-2-k}$$

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$$\leqq w^{[b]+1-k}(w - y)^{q'-m+\epsilon-2}(1 - r)^{q'-m+\epsilon-2}S(w, s)$$

where S is $(w(1 - s))^{p-q+m-\epsilon-[b]-1}$ or the same expression with $w(1 - s)$ replaced by w according as the power is <0 or $\geqq 0$. Then we obtain

$$Q_1^* = o(1) \quad \text{as } w \to \infty.$$

Similarly for Q_1^{**}. For the term R we put $u = ws$.

Then

$$R = w^{a+b} \int_0^1 (1 - s)^p s^{a+b-1} ds$$
$$\times \lim_{X\to\infty} \int_w^X (1 - t/X)^r (t - ws)^{-b-1} t^{1-a} p(t) dt.$$

Integrating the last integral by parts $[q] + 1$ times and then taking the limit in the sum of terms not involving an integral we obtain $R = S + T$ where

$$S = (-1)^{[b]+1}(b')_{[b]+1} \sum_{k=0}^{[q]} w^{-k} p_{k+1}(w)$$
$$\times \sum_{n=0}^{k} \binom{k}{n} (b + 1)_n (a - 1)_{k-n} B(p - b - n, a + b)$$

where B is the Beta function. This cancels with the term Q_1' by an identity (see [**5**, Lemma 5, p. 219]). For $[q] = -1$ both S and Q_1' are 0. Now

$$T = w^{a+b} \int_0^1 (1 - s)^p s^{a+b-1} ds$$
$$\times (-1)^{[q]} \lim_{X\to\infty} \int_w^X p_{[q]+1}(t)(\partial/\partial t)^{[q]+1} S(t, X, w, s) dt$$

where

$$S = (1 - t/X)^r (t - ws)^{-b-1} t^{1-a} \quad (r > [q] + 2).$$

Applying (11) and splitting the range $[0, t]$ into $[0, w] + [w, t]$ we get $T = T_1 + T_2$, say. On integrating T_1 by parts we then get

$$T_1 = C w^{a+b} p_{q+1}(w) \int_0^1 (1 - s)^p s^{a+b-1} ds$$
$$\times \lim_{X\to\infty} \int_w^X (t - w)^{-q'} (\partial/\partial t)^{[q]+1} S dt$$
$$+ C' w^{a+b} \int_0^w p_{q+1}(y) dy \int_0^1 (1 - s)^p s^{a+b-1} ds$$

$$\times \lim_{X\to\infty} \int_w^X (t - y)^{-q'-1}(\partial/\partial t)^{[q]+1} S dt$$

$$= T_1' + T_1'',$$

say. Performing the differentiation in T_1'' we get

$$T_1'' = \sum_{k=0}^{[q]+1} \sum_{\nu=0}^{k} c_{k\nu} w^{a+b} o(w^q) \int_0^1 (1 - s)^p s^{a+b-1} ds$$

$$\times \lim_{X\to\infty} X^{k-[q]-1} \int_w^X J(t) dt$$

where

$$J(t) = (1 - t/X)^{r-[q]-1+k}(t - w)^{-q'}(t - ws)^{-b-1-\nu} t^{1-a-k+\nu}.$$

Suppose $q \geqq 0$. The terms $1 \leqq k \leqq [q]$ are easily shown 0. For the terms $k = 0$, $k = [q] + 1$ we split $[w, X]$ into $[w, 2w] + [2w, X]$, giving $T_1'' = T_1^* + T_1^{**}$, say. Now T_1^{**} is easily shown to be $o(1)$ as $w \to \infty$; and in T_1^* we write

(12) $\quad (t - ws)^{-b-1-\nu} \leqq (t - w)^{q'-1+\epsilon}(w - ws)^{-b-\nu-q'-\epsilon}.$

Then T_1^* is seen to be $o(1)$ as $w \to \infty$.

Finally in T_2 we invert the order of the second and third integrals and then integrate by parts, obtaining $X + Y$, say, X being the term involving $p_{q+1}(w)$. Now X is seen to be just $-T_1'$ above, and in Y we integrate by parts and then perform the t-differentiation. Then

$$Y = \sum_{m=0}^{[q]+2} \sum_{n=0}^{m} c_{mn} w^{a+b} \int_0^1 (1 - s)^p s^{a+b-1} ds$$

$$\times \lim_{X\to\infty} X^{m-[q]-2} \int_w^X K(t) dt$$

where

$$K(t) = (t - ws)^{-b-1-n} t^{1-a-m+n} \int_w^t (t - y)^{-q'} p_{q+1}(y) dy,$$

the last integral being $o(t^q(t - w)^{1-q'})$ uniformly for $t > w$, as $w \to \infty$ for $q > -1$, the case $q < 0$ being dealt with by expanding

$$\left(1 - \frac{t - y}{y}\right)^q$$

in a power series. The cases $0 \leqq m \leqq [q] + 1$ are straightforward and give just 0. For $m = [q] + 2$ we split $[w, X]$ into $[w, 2w] + [2w, X]$. The second term is easily dealt with, and in the first we apply (12) with $\nu = n$ and q'

replaced by $q' - 1$; then the contribution to Y is again $o(1)$ as $w \to \infty$.

This completes the proof for $q \geqq 0$. For the case $q < 0$ of T_1'' a more careful analysis appears to be required; we write

$$\int_w^\infty (t - ws)^{-b-1}(t - y)^{-q'-1}t^{1-a}dt$$

as

$$\int_w^\infty (1 - (ws/t))^{-b-1}(1 - (y/t))^{-q'-1}t^{-b-q'-a-1}dt$$

and expand the binomials in power series. Writing

$$\int_0^w |p_{q+1}(y)|y^n dy < (A_\epsilon)^n + \epsilon w^{q+n+1}/(q + n + 1)$$

where A_ϵ is a constant depending only on ϵ, we get for the contribution B, say, to T_1'' of the second term,

$$B < C\epsilon \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(b + 1)_m(q' + 1)_n}{m!n!}(m + n + a + b + q')^{-1}$$

$$\times (q + n + 1)^{-1} \int_0^1 (1 - s)^p s^{a+b-1+m} ds.$$

After considerable simplifications and the use of the formula for a hypergeometric series $F(a, b; c, 1)$ with positive index ([**8**, p. 282]) we see that B is majorized by $C\epsilon$ times a hypergeometric series of type $F(a, b, c; d, e)$ of index $p - b - q$ (>0) and therefore convergent.

A similar, but simpler, argument goes through for the contribution A, say, to T_1'' from the term $(A_\epsilon)^n$ above.

For the negative part of the theorem we put

$$f(t) = t^{1-a}p(t)$$

where $p(t) = h(t)$ is used in the negative part of Lemma 4, with q for p and

$$Q = T_n^{[q]+1+b}.$$

A careful scan of the proof above (replacing $p > q + b$ by $p = q + b$) shows that for the chosen function $p(t)$ (or any of the functions a.e. equal to this) the term $m = n = [q] + 2$ in Y (above) 'blows up' and we get, after considerable details, and taking $w = T_n$, that

$$Y > CT_n^{[q]+b+2}$$

for all large n; an application of [**8**, p. 299, Example 18] is made in the final calculation, where a hypergeometric series has index $p - q - b = 0$. To ensure that f satisfies the conditions on $E(u, w)$ we may assume that $p_{q+1}(t)$ is differentiable $[q] + [b] + 4$ times at every t; cf. [**5**, p. 230 (87)].

THEOREM C. *Let* $a > 0$, b *fractional* > 0, $q > -1$, $-1 < r < b$. *Then*

$$(13) \quad \underset{(C,p)}{I^{a+b}} g(x) = \underset{(C,q)}{I^{a}} \ \underset{(C,r)}{I^{b}} g(x) \quad (p > q + b)$$

if the right side exists and g *is continuous on the right at* x. *The result is false with* $p = q + b$ *if* $q < a$ *and* $r > q + b$. (*If* b *is integral the result is true with* $r = p$, $q > -1$, $p = q + b$; *see* [**5**, p. 217, *line* 22].)

Proof. Put

$$f(u) = \underset{(C,r)}{I^{b}} g(u) \quad \text{a.e. for } u > x.$$

Then by Theorem A, $g(x) = D^b f(x)$ (since $r < b$), and also, by the proof of Theorem A with $k = b$ and the fact that (3) is $O(w - x)^{-k}$ as $x \to w-$, the conditions on $E(u, w)$ in Theorem B are satisfied. By hypothesis I^a $f(x)$ is summable (C, q) and the result then follows from Theorem B since

$$\underset{(C,O)^*}{D^b f(x)} = \underset{(C,q)^*}{D^b f(x)}$$

and g continuous on the right implies the same property for f.

For the negative part of Theorem C let f be a function chosen for the negative part of Theorem B above. Writing down (10) with this function f, putting

$$g(u) = \underset{(C,q)^*}{D^b f(u)}$$

and then splitting I^{a+b} into $I^a I^b$ by Lemma 1 we are then able to obtain

$$f(u) = \underset{(C,q+b+\epsilon)}{I^b g(u)} \quad \text{a.e. for } u > x \quad (\epsilon > 0)$$

by an application of Theorem A (using $q < a$). This gives that

$$\underset{(C,q)}{I^{a}} \ \underset{(C,q+b+\epsilon)}{I^{b} g(x)}$$

exists but $I^{a+b} g(x)$ does not exist in the $(C, q + b)$ sense.

Remarks. (a) In early attempts to prove Theorem C it was not realized that fractional derivatives would enter, since these do not appear explicitly, nor are they used in the proof of the converse, Lemma 1. From the development above, however, we see that $g(x)$ on the left side of (13) is 'really' $\underset{(C,q)^*}{D^b f(x)}$ which becomes $g(x)$ when

$$f(x) = \underset{(C,r)}{I^b g(x)} \quad \text{with } r < b.$$

(b) It would be of interest to obtain a counterexample to Theorem C in which $p = q + b$ but the order r is less than b.

(c) For results in fractional differences analogous to Lemma 1 and Theorem B see [**6**, Theorem 1, p. 431] and for results analogous to Theorem A (with $p = 0$) and Theorem C (with $r = 0$) see [**6**, Theorem 3] and [**1**, Theorem 1] or, for Theorem A with $p = 0$, [**7**, (11), p. 934]. The definitions for positive order in the fractional differences correspond to (7) rather than (1). It seems likely that results analogous to the general cases of Theorems A and C would hold.

(d) If, in the second integral of the Introduction, $g(t)dt$ is replaced by $dG(t)$, where $G(t)$ is of bounded variation on $x \leqq t \leqq X$ for every $X > x$, and if the total variation of the resulting integral is bounded on $x \leqq X < \infty$ then we say that

$$\int_x^\infty (t - x)^{k-1} dG(t)$$

is *summable* $|C, p|$ (this implies (C, p) to some value). If, in Lemma 1, $g(t)dt$ is replaced by $dG(t)$ as above, then Lemma 1 is true with $(C, \dots)$ replaced by $|C, \dots|$ throughout. The case b integral was given by David Borwein (see [**5**, pages 215-218] for discussion) and the case b fractional is given in unpublished notes by the author, the failure in the 'negative' case $q = p - b$ being particularly severe in that the left side of the conclusion can exist in the $|C, p|$ sense without the right side's being summable (C, q) $(q = p - b)$. It would be of interest to find a $|C|$ analogue of $(C, p)^*$ (see (9) above) and hence of Theorems A, B, C, with b fractional in Theorem C.

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