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A PSEUDOCOMPACT TYCHONOFF SPACE THAT IS NOT STAR LINDELÖF

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Abstract

Let P be a topological property. A space X is said to be *star* P if whenever $\mathcal{U}$ is an open cover of X, there exists a subspace $A \subseteq X$ with property P such that $X = \mathrm{St}(A, \mathcal{U})$, where $\mathrm{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. In this paper we construct an example of a pseudocompact Tychonoff space that is not star Lindelöf, which gives a negative answer to Alas *et al.* ['Countability and star covering properties', *Topology Appl.* **158** (2011), 620–626, Question 3].

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1. Introduction

By a 'space' we mean a topological space. Let X be a space and $\mathcal{U}$ a collection of subsets of X. For $A \subseteq X$, let $\mathrm{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$.

DEFINITION 1.1 [2, 9]. Let P be a topological property. A space X is said to be *star* P if whenever $\mathcal{U}$ is an open cover of X, there exists a subspace $A \subseteq X$ with property P such that $X = \mathrm{St}(A, \mathcal{U})$. The set A will be called a *star kernel* of the cover $\mathcal{U}$.

The term star P was coined in [9] and used in [1, 2], but certain star properties, specifically those corresponding to 'P = finite' and 'P = countable', were first studied by van Douwen *et al.* in [8] and later by many other authors. A survey of star covering properties with a comprehensive bibliography can be found in [4]. Here, we use the terminology from [2, 4]. In [8, 9] a star finite space is called starcompact and strongly 1-starcompact, and a star countable space is called star Lindelöf and strongly 1-star Lindelöf. In [7], a star σ-compact space is called σ-starcompact. From the definitions, we have the following diagram:

$$\textit{star countable} \longrightarrow \textit{star } \sigma\textit{-compact} \longrightarrow \textit{star Lindelöf}$$

In [2], Alas *et al.* studied the relationships of star P properties for $P \in$ {Lindelöf, σ-compact, countable} with other Lindelöf type properties and asked the following question.

QUESTION 1.2 [2, Question 3]. Is a pseudocompact Tychonoff space star Lindelöf?

The purpose of this note is to construct an example which gives a negative answer to this question.

Let $\mathfrak{c}$ denote the cardinality of the set of all real numbers. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow [3].

2. Main results

In this section we construct an example of a pseudocompact Tychonoff space that is not star Lindelöf. For a Tychonoff space X, let $\beta(X)$ denote the Čech–Stone compactification of X.

THEOREM 2.1. *There exists a pseudocompact Tychonoff space which is not star Lindelöf.*

PROOF. Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$ and let

$$X = (\beta(D) \times (\mathfrak{c} + 1)) \setminus ((\beta(D) \setminus D) \times \{\mathfrak{c}\})$$

be the subspace of $\beta(D) \times (\mathfrak{c} + 1)$. As was shown by Noble [5], X is pseudocompact; in fact, it has a countably compact, dense subspace $\beta(D) \times \mathfrak{c}$.

Next, we show that X is not star Lindelöf. For each $\alpha < \mathfrak{c}$, let

$$U_\alpha = \{d_\alpha\} \times [0, \mathfrak{c}].$$

Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{\beta(D) \times \mathfrak{c}\}$$

of X. Let A be a Lindelöf subset of X and let

$$\Lambda = \{\alpha : \langle d_\alpha, \mathfrak{c}\rangle \in A\}.$$

Then Λ is countable, since $\{\langle d_\alpha, \mathfrak{c}\rangle : \alpha < \mathfrak{c}\}$ is a discrete closed subset of X.

Let

$$A' = A \Big\backslash \bigcup \{U_\alpha : \alpha \in \Lambda\}.$$

If $A' = \emptyset$, then there exists an $\alpha_0 < \mathfrak{c}$ such that $A \cap U_{\alpha_0} = \emptyset$, hence $\langle d_{\alpha_0}, \mathfrak{c}\rangle \notin \mathrm{St}(A, \mathcal{U})$, since U_{α_0} is the only element of $\mathcal{U}$ containing the point $\langle d_{\alpha_0}, \mathfrak{c}\rangle$. On the other hand, if $A' \neq \emptyset$, then A' is closed in A and A' is Lindelöf and $A' \subseteq \beta(D) \times \mathfrak{c}$, hence $\pi(A')$ is a Lindelöf subset of a countably compact space $\mathfrak{c}$, where $\pi : \beta(D) \times \mathfrak{c} \to \mathfrak{c}$ is the projection. Hence, there exists $\alpha_1 < \mathfrak{c}$ such that $\pi(A') \cap (\alpha_1, \mathfrak{c}) = \emptyset$. Choose $\alpha < \mathfrak{c}$ such that $\alpha > \alpha_1$ and $\alpha \notin \Lambda$. Then $\langle d_\alpha, \mathfrak{c}\rangle \notin \mathrm{St}(A, \mathcal{U})$, since U_α is the only element of $\mathcal{U}$ containing $\langle d_\alpha, \mathfrak{c}\rangle$ and $U_\alpha \cap A = \emptyset$, which shows that X is not star Lindelöf, which completes the proof. □

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REMARK 2.2. Alas *et al.* [1] show there is an example of such a space by using the example due to Shakhmatov [6]. Shakhmatov's example is very complicated, but it has a point-countable base. The construction of Theorem 2.1 is simpler than theirs.

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