Can. J. Math., Vol. XXII, No. 3, 1970, pp. 666–673

PRODUCTS OF REFLECTIONS IN AN AFFINE MOUFANG PLANE

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Let $\mathfrak{A}$ be a Moufang plane. By specializing one line ω, the line at infinity, we obtain an affine Moufang plane $\mathfrak{A}_\omega$. The group generated by the shears of $\mathfrak{A}_\omega$ is called the *equiaffine group*. Veblen [**9**, § 52] asked whether every equiaffinity is a product of two affine reflections. He gave a proof which will work in an affine Pappian plane, using the following two properties.

Property 1. *If an equiaffinity fixes two distinct proper points of* $\mathfrak{A}_\omega$, *it fixes every point collinear with them.*

Property 2. *Let e be an equiaffinity and P a point such that* $PP^{e^2}P^{e^3}$ *is a triangle. Then the lines* $P^eP^{e^2}$ *and* PP^{e^3} *are parallel.*

Without using these properties, it will be proved that the answer to Veblen's question is "yes" if and only if the Moufang plane $\mathfrak{A}$ is Pappian.

1. Axial affinities. Let $\mathfrak{A}_\omega$ be an affine Moufang plane. A collineation fixing ω (while possibly permuting its points) is called an *affinity* (or an affine collineation). An *axis* of a collineation means a line whose points are all fixed. We call an affinity of $\mathfrak{A}_\omega$ *axial* if it is a homology or elation whose centre lies on ω. Thus if the axis is an ordinary line, the axial affinity is a strain or shear according as it is a homology or elation; it is a *translation* if its axis is ω. An affinity is called a *dilatation* if it is a homology with axis ω or a translation. We shall find it convenient to use "shear" both for ordinary shears and for translations.

Let B be a pencil of lines (concurrent or parallel). Let $\mathfrak{U}_B$ denote any group of affinities generated by axial affinities whose axes belong to B. Any axial affinity in $\mathfrak{U}_B$ will be called a generator if its axis belongs to B. We call $\mathfrak{U}_B$ a *B-group* if, for each pair of distinct points P and Q whose joining line PQ does not belong to B, $\mathfrak{U}_B$ has a generator a that transforms P into Q (that is, $P^a = Q$).

We shall find it convenient to use the same symbol B for the pencil and its centre (which is on ω if B is a pencil of parallels). Thus, for any other point P, the line PB is the member of B that passes through P.

Generally we will denote lines by lower case Greek letters, points by capital Latin letters, and axial affinities by lower case Latin letters. Occasionally we use the symbol $||$ for parallel.

Received April 28, 1969 and in revised form, September 15, 1969.

https://doi.org/10.4153/CJM-1970-073-6 Published online by Cambridge University Press

We investigate the following two statements.

1.1. THEOREM OF THE THREE AXIAL AFFINITIES. *Let* a_1, a_2, a_3 *be any generators of* $\mathfrak{U}_B$. *For any proper point* $P \neq B$, *let* β *denote the line* PB. *If* $P^{a_1a_2a_3} = P$ *and*

$$\beta \neq \beta^{a_1} \neq \beta^{a_1a_2} \neq \beta^{a_1a_2a_3} = \beta, \tag{1.11}$$

then $a_4 = a_1a_2a_3$ *is an axial affinity and* β *is its axis.*

1.2. DESARGUES' (B, ω)-THEOREM. *Let* $A_1A_2A_3B$ *and* $B_1B_2B_3B$ *be non-degenerate quadrangles such that*

(1.21) *each line* A_iB *coincides with* B_iB *and*

(1.22) $A_iA_{i+1} \| B_iB_{i+1}$ *for* $i = 1, 2$.

Then $A_3A_1 \| B_3B_1$.

1.3. THEOREM. *Let* $\mathfrak{U}_B$ *be a* B-*group. Then* 1.1 *holds for* $\mathfrak{U}_B$ *if and only if* 1.2 *holds for* B.

Proof. First, suppose that 1.2 holds for B; let $\mathfrak{U}_B$ be a B-group and let the assumptions of 1.1 be satisfied. Let B_1 be a point on β distinct from B; furthermore, let

$$A_1 = P, \qquad A_3 = A_2^{a_2} = A_1^{a_1a_2}, \qquad \text{and} \qquad B_3 = B_2^{a_2} = B_1^{a_1a_2}.$$

Then either the assumptions of 1.2 are satisfied or each of the triplets A_1, A_2, A_3 and B_1, B_2, B_3 is collinear. Thus either from 1.2 or trivially, $A_1A_3 \| B_1B_3$. Since

$$A_1 = P = P^{a_1a_2a_3} = A_3^{a_3},$$

we have

$$B_1 = B_3^{a_3} = B_1^{a_1a_2a_3}.$$

Hence β is an axis of $a_1a_2a_3$. This proves 1.1.

Secondly, let $\mathfrak{U}_B$ be a group satisfying 1.1. Suppose that B and A_i, B_i $(i = 1, 2, 3)$ satisfy the assumptions of 1.2. Then there exist generators a_1, a_2, a_3 of $\mathfrak{U}_B$ with

$$A_1 = A_3^{a_3} = A_2^{a_2a_3} = A_1^{a_1a_2a_3} \quad \text{and} \quad B_3^{a_3} = B_2^{a_2a_3} = B_1^{a_1a_2a_3}.$$

Put $\beta = A_1B$. Then (1.11) holds true since the quadrangle $A_1A_2A_3B$ is non-degenerate. Since $A_1^{a_1a_2a_3} = A_1$, 1.1 implies that β is an axis of $a_4 = a_1a_2a_3$. Thus we have $B_1 = B_1^{a_4}$ and $B_3^{a_3} = B_1$, and therefore $B_3B_1 \| A_3A_1$. This completes the proof of Theorem 1.3.

Our next goal is the following.

1.4. THEOREM. $\mathfrak{A}_\omega$ *is a Desarguesian plane if and only if*

(1.41) *for each* B *there exists a* B-*group* $\mathfrak{U}_B$ *and*

(1.42) 1.1 *holds for all* B *without* (1.11).

Consider the following lemma.

1.5. LEMMA. *Let a_1 and a_2 be generators of $\mathfrak{U}_B$, let $\beta \in B$, and let P be a proper point on β distinct from B. Suppose that*

$$P^{a_1 a_2} = P \qquad \textit{and} \qquad \beta^{a_1 a_2} = \beta.$$

Then β is an axis of $a_1 a_2$.

Proof. If $P^{a_1} = P^{a_2^{-1}} = P$, then β is an axis of $a_1 a_2$. Thus we may assume that $P^{a_1} = P^{a_2^{-1}} \neq P$. Then a_1 and a_2 have the same centre. Hence $a_1 a_2$ is axial with the fixed points P and B. Thus $\beta = PB$ is an axis of $a_1 a_2$ unless $\omega \in B$. Thus we may assume that $\omega \in B$.

Assume now that β is not an axis of $a_1 a_2$. Then since β is a fixed line, it must be a trace of $a_1 a_2$. Since a_1, a_2, and $a_1 a_2$ have the same centre (which is B since $\omega \in B$), β is also a trace of a_1 and a_2. Thus a_1, a_2, and therefore also $a_1 a_2$, are shears with centre B. Moreover, P is a fixed point of $a_1 a_2$. Hence β is an axis of $a_1 a_2$, contrary to the assumption. This proves the lemma.

Proof of Theorem 1.4. First suppose that for each B, 1.1 holds and a B-group exists. Then 1.3 yields 1.2 for every B. Hence $\mathfrak{A}_\omega$ is Desarguesian [**8**, § 3.2, *Satz* 27].

Secondly, let $\mathfrak{A}_\omega$ be Desarguesian. Then (1.41) holds and 1.3 yields 1.1 for each B-group $\mathfrak{U}_B$. We next show in three steps that 1.1 holds for each B-group without (1.11).

(a) 1.1 remains valid if (1.11) is replaced by

$$\beta \neq \beta^{a_1} \neq \beta^{a_1 a_2} = \beta^{a_1 a_2 a_3} = \beta. \tag{1.12}$$

We use the notation of 1.1, replacing (1.11) by (1.12). Let a be the strain with the axis β^{a_1} which maps $P^{a_1 a_2}$ into P. Let Q be any point on β distinct from B. We have

$$PP^{a_1} || QQ^{a_1} \qquad \text{and} \qquad P^{a_1}P^{a_1 a_2} || Q^{a_1}Q^{a_1 a_2};$$

furthermore, β^{a_1} is an axis of a and $\beta^{a_1 a_2} = \beta$. Hence $P^{a_1 a_2 a} = P$ implies $Q^{a_1 a_2 a} = Q$. Thus β is an axis of $a_1 a_2 a$. Since $a_1 a_2 a_3 = (a_1 a_2 a) \cdot (a^{-1} a_3)$, we have $P^{a^{-1} a_3} = P$ and $\beta^{a^{-1} a_3} = \beta$. Thus, by 1.5, β is an axis of $a^{-1} a_3$. Since β is also an axis of $a_1 a_2 a$, it must be an axis of $a_1 a_2 a_3$. This proves (a).

(b) 1.1 remains valid if (1.11) is replaced by

$$\beta = \beta^{a_1} = \beta^{a_1 a_2} = \beta^{a_1 a_2 a_3} = \beta. \tag{1.13}$$

We use once more the notation of 1.1, but replacing (1.11) by (1.13). Then (1.13) implies that $\beta^{a_1} = \beta^{a_2} = \beta^{a_3} = \beta$. Hence β is a trace or axis of each a_i. Since

$$a_1 a_2 a_3 = a_2 (a_2^{-1} a_1 a_2) a_3 = a_3 (a_3^{-1} a_1 a_3)(a_3^{-1} a_2 a_3), \tag{1.14}$$

we may assume without loss of generality that β is either an axis of a_1 or a trace of a_1, a_2, and a_3.

First, let β be an axis of a_1. Then $P^{a_2 a_3} = P$ and $\beta^{a_2 a_3} = \beta$; hence, by 1.5, β is an axis of $a_2 a_3$. Thus β is an axis of $a_1 a_2 a_3$.

Secondly, let β be a trace of a_1, a_2, and a_3. Then $a_1a_2a_3$ is axial and keeps P, β, and B fixed. Moreover, if $\omega \in B$, then B is the centre of a_1, a_2, and a_3, and therefore also of $a_1a_2a_3$, all of which are then shears. Thus β is an axis of $a_1a_2a_3$. This proves (b).

(c) 1.1 holds without (1.11). If $\beta^{a_1} = \beta^{a_2} = \beta^{a_3} = \beta$, then (1.13) holds and therefore 1.1 holds by (b). Suppose that (1.13) is false. On account of (1.14), we may assume that $\beta \neq \beta^{a_1}$. If $\beta^{a_1a_2} = \beta^{a_1a_3} = \beta^{a_1}$, we would have $\beta = \beta^{a_1a_2a_3} = \beta^{a_1a_3} = \beta^{a_1}$. Since $a_1a_2a_3 = a_1a_3(a_3{}^{-1}a_2a_3)$, we may even assume that $\beta \neq \beta^{a_1} \neq \beta^{a_1a_2}$. Thus (c) follows from 1.1, (a), and (b).

This proves that 1.1 holds without (1.11) for each B-group $\mathfrak{U}_B$. Since on account of (1.41) each group $\mathfrak{U}_B$ is contained in a B-group $\overline{\mathfrak{U}}_B$, 1.1 holds without (1.11) for each group $\mathfrak{U}_B$. This completes the proof of 1.4.

We next investigate the following statement.

1.6. Existence of the third axial affinity. *Let $\mathfrak{U}_B$ be maximal with respect to B. Let a_1 and a_2 be generators of $\mathfrak{U}_B$. Then for each $\beta \in B$ there exists a generator a_3 of $\mathfrak{U}_B$ such that β is an axis of $a_4 = a_1a_2a_3$.*

1.7. Theorem. *$\mathfrak{A}_\omega$ is Desarguesian if and only if*
(1.71) 1.6 *holds for each maximal $\mathfrak{U}_B$ and*
(1.72) *each maximal group $\mathfrak{U}_B$ is a B-group.*

Proof. First, let $\mathfrak{A}_\omega$ be Desarguesian and let $\mathfrak{U}_B$ be maximal. Let a_1 and a_2 be generators of $\mathfrak{U}_B$ and let $\beta \in B$. Further, let P be a proper point on β distinct from B. Since $\mathfrak{A}_\omega$ is Desarguesian, (1.72) holds, and some generator a_3 of $\mathfrak{U}_B$ will satisfy $P^{a_1a_2a_3} = P$. By 1.4, β is an axis of $a_1a_2a_3$. This proves (1.71).

Secondly, assume that (1.71) and (1.72) hold. Suppose that $\mathfrak{U}_B$, a_1, a_2, a_3, β, and P satisfy the assumptions of 1.1 without (1.11). Then by 1.6 there exists a generator a of the maximal group $\overline{\mathfrak{U}}_B$ containing $\mathfrak{U}_B$ such that β is an axis of a_1a_2a. By 1.5, β is also an axis of $a^{-1}a_3$. Thus β is an axis of $a_1a_2a_3$. This proves 1.1 without (1.11). Hence by 1.4, $\mathfrak{A}_\omega$ is Desarguesian. This completes the proof of 1.7.

2. Veblen's Theorem. Let $\mathfrak{A}_\omega$ be an affine Moufang plane of characteristic $\neq 2$, and let $\mathfrak{S}$ be the equiaffine group of $\mathfrak{A}_\omega$. Again B may be a pencil of lines. We call the group $\mathfrak{U}_B$ maximal in S if it is generated by all the shears with axes in B. Note that such a group is a B-group. Any group $\mathfrak{G}$ of affinities is called *bi-reflectional* if each element of G is a product of two reflections.

We wish to investigate the following theorem.

Veblen's Theorem. *The equiaffine group $\mathfrak{S}$ is bi-reflectional.*

We first prove the following result.

2.1. Lemma. *Let b_1 and b_2 be reflections. If b_1 and b_2 have the same centre or the same axis, then b_1b_2 is a shear. If b_1b_2 is axial, then b_1 and b_2 have the same centre or the same axis.*

Proof. Obviously, $b_1 b_2$ is a shear if b_1 and b_2 possess the same centre or the same axis.

Let β be an axis of $b_1 b_2$. If $P^{b_1} = P^{b_2} = P$ for each point P on β, then β is an axis of b_1 and b_2. If $P^{b_1} = P^{b_2} \neq P$ for some point P, then b_1 and b_2 have the same centre. This proves the lemma.

2.2. THEOREM. *If $\mathfrak{U}_B$ is bi-reflectional and maximal in $\mathfrak{S}$, then* 1.1 *holds for $\mathfrak{U}_B$ without* (1.11).

Proof. We use the notation of 1.1 (without assuming (1.11)). Since $\mathfrak{U}_B$ is bi-reflectional, there exist reflections b_1 and b_2 such that $a_1 a_2 a_3 = b_1 b_2$. The assumptions of 1.1 yield $P^{b_1 b_2} = P$ and $\beta^{b_1 b_2} = \beta$. Since the a_is are shears, $a_1 a_2 a_3$ is always a shear with axis β if B is a parallel class. We may therefore assume that B is a proper point.

Obviously, $B^{b_1 b_2} = B$. If $B^{b_1} = B^{b_2} \neq B$ or $P^{b_1} = P^{b_2} \neq P$, then b_1 and b_2 have the same centre and 2.1 implies that $a_1 a_2 a_3 = b_1 b_2$ is a shear. It trivially has the axis $BP = \beta$. Thus we may assume that $B^{b_1} = B^{b_2} = B$ and $P^{b_1} = P^{b_2} = P$. Then β is an axis of both b_1 and b_2 and hence of $a_1 a_2 a_3 = b_1 b_2$. This proves 2.2.

2.3. THEOREM. *If $\mathfrak{S}$ is bi-reflectional, $\mathfrak{A}_\omega$ is a Pappian plane.*

First proof. The groups $\mathfrak{U}_B$ which are maximal in $\mathfrak{S}$ are B-groups. Since $\mathfrak{S}$ is bi-reflectional, the groups $\mathfrak{U}_B$ contained in $\mathfrak{S}$ are bi-reflectional. 2.2 therefore implies 1.1 without (1.11). Thus 1.4 yields that $\mathfrak{A}_\omega$ is Desarguesian. Hence [**1**, Chapter IV, Theorem 4.2] the matrix

$$\begin{bmatrix} r \cdot s \cdot r^{-1} \cdot s^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

represents an element b of $\mathfrak{S}$ for any choice of r and s in the skew field of coordinates of $\mathfrak{A}_\omega$ and for a suitable basis. This basis may be chosen so that b becomes axial. Since $\mathfrak{S}$ is bi-reflectional, b is the product of two reflections. By 2.1, b is a shear. But the only shear that can be represented by such a matrix is the identity. Thus $r \cdot s \cdot r^{-1} \cdot s^{-1} = 1$, and the skew field of coordinates of $\mathfrak{A}_\omega$ is commutative. This completes the proof.

Second proof. Let B_1, B_2, B_3, and B be mutually distinct points on the line β and let D_1, D_2, D_3, and B be mutually distinct points on the line $\delta \neq \beta$. Consider the hexagon $B_1 D_2 B_3 D_1 B_2 D_3$. We assume that

(2.31) $B_1 D_2 || B_2 D_1$ and $B_3 D_2 || B_2 D_3$.

We have to show that $B_3 D_1 || B_1 D_3$. Let r_{ik} denote the reflection with the axis through B which maps B_i into D_k. Furthermore, denote by s_{ik} the strain with the axis δ which maps B_i into B_k. Then

(2.32) $\beta^{r_{ik}} = \delta$, $\delta^{r_{ik}} = \beta$ and

(2.33) $\beta^{s_{ik}} = \beta$, $\delta^{s_{ik}} = \delta$.

By Theorem 2.2, $a = r_{12}r_{32}s_{31}$ and $b = s_{13}r_{31}r_{11}$ are axial with axis β. Thus β is an axis of $r = a \cdot b = r_{12}r_{32}r_{31}r_{11}$.

Since r is the product of an even number of reflections, it belongs to $\mathfrak{S}$ [**4** or **6**]. Hence, $\mathfrak{S}$ being bi-reflectional, r is the product of two reflections. Since r has the axis β, it is a shear with that axis. By (2.32), $\delta^r = \delta$; hence r must be the identity. This yields $r_{11} = r_{12}r_{32}r_{31}$. Since, by (2.31), $r_{12} = r_{21}$ and $r_{23} = r_{32}$, we obtain

$$D_3^{r_{13}} = B_1 = B_1^{r_{11}r_{11}} = B_1^{r_{12}r_{32}r_{31}\cdot r_{21}r_{23}r_{31}} = D_3^{r_{31}}.$$

Hence $r_{31} = r_{13}$ and therefore $B_3D_1 \| B_1D_3$.

The next theorems are the groundwork for the proof that $\mathfrak{S}$ is bi-reflectional if $\mathfrak{A}_\omega$ is a Pappian plane.

2.4. Theorem. *Let $\mathfrak{A}_\omega$ be Desarguesian. Then every product b of axial affinities is the product $b = a_1a_2t$ of two axial affinities a_1 and a_2 and one translation t.*

Proof. Let B be a parallel class of lines and let $\mathfrak{U}_B$ be maximal with respect to B. Denote by $k = k(b)$ the smallest number such that

(2.41) $b \in a_1 \ldots a_k \cdot \mathfrak{U}_B$, where $a_1, a_2, \ldots$ are axial affinities.

Assume that $k > 1$. By (1.71), there must exist an axial affinity $\bar{a}_{k-1}$ such that $a_k^{-1}a_{k-1}^{-1}\bar{a}_{k-1} = \bar{a}_k^{-1}$ is axial with axis in B. Thus

$$a_1 \ldots a_{k-1}a_k\mathfrak{U}_B = a_1 \ldots a_{k-2}\bar{a}_{k-1}(\bar{a}_k\mathfrak{U}_B) = a_1 \ldots a_{k-2}\bar{a}_{k-1}\mathfrak{U}_B$$

and k would not be minimal. Hence $k \leqq 1$.

Next let $j = j(c)$ be the smallest number for the element c of $\mathfrak{U}_B$ such that

(2.42) $c^{-1} \cdot a_2 \ldots a_j$ is a translation for some axial affinities

$$a_2, a_3, \ldots, a_j \in \mathfrak{U}_B.$$

(1.71) similarly yields $j \leqq 2$.

Since $k \leqq 1$ and $j \leqq 2$, (2.41) and (2.42) together yield our assertion.

2.5. Theorem. *Let $\mathfrak{A}_\omega$ be an affine Moufang plane. Then the product b_1b_2t of two reflections b_1 and b_2 and one translation t is equal to a product of two reflections.*

Proof. Let β_1 and β_2 be the axes of b_1 and b_2, respectively.

First suppose that $\beta_1 \| \beta_2$. Let B_2 be the centre of b_2. Construct the reflection b with axis β_1 and centre B_2. Then $b_1b_2t = (b_1b)(bb_2t)$, where b_1b is a shear but not a translation different from the identity, and where bb_2 and therefore bb_2t are translations. If b_1b is the identity, then b_1b_2t is a translation and 2.5 holds trivially. If b_1b is not the identity, it is a product of two reflections with non-parallel axes. Thus this case will be included in the following case.

Let $\beta_1 \nparallel \beta_2$. There exist half turns H_1 and H_2 with centres A_1 and A_2, respectively, such that $t = H_1H_2$ and A_i is on β_i for $i = 1$ and 2. Since $b_1b_2H_2b_1$ and b_1H_1 are reflections, the splitting $b_1b_2t = (b_1b_2H_2b_1)(b_1H_1)$ completes the proof of 2.5.

2.6. THEOREM. *If $\mathfrak{A}_\omega$ is a Pappian plane, then $\mathfrak{S}$ is bi-reflectional.*

Proof. Let $b \in \mathfrak{S}$. By 2.4, there exist axial affinities a_1 and a_2 and one translation t such that $b = a_1 a_2 t$. Since $\mathfrak{A}_\omega$ is Pappian, $\mathfrak{S}$ is represented by a linear group over a commutative field. Thus we may use the theory of determinants (representing the elements of $\mathfrak{S}$ by matrices with determinant 1) [**1**, Chapter IV, Theorem 4.3]. Since $a_1 a_2 = bt^{-1} \in \mathfrak{S}$, $\det(a_1 a_2) = 1$. Hence $a_1 a_2$ is a dilatation if and only if it is a half turn or a translation, which implies that b is a half turn or a translation. Thus 2.6 holds trivially if $a_1 a_2$ is a dilatation. Hence, we may assume that $a_1 a_2$ is not a dilatation.

Let B be a pencil of lines containing the axes of a_1 and a_2, and let $\mathfrak{U}_B$ be maximal with respect to B. Since $a_1 a_2$ is not a dilatation, there exists a reflection $b_2 \in \mathfrak{U}_B$ satisfying $P^{a_1 a_2 b_2} = P$ for some proper $P \neq B$. By (1.42), $b_1 = a_1 a_2 b_2$ is axial. Since $\det b_1 = -1$, b_1 must be a reflection. Thus $b = b_1 b_2 t$, and 2.5 yields our assertion.

2.3 and 2.6 combined show that Veblen's Theorem holds if and only if $\mathfrak{A}_\omega$ is Pappian. Moreover, we show the following.

2.7. MAIN THEOREM. *If $\mathfrak{A}_\omega$ is an affine Moufang plane, the following statements are equivalent:*

(2.71) *$\mathfrak{A}_\omega$ is a Pappian plane;*

(2.72) *the equiaffine group $\mathfrak{S}$ is bi-reflectional;*

(2.73) *every equiaffinity is a product of three shears. Every equiaffinity that is not a half turn is a product of two shears (may be an ordinary shear and a translation);*

(2.74) *Properties* 1 *and* 2 *hold.*

By 2.3 and 2.6, (2.71) and (2.72) are equivalent; (2.72) implies (2.73) [**4**; **5**]; (2.74) implies (2.72) [**9**, § 52]. Thus we only have to show that (2.73) implies (2.74).

For the half turns, Properties 1 and 2 hold trivially. We may therefore assume that every equiaffinity which will occur below is a product of two shears.

Let e be an equiaffinity which is the product of the two shears s_1 and s_2, and let $P \neq Q$ satisfy $P^e = P$ and $Q^e = Q$. If $P^{s_1} = P^{s_2^{-1}} = P$ and $Q^{s_1} = Q^{s_2^{-1}} = Q$, then s_1, s_2, and e all have the axis PQ so that Property 1 holds for e. But if $P^{s_1} = P^{s_2^{-1}} \neq P$ or $Q^{s_1} = Q^{s_2^{-1}} \neq Q$, then s_1 and s_2 have the same centre which yields again that PQ is an axis of e. Hence Property 1 holds in either case. Coxeter [**3**, p. 42] used the Cayley-Hamilton Theorem to deduce Property 2 from (2.71). Since (2.71) and (2.72) are equivalent, it only remains to prove that (2.73) implies (2.72).

First, half turns are products of two reflections.

Secondly, each product of two shears with parallel axes is a shear, hence a product of two reflections.

Thirdly, let s_1 and s_2 be two shears with centres L_1 and L_2 and non-parallel axes β_1 and β_2, respectively. Denote by s_{ij} the reflection with centre L_i and axis β_j. Then the equation

$$s_1s_2 = (s_1s_{12})(s_{12}s_2)$$

splits the product s_1s_2 into the product of two reflections.

By the preceding discussion, (2.73) implies (2.72). This completes the proof of the Main Theorem.

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