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SIGNATURES AND SEMISIGNATURES OF ABSTRACT WITT RINGS AND WITT RINGS OF SEMILOCAL RINGS

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0. Introduction. This paper originated in an attempt to carry over the results of [**3**] from the case of a field of characteristic different from two to that of semilocal rings. To carry this out, we reverse the point of view of [**3**] and *do* assume a full knowledge of the theory of Witt rings of classes of non-degenerate symmetric bilinear forms over semilocal rings as given, for example, in [**10**; **11**]. It turns out that the rings W_T of [**3**] are just the residue class rings of $W(C)$, the Witt ring of a semilocal ring C, modulo certain intersections of prime ideals.

The first section of this paper deals with abstract Witt rings [**10**, Def. 3.12]. We generalize the usual notions of dimension, isotropy, and representability to these. Additionally, we study the homomorphisms, both as rings and as abelian groups, of abstract Witt rings to **Z**; the former are called signatures and certain of the latter, semisignatures. It turns out, for example, that an element of an abstract Witt ring is mapped to 0 by all signatures if and only if it is mapped to 0 by all semisignatures. The main result of Section 1 is a necessary and sufficient condition for the existence of a semisignature mapping a prescribed set of units to 1. This result may be viewed as a generalization of the main part of the implication (i) $\Rightarrow$ (ii) of [**8**, Thm. 5.7]. The section ends with consideration, still in the abstract case, of extensions of semisignatures and the Hasse-Minkowski property [**4**].

In Section 2, we consider a connected semilocal ring C all of whose residue class fields contain at least 3 elements. If $R = W(C)$ is the Witt ring of classes of nondegenerate symmetric bilinear C-forms, we show that the results of Section 1 are applicable to $\bar{R} = W(C)/I(Y)$, where $I(Y)$ is a special intersection of non-maximal minimal prime ideals of R. We show how the abstract notions of dimension, isotropy, and representability translate to $\bar{R}$ and also prove that $\bar{R}$ satisfies the necessary and sufficient condition for the existence of semisignatures established in Section 1. This section ends by giving a presentation of $\bar{R}$ by generators and relations.

In Section 3 we apply the results of the first two to translate and generalize results on semisignatures of R to results on semisignatures of $\bar{R}$. We deal with

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extensions of semisignatures, the Hasse-Minkowski property and with indefinite forms; that is, forms which are mapped by all semisignatures to an integer less than the form's rank in absolute value. Finally, we deduce [**8**, Thm. 5.13], in the case of trivial involution, from our results.

1. Abstract Witt rings. Let G be a group (necessarily abelian) of exponent 2, i.e. for all g in G, we have $g^2 = 1$. For a proper ideal K of the group ring $\mathbf{Z}[G]$, the ring $R = \mathbf{Z}[G]/K$ is called a *Witt ring* for G if the torsion subgroup, R_t, of the additive group of R is 2-primary [**10**, Def. 3.12]. R is called *reduced* if $R_t = 0$. By [**10**, Props. 3.15 and 3.16] a reduced Witt ring for G has no non-zero nilpotent elements. For g in G we denote the image of g in R by $\bar{g}$, although for the identity element of both G and R we often write 1. Clearly every element r of R may be written as $\sum_1^n \epsilon_i \bar{g}_i$ with $\epsilon_i = \pm 1$ for, not necessarily distinct, elements g_i of G. By G' we denote the multiplicative subgroup of $\mathbf{Z}[G]$ consisting of the elements $\pm g$, g in G, and write $\bar{g}'$ for $\pm \bar{g}$.

Definition 1.1. For r in R, $\dim_R r$, or $\dim r$ if there is no possibility of confusion, is the smallest natural number n such that $r = \sum_1^n \bar{g}_i'$, g_i' in G'. Clearly, for $r_1, \ldots, r_m$ in R, we always have $\dim (\sum_1^m r_j) \leqq \sum_1^m \dim r_j$.

If F is a field of characteristic not two and $R = W(F)$ is the Witt ring of equivalence classes of symmetric nondegenerate bilinear forms over F, then $W(F)$ is a Witt ring for $U(F)/(U(F))^2$, where $U(F)$ denotes the unit group, $F - \{0\}$, of F, [**10**, Ex. 3.11]. For r in $W(F)$, $\dim r$ is then the vector space dimension of the unique anisotropic representative of r [**12**, Thm. 1.7, p. 58].

Definition 1.2. For g' in G', the element r in R is said to *represent* the element g' of G', if there is a p in R with $r = \bar{g}' + p$ and $\dim p < \dim r$. The subset of $\mathbf{Z}[G]$ represented by r will be denoted by $D(r)$.

Definition 1.3. For $g_1', \ldots, g_n'$ in G', the element $\sum_1^n g_i'$ of $\mathbf{Z}[G]$ is called *anisotropic* for R if $\dim (\sum_1^n \bar{g}_i') = n$. Otherwise $\sum_1^n \bar{g}_i'$ will be called *isotropic* for R.

Since each r in R can be written as $\sum_1^n \bar{g}_i'$, it is clear that by choosing n minimal, we obtain an anisotropic representative in $\mathbf{Z}[G]$ of r. Furthermore, it is clear that these definitions coincide with the usual ones if $R = W(F)$, F a field of characteristic not two.

LEMMA 1.4. *For r in R, let $\sum_1^n g_i'$ be an anisotropic representative of r in $\mathbf{Z}[G]$. Then $\sum_1^n g_i' + g'$, for some g' in G', is an element of $\mathbf{Z}[G]$ isotropic for R if and only if $-g'$ is in $D(r)$.*

Proof. If $-g'$ is in $D(r)$, there exists an element p in R with $r = p - \bar{g}'$ and $\dim p < \dim r$. Thus $p = r + \bar{g}'$, and so $\dim (r + \bar{g}') < \dim r = n$. By Definition 1.3 this means that $\sum_1^n g_i' + g'$ is isotropic for R.

Conversely, if $\sum_1^n g_i' + g'$ is isotropic for R, there exist $h_1', \ldots, h_m'$ in G' with $\sum_1^m \bar{h}_i' = r + \bar{g}'$ and $m = \dim (r + \bar{g}') < n + 1$. Solving for r shows that

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$n = \dim r \leqq m + 1$. Thus $m = n$ or $n - 1$. Now, by definition of R, the element $\sum_1^m h_i' - (\sum_1^n g_i' + g')$ of $\mathbf{Z}[G]$ is in K, and so by [**10**, Thm. 3.9 (ii)], the natural number $n + m + 1$ is even. Hence $m = n - 1$, i.e., $\dim (r + \bar{g}') < n = \dim r$. Since $r = (r + \bar{g}') - \bar{g}'$, Definition 1.2 shows that $-g'$ is in $D(r)$.

Definition 1.5. If R is a Witt ring for G, the set of ring homomorphisms $R \to \mathbf{Z}$ is denoted by $X(R)$ and called the *set of signatures* of R.

Remark 1.6. For σ in $X(R)$, the ideal $\ker \sigma$ is a minimal non-maximal prime ideal of R and the mapping $\sigma \to \ker \sigma$ is a bijection of $X(R)$ onto the set of minimal non-maximal prime ideals of R [**10**, Lemma 3.1 and Remark 3.2]. Of course, by passing to inverse images in $\mathbf{Z}[G]$, the set $X(R)$ is also bijective with the set of minimal prime ideals of $\mathbf{Z}[G]$ containing K. By [**10**, Prop. 3.4], $X(R) \neq \emptyset$ if and only if $R_t = \operatorname{Nil} R$, the nilradical of R. Throughout the rest of this section we assume $X(R) \neq \emptyset$.

Definition 1.7 (cf. [**11**, Sec. 4]). (i) For any subset M of G' in $\mathbf{Z}[G]$, $V(M) = \{\sigma \text{ in } X(R) | \sigma(\bar{g}') = 1 \text{ for all } g' \text{ in } M\}$.

(ii) For $Y \subset X(R)$, we put $\Gamma(Y) = \{g' \text{ in } G' | \sigma(\bar{g}') = 1 \text{ for all } \sigma \text{ in } Y\}$.

(iii) A subset Y of $X(R)$ is *saturated* if $Y = V(\Gamma(Y))$.

(iv) For $Y \subset X(R)$, we put $I(Y) = \bigcap_{\sigma \in Y} \ker \sigma$, an ideal of R.

(v) For any subset M of G' in $\mathbf{Z}[G]$ we denote the (proper) ideal of R generated by $1 - \bar{g}'$, g' in M, by $\mathfrak{a}(M)$.

PROPOSITION 1.8 [cf. **11**, Lemma 4.15 and Corollary 4.16]. *For any subset M of G':*

(i) *$R/\mathfrak{a}(M)$ is again a Witt ring for G.*

(ii) *The radical of $\mathfrak{a}(M)$, written $(\mathfrak{a}(M))^{1/2}$, is $I(V(M))$, with the convention $I(\emptyset) = \mathfrak{M}_0(R)$, the unique maximal ideal of R containing* 2 [**10**, Lemma 2.13 and Theorem 3.9].

Proof. The proof of (i) is essentially the same as that of [**11**, Lemma 4.15]. As for (ii), since $\sigma(1 - \bar{g}') = 0$ if and only if $\sigma(\bar{g}') = 1$, it is clear that $\mathfrak{a}(M) \subset \ker \sigma$ if and only if σ is in $V(M)$. Since by [**10**, Lemma 3.1 and Theorem 3.9 (i)] the only prime ideals of R are either $\mathfrak{M}_0(R)$, a $\ker \sigma$, or maximal ideals properly containing a $\ker \sigma$, the result is clear if $V(M) \neq \emptyset$ since $\mathfrak{M}_0(R)$ contains all $\ker \sigma$ [**10**, Ex. 3.11]. If $V(M) = \emptyset$ and $\mathfrak{a}(M)$ were contained in a maximal ideal of R other than $\mathfrak{M}_0(R)$, the residue class ring $R/\mathfrak{a}(M)$ would not be a Witt ring for G [**10**, Theorem 3.9], contradicting (i). Thus (ii) is true also if $V(M) = \emptyset$.

LEMMA 1.9. *Let $M \neq \emptyset$ be a subset of G' in $\mathbf{Z}[G]$ and let $\tilde{M}$ be the multiplicative subgroup of G' generated by M.*

(i) *For r in $(\mathfrak{a}(M))^{1/2}$ and $\lambda : R \to \mathbf{Z}$ an additive homomorphism, constant on the cosets of $\bar{\tilde{M}}$ in $\bar{G}'$, we have $\lambda(r) = 0$.*

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(ii) *For r in R we have $\sigma(r) = 0$ for all σ in $V(M)$ if and only if $\lambda(r) = 0$ for all additive homomorphisms $R \rightarrow \mathbf{Z}$ constant on the cosets of $\bar{\tilde{M}}$ in $\bar{G}'$.*

Proof. It is readily verified that λ is constant on cosets of $\bar{\tilde{M}}$ in $\bar{G}'$ if and only if for every g' in G and h in M we have $\lambda(\bar{g}'\bar{h}) = \lambda(\bar{g}')$.

To prove (i), we note that by Proposition 1.8 (i), the ring $R/\mathfrak{a}(M)$ is again a Witt ring for G and thus its nilradical is torsion [**10**, Lemma 3.3]. Thus, for some natural number m, we see that mr is in $\mathfrak{a}(M)$. But the constancy of λ on the cosets of $\bar{\tilde{M}}$, forces $\lambda(\mathfrak{a}(M)) = 0$. Hence $0 = \lambda(mr) = m\lambda(r)$, which implies $\lambda(r) = 0$ since $\lambda(r)$ is in $\mathbf{Z}$.

To prove (ii), we note that every σ in $V(M)$ is constant on the cosets of $\bar{\tilde{M}}$ in $\bar{G}'$, so that one implication is clear. Conversely, if for every σ in $V(M)$ we have $\sigma(r) = 0$, then by Proposition 1.8 (ii), r is in $(\mathfrak{a}(M))^{1/2} = I(V(M))$. Thus, by (i), $\lambda(r) = 0$ for all additive homomorphisms $R \rightarrow \mathbf{Z}$, constant on cosets of $\bar{\tilde{M}}$ in $\bar{G}'$.

Remarks 1.10. (i) Let $M = \Gamma(X(R))$. Then $\tilde{M} = M$ and the constancy condition of Lemma 1.9 is automatically fulfilled for all additive homomorphisms $\lambda : R \rightarrow \mathbf{Z}$. For if for all σ in $X(R)$ we have $\sigma(\bar{h}') = 1$, let $r = \bar{g}'\bar{h}' - \bar{g}'$ in R, where $\bar{g}'$ is in $\bar{G}'$. Then for every σ in $X(R)$, we have $\sigma(r) = 0$. Hence by Remark 1.6, the element r is in Nil R and so by [**10**, Lemma 3.3 and Lemma 3.12], r is in R_t. But then clearly $\lambda(r) = 0$ for any additive homomorphism $\lambda : R \rightarrow \mathbf{Z}$, which is precisely the required constancy condition.

(ii) From (i) and Lemma 1.9 (ii) we see that for an element r of R, we have $\sigma(r) = 0$ for all σ in $X(R)$ if and only if $\lambda(r) = 0$ for all additive homomorphisms $\lambda : R \rightarrow \mathbf{Z}$.

PROPOSITION 1.11. *Let $Y \subset X(R)$ be a saturated set of signatures and $\bar{R} = R/I(Y)$. Then if λ is an additive homomorphism $R \rightarrow \mathbf{Z}$ constant on the cosets of $\overline{\Gamma(Y)}$ in $\bar{G}'$, we have $\lambda(I(Y)) = 0$ so that λ induces an additive homomorphism $\bar{\lambda}$ from the Witt ring $\bar{R}$ to $\mathbf{Z}$.*

Proof. Since Y is saturated, $Y = V(\Gamma(Y))$. Thus setting $M = \tilde{M} = \Gamma(Y)$, we obtain $(\mathfrak{a}(M))^{1/2} = I(Y)$ by Proposition 1.8 (ii). Proposition 1.8 (i) and [**10**, Remark 3.13 (ii)] then show that $\bar{R}$ is a Witt ring for G, and Lemma 1.9 (i) proves that $\lambda(I(Y)) = 0$.

Definition 1.12 (cf. [**8**, Def. 5.1]). Let $R = \mathbf{Z}[G]/K$ be a Witt ring for G. An additive homomorphism $\tau : R \rightarrow \mathbf{Z}$ is called a *semisignature* if for all g in G we have $\tau(\bar{g}) = \pm 1$. Note that since τ is additive, $-\tau(\bar{g}) = \tau(-\bar{g})$ and that since R is additively generated by the elements $\pm\bar{g}$, a semisignature is completely determined by its values on the elements $\bar{g}$, g in G.

Remarks 1.13. (i) Let $R = \mathbf{Z}[G]/K$ and $\bar{R} = \mathbf{Z}[G]/K'$ be two Witt rings for G such that $K \subset K'$. If π denotes the canonical projection $R \rightarrow \bar{R}$, then

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for any semisignature $\bar{\tau}$ of $\bar{R}$, there is, as usual a unique semisignature τ of R making

$$\begin{tikzcd} R \arrow[d, "\pi"'] \arrow[dr, "\tau"] & \\ \bar{R} \arrow[r, "\bar{\tau}"] & \mathbf{Z} \end{tikzcd}$$

commute. Clearly, τ is constant on the cosets of $\bar{M}$ in $\bar{G}'$, where

$$M = \{g' \text{ in } G' | g' \equiv 1 \bmod K'\}.$$

(ii) If $R = W(F)$, where F is a field of characteristic not two, and T' a Präordnung as defined in [**3**, 1.], then Lemma 1.9 (ii) with $M = T' - \{0\}$ immediately shows that the notions of "equivalence" and "strong equivalence" imply each other, which is Satz 7 of [**3**].

Definitions 1.14. Let $A \subset G' \subset \mathbf{Z}[G]$ be a subset of G' and R a Witt ring for G.

(i) The set A is said to be *anisotropic* for R if all elements $\sum_1^n a_i$ of $\mathbf{Z}[G]$ with a_i, not necessarily distinct, elements of A and arbitrary n, are anisotropic for R.

(ii) $D(A) = \{g' \text{ in } G' | g' \text{ in } D(\sum_1^n \bar{a}_i)$ for some, not necessarily distinct, a_i in A and arbitrary $n\}$.

LEMMA 1.15. (i) $A \subset D(A)$.

(ii) *If $A_1 \subset A_2$ then $D(A_1) \subset D(A_2)$.*

(iii) *If A_α is a totally ordered chain of subsets of G', then $\bigcup D(A_\alpha) = D(\bigcup A_\alpha)$.*

Proof. For any g' in G', we have $\dim \bar{g}' = 1$ since the unit g' of $\mathbf{Z}[G]$ cannot be in K. Moreover, in R we have $\bar{g}' + 0 = \bar{g}'$ so that g' is in $D(\bar{g}')$. This proves (i). Parts (ii) and (iii) follow from Definition 1.14 (ii).

LEMMA 1.16. *Let A be an anisotropic subset of G'. Then*

(i) $D(D(A)) = D(A)$.

(ii) $D(A) \cap -D(A) = \emptyset$.

(iii) *$D(A)$ is an anisotropic subset for R of G'.*

Proof. (i) By Lemma 1.15 (i), $A \subset D(A)$, so that by Lemma 1.15 (ii), $D(A) \subset D(D(A))$.

Now let g' be in $D(D(A))$. Then there are elements $d_1, \ldots, d_k$ in $D(A)$ and an element r in R such that $\dim r < k$ and, in R

$$\bar{g}' + r = \sum_1^k \bar{d}_i.$$

Since the d_i's are in $D(A)$, there are elements a_{ij}, $j = 1, \ldots, n_i$, in A and

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elements p_i in R with $\dim p_i < n_i$ such that

$$\bar{d}_i + p_i = \sum_{j=1}^{n_i} \bar{a}_{ij}.$$

Thus in R,

$$\bar{g}' + r + \sum_{i=1}^{k} p_i = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \bar{a}_{ij}.$$

Now, since A is anisotropic, the dimension of the right hand side is $\sum_1^k n_i$. As noted in Definition 1.1,

$$\dim \left(r + \sum_1^k p_i \right) \leqq \dim r + \sum_1^k \dim p_i \leqq (k-1) + \sum_1^k (n_i - 1)$$
$$= \left(\sum_1^k n_i \right) - 1.$$

Hence g' lies in $D(\sum_{i=1}^k \sum_{j=1}^{n_i} \bar{a}_{ij}) \subset D(A)$. Thus $D(D(A)) = D(A)$.

(ii) Suppose g' were in $D(A) \cap -D(A)$. Then there would exist elements $a_1, \ldots, a_n, b_1, \ldots, b_m$ in A and elements p, q in R, with $\dim p < n$, $\dim q < m$ and, in R

$$\bar{g}' + p = \sum_1^n \bar{a}_i, \quad \bar{g}' - q = -\sum_1^m \bar{b}_i.$$

These two equations yield, upon subtraction

$$p + q = \sum_1^n \bar{a}_i + \sum_1^m \bar{b}_i.$$

Since $\dim (p + q) \leqq \dim p + \dim q < n + m$, the last equality contradicts the anisotropy of A. Thus $D(A) \cap -D(A) = \emptyset$.

(iii) Let d_i, $i = 1, \ldots, n$, lie in $D(A)$ and suppose that the element $\sum_1^n d_i$ of $\mathbf{Z}[G]$ is isotropic for R. Since d_1 is anisotropic for R, there exists an integer t, $1 \leqq t < n$ such that $\sum_1^t d_i$ is anisotropic for R but $\sum_1^t d_i + d_{t+1}$ is isotropic for R. By Lemma 1.4, this implies $-d_{t+1}$ is in $D(\sum_1^t \bar{d}_i) \subset D(D(A)) = D(A)$, by (i). But then d_{t+1} lies in $D(A) \cap -D(A)$, contradicting (ii). Hence (iii) is proven.

THEOREM 1.17. *Let R be a Witt ring for G. Then the following are equivalent:*

(i) *For all r in R and all natural numbers m, we have* $\dim (mr) = m (\dim r)$.

(ii) *Let $A \subset G'$ be a subset with the property that all finite sums $\sum a_i$ with a_i distinct elements of A are anisotropic for R. Then there exists a semisignature τ of R with $\tau(\bar{a}) = 1$ for all a in A.*

(iii) *For any finite subset $g_1', \ldots, g_n'$ of G and natural numbers $n_i > 0$, if $\sum_1^n g_i'$ is anisotropic for R, then*

$$D\left(\sum_1^n \bar{g}_i' \right) = D\left(\sum_1^n n_i \bar{g}_i' \right).$$

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Proof. (i) $\Rightarrow$ (ii). Let $a_1, \ldots, a_n$ be distinct elements of A. By hypothesis the element $\sum_1^n a_i$ is anisotropic for R, i.e., has dimension n. Now for any positive natural numbers n_i, $i = 1, \ldots, n$, let m denote their maximum. By (i), $\dim\,(m(\sum_1^n \bar{a}_i)) = mn$ so that by Definition 1.1, the element $\sum_1^n n_i a_i$ is also anisotropic for R. Hence the set A is anisotropic for R in the sense of Definition 1.14 (i).

The remainder of the proof is inspired by [**8**, p. 63]. Consider the family of anisotropic subsets of G' containing $D(A)$ which are of the form $D(A_1)$ with A_1 again anisotropic for R. Since the union of a chain of anisotropic subsets of G' is again anisotropic for R, Lemma 1.15 (iii) shows that Zorn's lemma applies. Thus, let $D(A_0)$ be a maximal element of this family. If $D(A_0) \cup -D(A_0) \neq G'$ let g be an element of G such that neither g nor $-g$ is in $D(A_0)$. If the set $\{D(A_0), g\}$ were isotropic for R, there would exist elements $d_1, \ldots, d_n$ in $D(A_0)$ and a natural number l such that $\sum_1^n d_i + lg$ would be isotropic for R. But by Lemma 1.16 (iii), $D(A_0)$ is anisotropic for R, and so if $\sum_1^n d_i + g$ were isotropic for R, Lemma 1.4 would yield $-g$ in

$$D\left(\sum_1^n \bar{d}_i\right) \subset D(\{d_i\}, i = 1, \ldots, n) \subset D(D(A_0)) = D(A_0)$$

by Lemma 1.16 (i). Hence $\sum_1^n d_i + g$ is anisotropic for R. By (i) of the theorem so is $l \sum_1^n d_i + lg$, and from Definition 1.1, it is clear that then $\sum_1^n d_i + lg$ is also anisotropic for R, a contradiction. Thus, the set $\{D(A_0), g\}$ is anisotropic for R and $D(\{D(A_0), g\}) \supsetneqq D(A_0)$, contradicting the maximality of $D(A_0)$. Hence $G' = D(A_0) \cup -D(A_0)$.

Next, we define an additive homomorphism $\tau_0: \mathbf{Z}[G] \rightarrow \mathbf{Z}$ by $\tau_0(g') = 1$ if g' is in $D(A_0)$ and $\tau_0(g') = -1$ if g' is in $-D(A_0)$. Since, by Lemma 1.16 (ii), $D(A_0) \cap -D(A_0) = \emptyset$, the homomorphism τ_0 is well-defined. Let $\sum_1^n g_i'$ be an element of K, the kernel of the ring surjection $\mathbf{Z}[G] \twoheadrightarrow R$. We reindex so that $g_1', \ldots, g_k'$ lie in $D(A_0)$ and $g_{k+1}', \ldots, g_n'$ lie in $-D(A_0)$. Now in R we have

$$\sum_1^k \bar{g}_i' = \sum_{k+1}^n (-\bar{g}_i').$$

But both g_i', $i = 1, \ldots, k$ and $-g_i'$, $i = k + 1, \ldots, n$ are elements of $D(A_0)$, which, by Lemma 1.16 (iii) is anisotropic. Hence $k = n - k$, so that $\tau_0(K) = 0$ and τ_0 induces a semisignature on R which is 1 on $\overline{D(A_0)} \supset \bar{A}$.

(ii) $\Rightarrow$ (i). Let r be in R and $\dim r = n$ so that $r = \sum_1^n \bar{h}_i'$. The sums $\sum h_i'$ extended over all subsets of $A = \{h_1', \ldots, h_n'\}$ are then clearly anisotropic for R. Thus by (ii) there exists a semisignature $\tau_1 : R \rightarrow \mathbf{Z}$ with $\tau_1(\bar{h}_i') = 1$. Now, clearly for all p in R and any semisignature τ of R, we always have $|\tau(p)| \leqq \dim p$. Thus $m(\dim r) = m\tau_1(r) = \tau_1(mr) \leqq \dim\,(mr)$. Since the opposite inequality has already been noted in Definition 1.1, the implication (ii) $\Rightarrow$ (i) is proven.

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(iii) $\Rightarrow$ (i). Suppose (i) is false. Then there exists an element r in R with $\dim (mr) < m(\dim r)$ for some natural number m. Let r be an element of minimal dimension with this property. Since for all σ in $X(R)$, we have $\sigma(m\bar{g}') = \pm m$, and if $\dim(m\bar{g}') < m$, we would have $|\sigma(m\bar{g}')| < m$, we must have, since $X(R)$ is assumed non-empty, $\dim r > 1$. Thus let $r = \sum_1^n \bar{g}_i'$ with $n = \dim r > 1$. Since the element $\sum_1^n m\bar{g}_i'$ is isotropic for R, there exist natural numbers n_i, $0 < n_i \leqq m$, with $n_i < m$ for at least one index, such that $\sum_1^n n_i g_i'$ is anisotropic for R, but for some k, $1 \leqq k \leqq n$, the element $\sum_1^n n_i g_i' + g_k'$ is not. By Lemma 1.4 this means that $-g_k'$ lies in $D(\sum_1^n n_i\bar{g}_i') = D(r)$ by (iii) of the theorem. Without loss of generality, we assume $k = 1$.

Let $p = \sum_1^{n-1} \bar{g}_i'$. By the hypothesis on r, $\dim (2p) = 2(\dim p) = 2(n-1)$ so that $\sum_1^{n-1} 2g_i'$ is anisotropic for R. Therefore, $2g_1' + \sum_2^{n-1} g_i'$ is also anisotropic for R, whence $\dim (\bar{g}_1' + p) = n$. Now $-g_1'$ in $D(r)$ implies the existence of an element w in R, with $\dim w < n$, such that $r = -\bar{g}_1' + w$. Since

$$2\bar{g}_1' + \sum_2^{n-1} \bar{g}_i' = \bar{g}_1' + p = \bar{g}_1' + (r - \bar{g}_n') = -\bar{g}_n' + w,$$

Definition 1.2 shows that $-g_n'$ lies in $D(\bar{g}_1' + p)$. Again, by (iii) of the theorem $-g_n'$ lies in $D(p)$. But then by Lemma 1.4, the element $\sum_1^n g_i'$ is isotropic for R, a contradiction. Thus (i) must hold.

(i) $\Rightarrow$ (iii). As in the proof of (i) $\Rightarrow$ (ii), if $\sum_1^n g_i'$ is anisotropic for R, so is $\sum_1^n n_i g_i'$ for any $n_i > 0$. From Definitions 1.1 and 1.2 it is clear that if $p = p' + r$ in R with $\dim p = \dim p' + \dim r$ then $D(p') \subset D(p)$, so that $D(\sum_1^n \bar{g}_i') \subset D(\sum_1^n n_i\bar{g}_i')$.

Let g' be in $D(\sum_1^n n_i\bar{g}_i')$ and denote $\sum_1^n \bar{g}_i'$ by p'. If $m = \max (n_i)$, then g' is in $D(mp')$. If g' were not in $D(p')$, Lemma 1.4 implies that the element $\sum_1^n g_i' - g'$ of $\mathbf{Z}[G]$ is anisotropic for R, or $\dim (p' + (-\bar{g}')) = 1 + n$. By (i), then, $\dim (mp' + m(-\bar{g}')) = m + mn$ so that the element $\sum_1^n mg_i' + (-g')$ of $\mathbf{Z}[G]$ is anisotropic for R. But, again by Lemma 1.4, this contradicts g' in $D(mp')$. Hence g' is in $D(p')$ and $D(\sum_1^n n_i\bar{g}_i') \subset D(\sum_1^n \bar{g}_i')$, which proves (iii).

Definition 1.18. A Witt ring for G is called *dimensional* if it satisfies one of the conditions of Theorem 1.17. Note that by 1.17 (i) such a ring is reduced.

Remark 1.19. In Section 2 we shall show that the class of dimensional Witt rings includes the residue class rings of Witt rings of classes of symmetric nondegenerate bilinear forms over semilocal rings modulo the radical. However, not all reduced Witt rings for groups of exponent two are dimensional, nor, since $\mathbf{Z}[G]$ is clearly dimensional, are surjective images of dimensional Witt rings necessarily dimensional, as is shown by the following example:

Let G be the direct product of 8 groups of order 2, i.e. $G = \prod_1^8 \{1, g_i\}$ with $g_i^2 = 1$. Let

$$p_0 = g_1 + g_2 + g_3 - g_1g_2g_3 \quad \text{and} \quad q_0 = g_4 + g_5 + g_6 + g_7 + g_8 - g_4g_5g_6g_7g_8$$

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in $\mathbf{Z}[G]$ and let K be the principal ideal in $\mathbf{Z}[G]$ generated by $2p_0 - q_0$. A character of G is given by mapping the g_i arbitrarily to ± 1. It is then easily verified that for any character χ of G, the induced ring homomorphism ψ_χ [**10**, p. 135] sends p_0 to ± 2 and q_0 to 0, ± 4. Hence for all characters χ of G, we obtain $\psi_\chi(2p_0 - q_0) = 0, \pm 4, \pm 8$. Therefore, by [**10**, Theorem 3.9 (ii)] the ring $R = \mathbf{Z}[G]/K$ is a Witt ring for G, and, thus [**10**, Remark 3.13 (iii)] so is the reduced ring $\bar{R} = \mathbf{Z}[G]/K^{1/2}$, where as in Proposition 1.8 (ii), the radical of K is denoted by $K^{1/2}$. We also note if χ_0 is the identity character, χ_i the character defined by $\chi_i(g_j) = (-1)^{\delta_{ij}}$, and χ_9, the character defined by $\chi_9(g_i) = -1$, then $\psi_{\chi_j}(K^{1/2}) = \psi_{\chi_j}(K) = 0, j = 0, 1, \ldots, 9$. Thus $X(\bar{R}) \neq \emptyset$ and contains, at least, the signatures σ_j induced by $\psi_{\chi_j}, j = 0, 1, \ldots, 9$.

Let the images of p_0, q_0 in $\bar{R}$ be denoted by p, q. Then $2p = q$ in $\bar{R}$, and $\sigma_j(p) = 2$, $\sigma_j(q) = 4$, $j = 0, 1, \ldots, 8$, and $\sigma_9(p) = -2$ and $\sigma_9(q) = -4$. Clearly dim $2p$ = dim $q \leqq 6$. Therefore, to show that R is not dimensional, it suffices to prove that dim $p = 4$.

Since $\sigma_0(p) = 2$, the element p of $\bar{R}$ is not 0 and so dim $p \geqq 1$. If dim $p = 1$, then for all σ in $X(\bar{R})$ we would have $\sigma(p) = \pm 1$, contradicting $\sigma_0(p) = 2$. If dim $p = 3$, then for all σ in $X(\bar{R})$, we would have $\sigma(p) = \pm 1 \pm 1 \pm 1 = \pm 3, \pm 1$, again contradicting $\sigma_0(p) = 2$.

Suppose now dim $p = 2$. Then $p = \pm \bar{w}_1 \pm \bar{w}_2$, where $w_1 = \prod_1^8 g_i^{\epsilon_i}$, $w_2 = \prod_1^8 g_i^{\eta_i}$, with $\epsilon_i, \eta_i = 0$ or 1, are elements of G. Applying σ_0, yields $2 = \pm 1 \pm 1$. Hence $p = \bar{w}_1 + \bar{w}_2$. Applying σ_i, $i = 1, \ldots, 8$, shows $2 = (-1)^{\epsilon_i} + (-1)^{\eta_i}$, so that $\epsilon_i = \eta_i = 0$, and $w_1 = w_2 = 1$. But $p = \bar{1} + \bar{1}$ contradicts $\sigma_9(p) = -2$, so that dim $p \neq 2$. Hence dim $p = 4$ and $\bar{R}$ is a reduced Witt ring which is not dimensional.

COROLLARY 1.20. *Let R be a dimensional Witt ring for G, A a subset of G' anisotropic for R, and $Z = Z(A)$, the (non-empty) subset of semisignatures τ of R with $\tau(\bar{a}) = 1$ for all a in A. Then*

$$\bigcap_{\tau \in Z} \tau^{-1}(1) = \overline{D(A)},$$

the image of $D(A)$ in R, where $\tau^{-1}(1)$ denotes all $\bar{g}'$ in R, g' in G', with $\tau(\bar{g}') = 1$.

Proof. Let g' be an element of G' not in $D(A)$. By Lemma 1.4, for any $a_1, \ldots, a_n$ in A, the element $\sum_1^n a_i + (-g')$ of $\mathbf{Z}[G]$ is anisotropic for R, as is $\sum_1^n n_i a_i + l(-g')$, n_i, l natural numbers, since R is dimensional. Hence the subset $A \cup \{-g'\}$ is anisotropic for R, and so by Theorem 1.17, there exists a semisignature τ_0 of R with $\tau_0(\bar{a}) = \bar{\tau}_0(-\bar{g}') = 1$ for all a in A. Since τ_0 is in Z and $\tau_0(\bar{g}') = -\tau_0(\underline{-\bar{g}'}) = -1$, the element $\bar{g}'$ does not lie in $\bigcap_{\tau \in Z} \tau^{-1}(1)$. If now $\bar{g}'$ does lie in $\overline{D(A)}$, then by Definition 1.2, there are $a_1, \ldots, a_n$ in A and an element r of R with dim $r < n$ and $\bar{g}' + r = \sum_1^n \bar{a}_i$. Now for any semisignature τ of R, we have $|\tau(r)| < n$ and for all τ in Z, $\tau(\bar{g}' + r) = \tau(\bar{g}') + \tau(r) = n$. Hence for all τ in Z, $\tau(\bar{g}') = 1$ and $\tau(r) = n - 1$. Thus $\bar{g}'$ lies in $\bigcap_{\tau \in Z} \tau^{-1}(1)$, proving Corollary 1.20.

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PROPOSITION 1.21. *Let* $G_1 \xrightarrow{\varphi} G_2$ *be a homomorphism of groups of exponent* 2. *Let* $R_i = \mathbf{Z}[G_i]/K_i$, $i = 1, 2$, *be Witt rings for* G_i, *and assume that the induced homomorphism* $\mathbf{Z}[G_1] \to \mathbf{Z}[G_2]$ *sends* K_1 *into* K_2. *Denote the resulting ring homomorphism* $R_1 \to R_2$ *by* φ *also. Then, if* R_2 *is dimensional, a semisignature* τ_1 *of* R_1 *can be extended to a semisignature* τ_2 *of* R_2 *if and only if any element* $\sum \varphi(g_i')$ *of* $\mathbf{Z}[G_2]$ *is anisotropic for* R_2 *for any* g_i' *with* $\tau_1(\bar{g}_i') = 1$.

Proof. To say that τ_2 extends τ_1 means, of course, that for all r_1 in R_1, we have $\tau_1(r_1) = \tau_2(\varphi(r_1))$. Hence, if τ_1 can be extended to τ_2 and $\tau_1(\bar{g}_i') = 1$, then $\tau_2(\sum_1^n \varphi(\bar{g}_i')) = n$, so that $\sum_1^n \varphi(g_i')$ must be anisotropic for R_2.

Conversely, let $A \subset G_1'$ be defined by $\tau_1(\bar{a}) = 1$. Then $G_1' = A \cup -A$. By hypothesis, $\varphi(A)$ is anisotropic for R_2 and so by Theorem 1.17 there is a semisignature τ_2 of R_2 with $\tau_2(\overline{\varphi(A)}) = 1 = \tau_1(\bar{A})$. Now since τ_2 is a homomorphism of additive groups, $\tau_2(\overline{-\varphi(A)}) = \tau_2(\overline{\varphi(-A)}) = -1 = \tau_1(-\bar{A})$. Since any r_1 in R_1 is a sum of elements in A and $-A$, this proves $\tau_2(\varphi(r)) = \tau_1(r)$.

Definition 1.22. A Witt ring R for G satisfies the *Hasse-Minkowski principle* (HMP) if for every r in R, there exists a σ in $X(R)$ with $|\sigma(r)| = \dim r$. Note that if R satisfies HMP, it is necessarily dimensional, and hence reduced.

PROPOSITION 1.23. [**13**, Theorem 2.12, p. 32]. *Let* R *be a dimensional Witt ring for* G. *Then the following are equivalent:*

(i) *R satisfies HMP.*

(ii) *For all* a, b, *in* G', *the element* $1 + a + b - ab$ *of* $\mathbf{Z}[G]$ *is isotropic for* R.

(iii) *Every semisignature* τ *of* R, *with* $\tau(1) = 1$, *is a signature.*

Proof. (i) $\Rightarrow$ (ii). By examining the various possibilities for $\sigma(\bar{a})$, $\sigma(\bar{b})$ it is easily seen that $\sigma(\bar{1} + \bar{a} + \bar{b} - \bar{a}\bar{b}) = \pm 2$. Thus $\dim(\bar{1} + \bar{a} + \bar{b} - \bar{a}\bar{b}) = 2$ and $1 + a + b - ab$ is isotropic for R.

(ii) $\Rightarrow$ (iii) Let τ be a semisignature of R with $\tau(1) = 1$ which is *not* a signature. Then there exist elements a, b in G' with $\tau(\bar{a}) = \tau(\bar{b}) = 1$ but $\tau(\bar{a}\bar{b}) = -1$. Hence $\tau(\bar{1} + \bar{a} + \bar{b} - \bar{a}\bar{b}) = 4$ and so $\dim(\bar{1} + \bar{a} + \bar{b} - \bar{a}\bar{b}) = 4$, contradicting (ii).

(iii) $\Rightarrow$ (i) Let r be an element of R, $\dim r = n$, and $r = \sum_1^n \bar{g}_i'$ for g_i' in G'. Since R is dimensional Theorem 1.17 applies to yield a semisignature τ of R with $\tau(\bar{g}_i') = 1$, $i = 1, \ldots, n$. If $\tau(1) = 1$, then τ is a signature with $\tau(r) = n = \dim r$. If $\tau(1) = -1$, then $-\tau$ is a signature with $|-\tau(r)| = n = \dim r$, completing the proof.

Remark 1.24. As in [**11**, Lemma 3.3], $X(R)$ carries a natural topology induced by the Zariski topology of Spec R. R is said to satisfy *SAP* if every closed and open subset of $X(R)$ is of the form $V(g')$ for g' in G'. The referee has kindly pointed out that for dimensional Witt rings for G, *SAP* and *HMP* are equivalent and we gratefully present his proof here. We first note the following.

LEMMA. *Let* R *be a Witt ring for* G *and* $r = \sum_1^n \bar{g}'$ *an element of* R *with* $\dim_R r = m$. *Then* $n \equiv m \pmod 2$.

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Proof. There exist h_i' in G' such that $r = \sum_1^m \bar{h}_i'$, so that the element $\sum_1^n g_i' - \sum_1^m h_i'$ of $\mathbf{Z}[G]$ lies in $K = \operatorname{Ker}(\mathbf{Z}[G] \to R)$. Let M_0 be the unique maximal ideal of $\mathbf{Z}[G]$ containing 2 [**10**, Lemma 2.13], then by [**10**, Theorem 3.9 (iv)] we have $K \subset M_0$ and so $n - m \equiv 0 \pmod 2$ since $\mathbf{Z}[G] \to \mathbf{Z}[G]/M_0$ is given by setting the elements of G equal to one and reducing mod 2.

PROPOSITION. *Let R be a reduced Witt ring for G. Then R satisfies HMP if and only if R satisfies SAP.*

Proof. Suppose R satisfies *SAP* and that r is an element of R with $\dim_R r = n$. If for all σ in $X(R)$ we have $|\sigma(r)| < n$, then just as in the proof of [**15**, Theorem 3.1] there exist $h_1', \ldots, h_{n-2}'$ in G' with $\sigma(r) = \sigma(\sum_1^{n-2} \bar{h}_i')$ for all σ in $X(R)$. Since R is reduced, $r = \sum_1^{n-2} \bar{h}_i'$ and $\dim_R r \leqq n - 2$. This contradiction shows that for some σ in $X(R)$ we have $|\sigma(r)| = n$ so that R satisfies *HMP*.

Conversely, suppose R satisfies *HMP*. Since the sets $V(g')$, g' in G', form a subbasis of the topology on $X(R)$ and $V(-g') = X(R) - V(g')$, it suffices, as in the proof of Theorem 2.2 of [**15**], to show that for any a, b in G' there exists an element c in G' with $V(a) \cap V(b) = V(c)$. By Proposition 1.23, for any a, b in G', the element $-(1 - a - b - (-a)(-b)) = -1 + a + b + ab$ of $\mathbf{Z}[G]$ is isotropic for R. If $\dim_R(\bar{a} + \bar{b} + \bar{a}\bar{b}) < 3$, then by the lemma, there exists an element d in G' with $\bar{a} + \bar{b} + \bar{a}\bar{b} = \bar{d}$ in R. But then for any signature σ in $V(a) \cap V(b)$, we would get $\sigma(\bar{d}) = 3$, which is impossible. Hence $\emptyset = V(a) \cap V(b) = V(-1)$, in this case. If $\dim_R(\bar{a} + \bar{b} + \bar{a}\bar{b}) = 3$, then by Lemma 1.4, the element 1 lies in $D(\bar{a} + \bar{b} + \bar{a}\bar{b})$. Thus there exists p in R with $1 + p = \bar{a} + \bar{b} + \bar{a}\bar{b}$ and $\dim_R p < 3$. Now, $\dim_R p = 2$, else $\dim_R(\bar{a} + \bar{b} + \bar{a}\bar{b}) < 3$. Thus there are elements c, d in G' with $p = \bar{c} + \bar{d}$ and $1 + \bar{c} + \bar{d} = \bar{a} + \bar{b} + \bar{a}\bar{b}$. Then clearly $V(a) \cap V(b) \subset V(c)$. If σ is any signature not in $V(a) \cap V(b)$, then it is immediate that $\sigma(\bar{a} + \bar{b} + \bar{a}\bar{b}) = -1 = 1 + \sigma(\bar{c}) + \sigma(\bar{d})$, so that $\sigma(\bar{c}) = \sigma(\bar{d}) = -1$. Hence $V(c) \subset V(a) \cap V(b)$, proving the proposition.

PROPOSITION 1.25. *Let Y be a saturated set of signatures for a Witt ring R for G and let $\bar{R} = R/I(Y)$. If $\bar{R}$ is dimensional, then for an element r of R the inequality $|\tau(r)| < \dim_R r$ holds for all semisignatures τ of R that are constant on the cosets of $\overline{\Gamma(Y)}$ in $\bar{G}'$ if and only if $\dim_{\bar{R}}(r + I(Y)) < \dim_R r$.*

Proof. Let $n = \dim r$ and $r = \sum_1^n \bar{g}_i'$. Denoting residue classes modulo $I(Y)$ by $\sim$, we have $\tilde{r} = \sum_1^n \tilde{\bar{g}}_i'$. Thus, if $\dim_{\bar{R}} \tilde{r} = n$, the element $\sum_1^n g_i'$ of $\mathbf{Z}[G]$ is anisotropic for $\bar{R}$. Since $\bar{R}$ is dimensional, Theorem 1.17 yields a semisignature $\bar{\tau}$ of $\bar{R}$ with $\bar{\tau}(\tilde{r}) = n$. By Remark 1.13 (i), there is then a semisignature τ of R satisfying the constancy condition with $\tau(r) = n$. This contradiction shows $\dim_{\bar{R}} \tilde{r} < \dim_R r$. Conversely, suppose there exists a semisignature τ of R constant on cosets of $\overline{\Gamma(Y)}$ in $\bar{G}'$ with $\dim_R r = |\tau(r)|$. By Proposition 1.11 τ then induces a semisignature $\bar{\tau}$ of $\bar{R}$, with $|\bar{\tau}(\tilde{r})| = \dim_R r$ so that $\dim_{\bar{R}} \tilde{r} = \dim_R r$, a contradiction.

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2. The semilocal case. In this section we prove some properties of Witt rings of bilinear forms over semilocal rings that enable us to apply the results of Section 1. Throughout the rest of this paper C will denote a connected semilocal ring and $U(C)$ its group of units. By a *space* over C we shall mean a pair (E, B) where E is a finitely generated projective (whence free) left C-module and B is a symmetric nondegenerate bilinear form on E. Isometries will be written as $\cong$ and for any natural number m, the space $E \perp \ldots \perp E$ (m times) will be denoted by mE. An element e of E is called *primitive* if it can be augmented to a basis of E. A space (E, B) is *isotropic* if there is a primitive element e in E with $B(e, e) = 0$, and *weakly isotropic* if for some natural number m, mE is isotropic. The space $Ce_1 \perp \ldots \perp Ce_n$ with $B(e_i, e_i) = a_i$ in $U(C)$ will, as usual, be denoted by $\langle a_1, \ldots, a_n \rangle$. The Witt ring of equivalence classes of C-spaces will be denoted by $W(C)$ and the class of a space (E, B) in $W(C)$ by $[E]$. For any space (E, B) there always exist $a_1, \ldots, a_n$ in $U(C)$ with $[E] = [\langle a_1, \ldots, a_n \rangle]$, [**10**, Theorem 1.16].

It will also be necessary to consider *quadratic* C-spaces [**12**, pp. 110–111] and the left $W(C)$-module $W_q(C)$ of equivalence classes of quadratic C-spaces [**12**, pp. 110–111 or **1**, Kapitel I]. We shall use similar notations for quadratic spaces as for spaces.

As pointed out in [**8**, p. 49], there always is a natural number h with both $2h - 1$ and $4h - 1$ in $U(C)$. Following one of the ideas of [**8**] we note:

Remark 2.1. Let (F, q) be the quadratic space $Cf_1 \oplus Cf_2$, $q(f_1) = 1$, $q(f_2) = h$, $q(f_1 + f_2) - q(f_1) - q(f_2) = 1$. If (E, B) is a C-space, then $(E \otimes F, Q)$ is a quadratic C-space with $Q(e_1 \otimes f_1 + e_2 \otimes f_2) = B(e_1, e_1) + B(e_1, e_2) + B(e_2, e_2)h$ for e_1, e_2 in E. (The unadorned tensor product sign always means tensor product over C.)

LEMMA 2.2. *Let E, E' be two C-spaces with* rank $E = n > m =$ rank E'. *If, in $W(C)$, the equation $[E] = [E']$ holds, then $6E$ is isotropic; if in addition, 2 is in $U(C)$, then E is isotropic.*

Proof. If 2 is in $U(C)$ this is immediate from the definitions and the Witt cancellation theorem [**14**]. In general, $[E \otimes F] = [E' \otimes F]$ in $W_q(C)$. Now this means that there exist natural numbers k, k' with $E \otimes F \perp k\mathbf{H} \cong E' \otimes F \perp k'\mathbf{H}$, where $\mathbf{H}$ is the quadratic C-module $Cg_1 \oplus Cg_2$ with quadratic form q' given by $q'(g_1) = q'(g_2) = 0$ and $q'(g_1 + g_2) - q'(g_1) - q'(g_2) = 1$ [**1**, p. 31]. But the Witt cancellation theorem holds for quadratic C-modules [**1**, p. 109; **5**, Satz 0.1], and thus since $n + 2k = m + 2k'$ and $n > m$, we obtain $E \otimes F \cong E' \otimes F \perp (k' - k)\mathbf{H}$, with $k' - k > 0$. Hence $E \otimes F$ is isotropic and by [**8**, Lemma 5.14] the space $6E$ is also.

By [**10**, Corollary 1.21], $R = W(C)$ is a Witt ring for the group $U(C)/(U(C))^2$. We shall view the signatures of R as defined in Section 1 either as homomorphisms of R to $\mathbf{Z}$ or as homomorphisms of $U(C)$ to $\{\pm 1\}$ sending $(U(C))^2$ to 1. If Y is a set of signatures of $W(C)$ (or C) we shall slightly alter one of the nota-

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tions of Section 1 and consider $\Gamma(Y)$ as a subset of $U(C)$ instead of $(U(C)/(U(C))^2)'$.

LEMMA 2.3. *Let Y be a saturated set of signatures of $R = W(C)$.*

(i) *If for two spaces E and E'*

$$[E] \equiv [E'] \quad \text{modulo } I(Y),$$

there exists a Pfister form

$$P = \bigotimes_1^k \langle 1, t_i \rangle, \quad t_i \in T = \Gamma(Y),$$

such that in R the equation $[E][P] = [E'][P]$ holds.

(ii) *If, in addition,* rank $E >$ rank E' *then there also exists a Pfister form*

$$P_1 = \bigotimes_1^{k_1} \langle 1, t_{1i} \rangle, \quad t_{1i} \in T,$$

with $E \otimes P_1$ isotropic.

(iii) *A space E is weakly isotropic if and only if there exists a Pfister form*

$$P = \bigotimes_1^k \langle 1, t_i \rangle, \quad t_i \in \Gamma(X(R))$$

such that $E \otimes P$ is isotropic.

Proof. (i) Since Y is saturated, $Y = V(\Gamma(Y))$. In particular, therefore $I(V(T)) = I(V(\Gamma(Y))) = I(Y)$. By Proposition 1.8 (ii) or [**11**, Corollary 4.16] then, $I(Y)$ is the radical of the ideal of $W(C)$ generated by $[\langle 1, -t \rangle]$ for all t in T. Now by [**11**, Lemma 4.17] this is precisely the union of all anihilators in R of the elements $[\bigotimes_1^k \langle 1, t_i \rangle]$, t_i in T, proving (i).

(ii) By (i) we have $[E \otimes P] = [E' \otimes P]$ in R with

$$\text{rank } (E \otimes P) = 2^k \text{ rank } E > 2^k \text{ rank } E' = \text{rank } (E' \otimes P).$$

Hence by Lemma 2.2 the space $6(E \otimes P) \cong E \otimes 6P$ is isotropic. Now since 1 is in T, the space $(\bigotimes_1^3 \langle 1, 1 \rangle) \otimes P = P_1$ is a Pfister form of the desired kind and clearly $E \otimes P_1 \cong E \otimes 6P \perp E \otimes 2P$ is isotropic.

(iii) If mE is isotropic and $2^k \geqq m$ then $2^k E = E \otimes (\bigotimes_1^k \langle 1, 1 \rangle)$ is also isotropic, and, since 1 is in $\Gamma(X(R))$, the implication one way is proved. Conversely, suppose $E \otimes P$ is isotropic for $P = \bigotimes_1^k \langle 1, t_i \rangle$, t_i in $\Gamma(X(R))$. Then for any σ in $X(R)$, we have $\sigma([P]) = 2^k$ and so if $x = [E \otimes P] - [2^k E]$ in R, then $\sigma(x) = 0$ for all σ in $X(R)$. Thus x is a nilpotent element of R. By [**10**, Ex. 3.11], there is, therefore, a natural number s such that $2^s x = 0$. Hence $[2^s(E \otimes P)] = [E \otimes 2^s P] = [2^{k+s} E]$. Since $E \otimes 2^s P$ is still isotropic, [**6**, Satz 3.2.1, p. 18] shows that $E \otimes 2^s P \cong E' \perp M$ with M metabolic and of rank at least 2. Thus rank $E \otimes 2^s P =$ rank $2^{s+k} E >$ rank E'. Therefore Lemma 2.2 applied to the equality $[2^{k+s} E] = [E']$ in R, shows that $3 \cdot 2^{k+s+1} E$ is isotropic.

Remark 2.4. Let $\bar{R} = W(C)/I(Y)$ for a set Y of signatures of $R = W(C)$,

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and set $T = \Gamma(Y)$. Since $[\langle 1 \rangle] \equiv [\langle t \rangle]$ mod $I(Y)$ for all t in T, it is clear that for any subset $\{t_1, \ldots, t_n\}$ of T the images of $[\langle a_1, \ldots, a_n \rangle]$ and $[\langle a_1 t_1, \ldots, a_n t_n \rangle]$ in $\bar{R}$ are the same.

Definition 2.5. For a set of signatures Y of $R = W(C)$ with $T = \Gamma(Y)$ two spaces $\langle a_1, \ldots, a_n \rangle$, $\langle b_1, \ldots, b_m \rangle$ are *T-related* if there exist $t_1, \ldots, t_n, t_1', \ldots, t_m'$ in T such that

$$\sum_1^n [\langle a_i t_i \rangle] = \sum_1^m [\langle b_i t_i' \rangle].$$

COROLLARY 2.6. *Let $E = \langle a_1, \ldots, a_n \rangle$, $E' = \langle b_1, \ldots, b_m \rangle$. Then $[E] \equiv [E']$ mod $I(Y)$ for a saturated set of signatures Y of $R = W(C)$ if and only if there is a natural number l such that lE and lE' are T-related.*

Proof. If $[E] \equiv [E']$ modulo $I(Y)$, Lemma 2.3 (i) shows that $[E \otimes P] = [E' \otimes P]$ with $P = \bigotimes_1^k \langle 1, t_i \rangle$, t_i in T. Then writing P in diagonal form, immediately shows that $2^k E$ and $2^k E'$ are T related since T is a subgroup of $U(C)$.

Conversely, suppose lE and lE' are T-related. By Remark 2.4, it is clear that $l[E] \equiv l[E']$ modulo $I(Y)$. Now $\bar{R} = R/I(Y)$ is an abstract Witt ring with no nonzero nilpotent elements by Proposition 1.8 applied to $I(Y) = I(V(\Gamma(Y)))$. Hence it is torsion free [**10**, Theorem 3.9 and Proposition 3.15] and so $[E] \equiv [E']$ modulo $I(Y)$.

Definition. A commutative semilocal ring C will be said to satisfy (*) if it is connected and all of its residue class fields contain at least 3 elements.

LEMMA 2.7. *Let C denote a semilocal ring satisfying* (*), *Y a nonempty set of signatures of $R = W(C)$, a_i, $i = 1, \ldots, n$ elements of $U(C)$, and t_{ij}', $j = 1, \ldots, m_i$, elements of $T = \Gamma(Y)$. Let*

$$E = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \langle a_i t_{ij}' \rangle,$$

F the quadratic space defined in Remark 2.1, *and denote the quadratic form on $E \otimes F$ by Q.*

(i) *If there is a primitive element z of $E \otimes F$ with $Q(z) = u$ in $U(C)$, then there exists elements $t_1, \ldots, t_n$ in T with $\sum_1^n a_i t_i = u$ and $\sum_1^l a_i t_i$ a unit for all $l < n$.*

(ii) *If $E \otimes F$ is isotropic, then $n > 1$ and there exists elements $t_1, \ldots, t_n$ in T with $\sum_1^n a_i t_i = 0$ and $\sum_1^l a_i t_i$ a unit for all $l < n$.*

(iii) *If E is isotropic, the same conclusions as in* (ii) *hold.*

(iv) *If there is a primitive element e in $E = (E, B)$ with $B(e, e) = u$ in $U(C)$, the same conclusions as in* (i) *hold.*

Proof. (i). Let $z = x \otimes f_1 + y \otimes f_2$ with x and y in E. Then $u = B(x, x) + B(x, y) + B(y, y)h$. We proceed by induction on n. If $n = 1$, then

$$u = a_1 \left(\sum_1^{m_1} t_{1j}'(x_j^2 + x_j y_j + y_j^2 h) \right)$$

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for elements x_j, y_j of C. Then

$$a_1^{-1}u = \sum_1^{m_1} t_{1j}'(x_j^2 + x_jy_j + y_j^2h)$$

is a unit of C with $\sigma(a_1^{-1}u) = 1$ for each σ in $X(R)$ with $\sigma(t_{1j}') = 1$ [**8**, Theorem 2.5]. In particular, therefore $\sigma(a_1^{-1}u) = 1$ for all σ in Y, whence $a_1^{-1}u$ lies in T, proving the statement in case $n = 1$. Suppose that (i) holds for $n - 1$. Noting that $E \otimes F = E_n \otimes F \perp E' \otimes F$ where $E_n = \langle a_nt_{n1}', \ldots, a_nt_{nm_n}' \rangle$ and

$$E' = \sum_{i=1}^{n-1} \sum_{j=1}^{m_i} \langle a_it_{1j}' \rangle$$

we may apply [**2**, Satz 2.7(b)] since both rank $E_n \otimes F$ and rank $E' \otimes F$ are at least two, to obtain the existence of primitive elements z_n in $E_n \otimes F$ and z' in $E' \otimes F$ such that $u = Q(z_n) + Q(z')$ and $Q(z_n)$, $Q(z')$ are units of C. Just as for the case $n = 1$, the unit $a_n^{-1}Q(z_n) = t_n$ lies in T. By induction hypothesis there are elements $t_1, \ldots, t_{n-1}$ in T with $Q(z') = \sum_1^{n-1} a_it_i$ and $\sum_1^l a_it_i$ a unit for all $l < n - 1$. Hence $u = \sum_1^n a_it_i$ and $\sum_1^l a_it_i$ is a unit for all $l < n$.

(ii) Suppose $n = 1$ and $E \otimes F$ is isotropic. If $m_1 > 1$ the quadratic space

$$\perp_1^{m_1} a_1t_{1j}'F,$$

and, in case $m_1 = 1$, the quadratic space $a_1t_{11}'F \perp a_1t_{11}'F$, is isotropic. Again by [**2**, Satz 2.7(b)] we find that a sum of units of the form $t_{1j}'(x_j^2 + x_jy_j + y_j^2h)$, x_j, y_j in C, is zero. Thus

$$-t_{11}'(x_1^2 + x_1y_1 + y_1^2h) = \sum_2^{m} t_{1j}'(x_j^2 + x_jy_j + y_jh^2)$$

$$\text{(or } t_{11}'(x'^2 + x'y' + y'^2h) \text{ in case } m_1 = 1).$$

However by [**8**, Theorem 2.5], for all σ in $X(R)$ with $\sigma(t_{1j}') = 1$ we must have $\sigma(-t_{11}'(x_1^2 + x_1y_1 + y_1^2h)) = 1$, which contradicts, again by [**8**, Theorem 2.5] the fact that $\sigma(t_{11}'(x_1^2 + x_1y_1 + y_1^2h)) = 1$. Thus $n \geqq 2$.

If $n \geqq 2$ we again write $E \otimes F = E_n \otimes F \perp E' \otimes F$ and apply [**2**, Satz 2.7(b)] to yield units $Q(z_n)$, $Q(z')$ such that $0 = Q(z_n) + Q(z')$. Just as in the proof of (i), there is a unit t_n in T with $Q(z_n) = a_nt_n$ and by (i) there are elements $t_1, \ldots, t_{n-1}$ in T with $Q(z') = \sum_1^{n-1} a_it_i$ and $\sum_1^l a_it_i$ a unit for all $l < n - 1$. Since $0 = \sum_1^n a_it_i$, the proof of (ii) is done.

(iii) Let e be a primitive iostropic element of E. Then $e \otimes f_1$ is a primitive isotropic element of $E \otimes F$, and (ii) yields the desired result.

(iv) This follows immediately from (i) since $Q(e \otimes f_1) = B(e, e) = u$.

Remark 2.8. By applying [**2**, Satz 2.7(c)] repeatedly under the hypotheses of Lemma 2.7(ii) it is possible to show, with no hypotheses on the residue class fields of C, that there are units $t_1, \ldots, t_n$ in T such that $\langle a_1t_1, \ldots, a_nt_n \rangle \otimes F$ is

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isotropic, i.e. that there exist x_i, y_i in C with

$$\sum_1^n a_i t_i (x_i^2 + x_i y_i + y_i^2 h) = 0.$$

Indeed, for one index i one can assume $x_i = 1$, $y_i = 0$. However, using this we are unable to prove the if part of Theorem 2.11.

THEOREM 2.9. *Let C be a semilocal ring satisfying* (*), *Y a saturated set of signatures of $R = W(C)$, $T = \Gamma(Y)$ and $E = \langle a_1, \ldots, a_n \rangle$. If in R, the element $[E]$ is congruent modulo $I(Y)$ to the class of a space of rank less than n, there exist $t_1, \ldots, t_n$ in T such that $\sum_1^n a_i t_i = 0$ and $\sum_1^l a_i t_i$ is in $U(C)$ for all $l < n$.*

Proof. Lemma 2.2 and Corollary 2.6 show that there is a natural number s such that sE is T-related to a space E_T with $6E_T$ isotropic. Since $6sE$ is T-related to $6E_T$, there exist t_{ij}', $j = 1, \ldots, s$, in T with $6 \sum_{i=1}^n \sum_{j=1}^s \langle a_i t_{ij}' \rangle$ isotropic. Then Lemma 2.7 (iii) finishes the proof.

LEMMA 2.10. *Let $b_1, \ldots, b_m$ be in $U(C)$, where C is an arbitrary semilocal ring. Assume $\sum_1^m b_i = 0$ and $\sum_1^{m-2} b_i$ is a unit. Then in $W(C)$ we have $[\langle b_1, \ldots, b_m \rangle] = [\langle c_1, \ldots, c_{m-2} \rangle]$ for some units $c_1, \ldots, c_{m-2}$ in C.*

Proof. For e, e' in $\langle b_1, \ldots, b_m \rangle$ we abbreviate $B(e, e')$ by $e \cdot e'$. Let $e_1, \ldots, e_m$ be the orthogonal basis of $\langle b_1, \ldots, b_m \rangle$ with $e_i \cdot e_i = b_i$. Consider the C-module

$$S = Ce_m \oplus C \sum_1^{m-1} (e_j) \subset \langle b_1, \ldots, b_m \rangle.$$

Since $\sum_1^m b_j = 0$ we have $S \cong \langle b_m, -b_m \rangle$. Thus S is nondegenerate. Further, since $\sum_1^{m-1} e_j, e_2, \ldots, e_m$ is still a basis of $\langle b_1, \ldots, b_m \rangle$, the submodule S is a direct summand of $\langle b_1, \ldots, b_m \rangle$. By [**10**, Lemma 1.1] therefore $\langle b_1, \ldots, b_m \rangle = S \perp S^\perp$. The element

$$f = \sum_1^{m-2} e_j - (1/b_{m-1}) \left(\sum_1^{m-2} b_j \right) e_{m-1}$$

is easily seen to be in $S^\perp$. Moreover an easy computation shows that

$$f \cdot f = \left(\sum_1^{m-2} b_j \right) (-b_m / b_{m-1})$$

so that $S^\perp$ is proper. By [**10**, Lemma 1.12], then $S^\perp = \langle c_1, \ldots, c_{m-2} \rangle$, and since $[S] = 0$ in $W(C)$, we obtain $[\langle b_1, \ldots, b_m \rangle] = [\langle c_1, \ldots, c_{m-2} \rangle]$.

THEOREM 2.11. *Let C be a semilocal ring satisfying* (*), *Y a saturated set of signatures of $R = W(C)$, $T = \Gamma(Y)$ and $\bar{R} = R/I(Y)$. For a_i, $i = 1, \ldots, n$, the element $x = \sum_1^n a_i (U(C))^2$ of $\mathbf{Z}[U(C)/(U(C))^2]$ is isotropic for $\bar{R}$ in the sense of Definition* 1.3 *if and only if there exist elements $t_1, \ldots, t_n$ in T with $\sum_1^n a_i t_i = 0$.*

Moreover, if x is anisotropic for $\bar{R}$ and $\bar{x}$ denotes its image in $\bar{R}$, the set $D(\bar{x})$

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defined in Definition 1.2 *consists of the images in* $\mathbf{Z}[U(C)/(U(C))^2]$ *of all units of* C *of the form* $\sum_1^n a_it_i$ *with* t_i *in* T.

In both instances, there always exist $t_1', \ldots, t_n'$ *in* T *with* $\sum_1^n a_it_i' = \sum_1^n a_it_i$ *and* $\sum_1^l a_it_i'$ *a unit for all* $l < n$.

Proof. Let $E = \langle a_1, \ldots, a_n\rangle$. According to Definition 1.3 the element x is isotropic for $\bar{R}$, if in $\bar{R}$ an equation $[\overline{E}] = [\overline{\langle b_1, \ldots, b_m\rangle}]$ with $m < n$ holds. Then Theorem 2.9 yields the desired conclusion.

Conversely, if there are $t_1, \ldots, t_n$ in T with $\sum_1^n a_it_i = 0$, Lemma 2.7 (iii) allows us to assume, in addition, that $\sum_1^l a_it_i$ is in $U(C)$ for all $l < n$. Hence Lemma 2.10 yields the existence of units $c_1, \ldots, c_{n-2}$, in C with $[\langle a_1t_1, \ldots, a_nt_n\rangle] = [\langle c_1, \ldots, c_{n-2}\rangle]$ in R. By Remark 2.4, therefore $[E] \equiv [\langle a_1t_1, \ldots, a_nt_n\rangle] \equiv [\langle c_1, \ldots, c_{n-2}\rangle]$ modulo $I(Y)$, which proves that x is isotropic for $\bar{R}$ in the sense of Definition 1.3.

Finally, by Lemma 1.4, for an element u of $U(C)$, the element $u(U(C))^2$ is represented by $\bar{x}$ if and only if

$$\sum_1^n a_i(U(C))^2 - u(U(C))^2$$

is isotropic for $\bar{R}$. By the first part of this theorem, this occurs if and only if there exist elements $t, t_1, \ldots, t_n$ in T with $\sum_1^n a_it_i - ut = 0$. Since $T = \Gamma(Y)$ is a subgroup of $U(C)$, this is equivalent to $u = \sum_1^n a_it_i'$ with t_i' in T. Clearly by Lemma 2.7 (iv) the t_i' may be chosen so that $\sum_1^l a_it_i'$ is a unit for all $l < n$, completing the proof.

If C is a field of characteristic not 2, Theorem 2.11 shows that our definition of $D(\bar{x})$ coincides with the definition of $D_T(\rho)$ given in [**3**] just before Lemma 2 of [**3**].

COROLLARY 2.12. *For any semilocal ring* C *satisfying* (*) *and a saturated set of signatures* Y *the ring* $\bar{R} = W(C)/I(Y)$ *is dimensional in the sense of Definition* 1.18.

Proof. Let r be an element of $\bar{R}$ with $\dim_{\bar{R}} r = n$. From Definitions 1.1 and 1.3 and Theorem 2.11 it is clear that there are units $a_1, \ldots, a_n$ in C such that $[\langle a_1, \ldots, a_n\rangle]$ is a representative of r in $W(C)$ and $\sum_1^n a_it_i \neq 0$ for all n-tuples $t_1, \ldots, t_n$ of elements of T. Now if for some natural number m, $m \geqq 2$, we had $\dim_{\bar{R}}(mr) < mn$, the space $m\langle a_1, \ldots, a_n\rangle$ would satisfy the hypotheses of Theorem 2.9. Lemma 2.7 (iii) then produces a contradiction.

We end this section by showing that for a saturated set of signatures Y, the ring $W(C)/I(Y)$ coincides with the ring $W_{T'}$ introduced in [**3**] for $T' = \Gamma(Y) \cup \{0\}$ and C a field of characteristic not 2. The first lemma is well known in this case (and relation (iii) is not needed then).

LEMMA 2.13. *Let* C *be a semilocal ring satisfying* (*). *Then* $W(C)$ *is isomorphic to the commutative ring on generators* $\{u\}$, u *in* $U(C)$, *subject only to:*

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(i) $\{1\} = 1$, (ii) $\{1\} + \{-1\} = 0$, (iii) $\{uv^2\} = \{u\}$,
(iv) $\{u\}\{v\} = \{uv\}$, (v) $\{1\} + \{u\} = \{x^2 + y^2u\}(\{1\} + \{u\})$
where (iii) *and* (iv) *hold for all* u, v *in* $U(C)$ *and* (v) *is valid for all* u *in* $U(C)$ *and all* x, y *in* C *such that* $x^2 + y^2u$ *is in* $U(C)$.

Proof. Let H denote the commutative ring described in the lemma. Clearly, (i)-(iv) hold in $W(C)$ if $\{u\}$ is replaced by $[\langle u\rangle]$ throughout. Let e, f denote the canonical basis of $\langle 1, u\rangle$ with $B(e, e) = 1$, $B(e, f) = 0$, $B(f, f) = u$. Provided that $x^2 + y^2u$ is a unit, $\langle 1, u\rangle$ has also $xe + yf$, $yue - xf$ as another orthogonal basis with

$$B(xe + yf, xe + yf) = x^2 + y^2u, \quad B(yue - xf, yue - xf) = u(x^2 + y^2u),$$

i.e. $\langle 1, u\rangle \cong \langle x^2 + y^2u, (x^2 + y^2u)u\rangle$. Thus (v) is also valid in $W(C)$ with $\{\ \ \}$ replaced by $[\langle\ \ \rangle]$ throughout. Hence by universality $\varphi : H \to W(C)$ defined by $\varphi(\{u\}) = [\langle u\rangle]$ and additivity is a ring surjection.

Let $G = U(C)/(U(C))^2$. Relations (i), (iii), and (iv) show that $\psi_0 : \mathbf{Z}[G] \to H$ defined by

$$\psi_0(\textstyle\sum \pm u(U(C))^2) = \sum \pm \{u\}$$

is a ring surjection of $\mathbf{Z}[G]$ onto H. Since (ii) and (v) hold in H, for any u in $U(C)$ and x, y in C with $x^2 + y^2u$ in $U(C)$, we have

$$\begin{aligned}\psi_0((1 + u(U(C))^2)(1 - (x^2 + y^2u)(U(C))^2)) &= 0 \\ &= \psi_0((U)(C))^2) + \psi_0(-(U(C))^2).\end{aligned}$$

But [**10**, Theorem 1.16, Corollary 1.17 and Lemma 1.19] state that the kernel of the projection $\mathbf{Z}[G] \twoheadrightarrow W(C)$ is generated precisely by these elements of $\ker \psi_0$. Thus ψ_0 factors through the homomorphism $\psi : W(C) \to H$ defined by $\psi([\langle u\rangle]) = \{u\}$ and additivity. Now, clearly $\varphi\psi = \psi\varphi = 1$, proving Lemma 2.13.

LEMMA 2.14. *Let* C *be a semilocal ring satisfying* (*). *Let* Y *denote a saturated set of signatures of* $R = W(C)$ *and* $T = \Gamma(Y)$. *Then* $I(Y)$ *is generated as an ideal of* R, *by* $[\langle 1, -t\rangle]$ *for* t *in* T.

Proof. For all σ in Y and t in T, we have $\sigma([\langle 1, -t\rangle]) = 0$, hence $[\langle 1, -t\rangle]$ is in $I(Y)$. Conversely let r be in $I(Y)$. Since Y is saturated, $I(Y) = I(V(\Gamma(Y))) = I(V(T))$. Hence by [**11**, Corollary 4.16 and Lemma 4.17], $I(Y)$ is the union of the annihilators in R of the elements

$$[P] = \left[\bigotimes_{1}^{k} \langle 1, t_i\rangle\right], \quad k \geqq 2, t_i \in T.$$

By [**7**, Theorem 4.1], if F is the quadratic space defined in Remark 2.1, the annihilators in R of the class of the quadratic space $F' = P \otimes F$ is generated as an ideal of R by $[\langle 1, -u\rangle]$ where u in $U(C)$ has the property $uF' \cong F'$. Since F' represents 1, it also represents u and so by the equivalence of (i) and

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(ii) of Theorem 2.5 of [**8**], u lies in

$$\Gamma(V(t_1, \ldots, t_k)) \subset \Gamma(V(T)) = \Gamma(V(\Gamma(Y))) = \Gamma(Y) = T.$$

Now $r[P] = 0$ in $W(C)$ certainly implies $r[F'] = 0$ in $W_q(C)$. Thus r lies in the ideal generated by $[\langle 1, -t\rangle]$, completing the proof.

THEOREM 2.15. *Let C be a semilocal ring satisfying* (*), *Y a saturated set of signatures of $R = W(C)$, $T = \Gamma(Y)$, and $\bar{R} = R/I(Y)$. Then $\bar{R}$ is isomorphic to the commutative ring generated by $\{u\}$ for u in $U(C)$ subject only to*

(i) $\{1\} = 1$, (ii) $\{1\} + \{-1\} = 0$, (iii) $\{ut\} = \{u\}$,

(iv) $\{uv\} = \{u\}\{v\}$, (v) $\{1\} + \{u\} = \{x^2 + y^2u\}(\{1\} + \{u\})$

for all u, v in $U(C)$, all t in T, and all x, y in C such that $x^2 + y^2u$ is a unit.

Proof. By Lemma 2.13 there is an isomorphism $\varphi : R \to H$ where H is the ring defined in Lemma 2.13. Since Lemma 2.14 shows that $I(Y)$ is generated by $[\langle 1, -t\rangle]$ for all t in T, the ideal $\varphi(I(Y))$ in H is generated by $\{1\} - \{t\}$ for all t in T. Hence $\bar{R} \cong H/\varphi(I(Y))$ is isomorphic to the ring described in Theorem 2.15.

In Satz 9 of [**3**] it is shown that the ring $W_{T'}$ studied in that paper has the presentation given in Theorem 2.15. Thus $W_{T'} \cong W(C)/I(Y)$, where C is a field of characteristic not 2, $T' = \Gamma(Y) \cup \{0\}$ and Y a saturated set of signatures of $W(C)$.

3. Applications. In this section the results of the previous two are combined to yield information about $W(C)$ for C semilocal satisfying (*).

Definitions 3.1. (i) A subset $T \subset U(C)$ is called *saturated* if $T = \Gamma(V(T))$, with the notation of Definitions 1.7, and the convention introduced before Lemma 2.3, that $\Gamma(Y)$ is considered as a subset of $U(C)$ instead of $U(C)/(U(C))^2$.

(ii) For A, T subsets of $U(C)$, the subset of elements of C of the form $b = \sum_1^n a_i t_i$, a_i in A, t_i in T, with b a unit or zero, and $\sum_1^l a_i t_i$ a unit for all $l < n$, will be denoted by $S(A, T)$. The set of all semisignatures τ of $W(C)$ (Definition 1.12) satisfying

$$\tau(Tu) = \tau(u) \quad \text{for all } u \text{ in } U(C), \text{ and}$$

$$\tau(a) = 1 \quad \text{for all } a \text{ in } A$$

will be denoted by $Z(A, T)$.

If τ is a semisignature of $W(C)$, we have written in Definition 3.1, and shall continue to write, $\tau(u)$ for $\tau([\langle u\rangle])$; the set of units u of C with $\tau(u) = 1$ will be denoted by $\tau^{-1}(1)$.

THEOREM 3.2. *Let C denote a semilocal ring satisfying* (*), *A and T subsets of $U(C)$, and let T be saturated. Then,*

(i) *$S(A, T)$ does not contain* 0 *if and only if $Z(A, T) \neq \emptyset$.*

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(ii) *If* $Z(A, T) \neq \emptyset$, *then*

$$S(A, T) = \bigcap_{\tau \in Z(A,T)} \tau^{-1}(1).$$

(iii) *If* $Z(A, T) = \emptyset$, *then* $U(C) \cup \{0\} = S(A, T)$.

Proof. (i) Let $R = W(C)$ and $Y = V(T)$. Since $T = \Gamma(V(T))$ we have $V(\Gamma(Y)) = V(\Gamma(V(T))) = V(T) = Y$, so that Y is also saturated. If $\bar{R} = R/I(Y)$, as already noted in the proof of Corollary 2.6, $\bar{R}$ is a reduced Witt ring for $G = U(C)/(U(C))^2$. Furthermore, in case $S(A, T)$ does not contain 0, Theorem 2.11 shows that the image of A in $\mathbf{Z}[G]$ is anisotropic for $\bar{R}$. Since, by Corollary 2.12 the ring $\bar{R}$ is dimensional, Theorem 1.17 yields a semisignature $\bar{\tau}$ of $\bar{R}$ with $\bar{\tau}([\langle a \rangle] + I(Y)) = 1$. By Remark 1.13 (i), $\bar{\tau}$ lifts to an element τ of $Z(A, T)$.

Conversely, let τ be an element of $Z(A, T)$ and for $a_1, \ldots, a_n$ in A let $E = \langle a_1, \ldots, a_n \rangle$. Now by Proposition 1.11, the semisignature τ induces a semisignature $\bar{\tau}$ on $\bar{R} = R/I(Y)$ with $\bar{\tau}([\bar{E}]) = \tau([E]) = n$, so that $\{a(U(C))^2 | a \text{ in } A\}$ is a subset of $\mathbf{Z}[G]$ anisotropic for $\bar{R}$. Consequently by Theorem 2.11, the set $S(A, T)$ does not contain zero.

(ii) By Remark 1.13 (i) and Proposition 1.11 every semisignature $\bar{\tau}$ of $\bar{R}$ for which $\bar{\tau}(\overline{[\langle a \rangle]}) = 1$ for all a in A is induced by an element τ of $Z(A, T)$. Denoting this set of $\bar{\tau}$'s by $\bar{Z}$, Corollary 1.20 shows that $\overline{D(A)}$, the image of $D(A)$ in $\bar{R}$, is $\bigcap_{\bar{\tau} \in \bar{Z}} \bar{\tau}^{-1}(1)$. If for u in $U(C)$, the element $\overline{[\langle u \rangle]}$ is in $\overline{D(A)}$, then by Theorem 2.11 there is a unit v in $S(A, T)$ with $\overline{[\langle u \rangle]} = \overline{[\langle v \rangle]}$. Hence for all σ in Y we have $\sigma(uv^{-1}) = 1$, i.e., uv^{-1} is in T, so that u is in $S(A, T)$ also. Thus, the inverse image of $\overline{D(A)}$ in $U(C)$ is $S(A, T)$. Hence $\bigcap_{\tau \in Z(A,T)} \tau^{-1}(1)$ lies in $S(A, T)$. On the other hand, if $\sum_1^n a_i t_i$ lies in $S(A, T)$, it must be a unit by (i). Then since for all τ in $Z(A, T)$ we have $\tau(a_i t_i) = \tau(a_i) = 1$ the proof of [**11**, Lemma 2.3 (ii)] can easily be adapated to show that $\tau(\sum_1^n a_i t_i) = 1$, i.e.,

$$S(A, T) \subset \bigcap_{\tau \in Z(A,T)} \tau^{-1}(1).$$

Thus the proof of (ii) is complete.

(iii) Let u be in $U(C)$. Since $Z(A, T)$ is empty, by (i), $S(A, T)$ contains 0, i.e., there exist elements $a_1, \ldots, a_n$ in A $t_1', \ldots, t_n'$ in T with $\sum_1^n a_i t_i' = 0$. Exactly as in the proof of Lemma 2.10, the space $\langle a_1 t_1', \ldots, a_n t_n' \rangle$ is isotropic, as is, therefore $\langle -u, a_1 t_1', \ldots, a_n t_n' \rangle$. Since T is saturated we have $\Gamma(Y) = \Gamma(V(T)) = T$. By Lemma 2.7 (iii) then, there are elements $t, t_1, \ldots, t_n$ in T with $ut = \sum_1^n a_i t_i$ and $\sum_1^l a_i t_i$ in $U(C)$ for all $l < n$. This completes the proof of (iii) since $T = \Gamma(Y)$ is a subgroup of $U(C)$, and $S(A, T) \subset U(C) \cup \{0\}$ by definition.

LEMMA 3.3. *Let C be a semilocal ring satisfying* (*), $R = W(C)$ *and* $a_1, \ldots, a_n$ *in* $U(C)$. *If* $E = \langle a_1, \ldots, a_n \rangle$, *then E is weakly isotropic if and only if* 0 *lies in* $S(\{a_1, \ldots, a_n\}, \Gamma(X(R)))$.

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Proof. Clearly $V(\Gamma(X(R))) = X(R)$ so that $\Gamma(X(R))$ is a saturated subset of $U(C)$ containing 1. If mE is isotropic for some natural number m, Lemma 2.7(iii) shows that 0 is in $S(\{a_1, \ldots, a_n\}, \Gamma(X(R)))$. Conversely if 0 lies in this set, there exist $t_1, \ldots, t_n$ in $\Gamma(X(R))$ such that $\sum_1^n a_i t_i = 0$ with $\sum_1^l a_i t_i$ a unit for $l < n$. Let $P = \bigotimes_1^n \langle 1, t_i \rangle$; then $E \otimes P = \langle a_1 t_1, \ldots, a_n t_n \rangle \perp E'$ and thus, as in the proof of Lemma 2.10, is isotropic. Hence by Lemma 2.3 (iii), the space E is weakly isotropic.

PROPOSITION 3.4 [**13**, Lemma 1.24, p. 16]. *Let $C_1 \xrightarrow{\varphi} C_2$ be a homomorphism of connected semilocal rings. Let τ_1 be a semisignature of $W(C_1)$, and denote $\tau_1^{-1}(1) \subset U(C_1)$ by A, and $W(C_2)$ by R. If C_2 satisfies* (*), *τ_1 extends to a semisignature τ_2 of R, i.e., $\tau_1 = \tau_2 W(\varphi)$, if and only if all spaces $\langle \varphi(a_1), \ldots, \varphi(a_n) \rangle$ for a_i in A are anisotropic.*

Proof. By [**10**, Proposition 3.15], $I(X(R))$ is the torsion subgroup of R. Clearly τ_1 extends to R if and only if it extends to $\bar{R} = R/I(X(R))$. By Corollary 2.12, the latter is dimensional and thus Proposition 1.21 shows that τ_1 extends if and only if all elements $\sum \varphi(a_i)(U(C))^2$, a_i in A, are anisotropic for $\bar{R}$. By Theorem 2.11 this is equivalent to 0 not being in $S(\varphi(A), \Gamma(X(R)))$. Lemma 3.3 then completes the proof.

LEMMA 3.5. *Let R be a Witt ring for G and Y a saturated set of signatures of R. Then all the signatures of $\bar{R} = R/I(Y)$ are induced by the elements of Y.*

Proof. Let $\bar{\sigma} : \bar{R} \to \mathbf{Z}$ be a signature and $\sigma : R \to \bar{R} \to \mathbf{Z}$ the lifted signature of R. Then $\ker \sigma \supset I(Y) \supset \mathfrak{a}(\Gamma(Y))$. Thus $\sigma(t) = 1$ for all t in $\Gamma(Y)$, i.e., σ is in $V(\Gamma(Y)) = Y$.

Definition 3.6. Let C be a semilocal ring and Y a set of signatures of $R = W(C)$. The set Y is said to satisfy *HMP* if the following holds: If for every σ in Y we have $|\sigma(E)| <$ rank E for a C-space E, then there exists a Pfister form $P = \prod_1^m \langle 1, t_i \rangle$ with t_i in $\Gamma(Y)$ such that $E \otimes_C P$ is isotropic. If $Y = X(W(C))$, then C is said to satisfy *HMP*, and by Lemma 2.3(iii), C satisfies *HMP* if and only if every C-space for which $|\sigma(E)| <$ rank E for all σ in $X(W(C))$ is weakly isotropic, the usual definition of *HMP* given in [**4**] and [**15**].

PROPOSITION 3.7. *Let C be a semilocal ring satisfying* (*), *$R = W(C)$, and Y a saturated set of signatures of R. Then Y satisfies HMP if and only if $\bar{R} = R/I(Y)$ satisfies HMP in the sense of Definition* 1.22.

Proof. Assume first that $\bar{R}$ satisfies *HMP* and that for a C-space E we have $|\sigma(E)| <$ rank E for all σ in $X(\bar{R})$, which by Lemma 3.5 we identify with Y. Then if $\bar{E}$ denotes the image of $[E]$ in $\bar{R}$ we must have $\dim_{\bar{R}} \bar{E} <$ rank E, since by Definition 1.22 there is a signature σ_0 in Y with $|\sigma_0(E)| = \dim_{\bar{R}} \bar{E}$. According to Definition 1.1, this means that there exist units $b_1, \ldots, b_k$ in C with $k <$ rank E, such that $[E] \equiv [\langle b_1, \ldots, b_k \rangle] \bmod I(Y)$. But then by Lemma 2.3 (ii), there exists a Pfister form $P = \prod_1^m \langle 1, t_i \rangle$, t_i in $\Gamma(Y)$, such that $E \otimes_C P$ is isotropic.

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Conversely, assume that Y satisfies HMP and let the element $\sum_1^n a_i(U(C))^2$ of $\mathbf{Z}[U(C)/(U(C))^2]$ be anisotropic for $\bar{R}$. By Theorem 2.11 this means that for all $t_1, \ldots, t_n$ in $\Gamma(Y)$, all the elements $\sum_1^n a_i t_i$ of C are different from 0. Let $E = \langle a_1, \ldots, a_n \rangle$, then by Lemma 2.7(iii), the spaces $E \otimes_C P$ are anisotropic for all Pfister forms $P = \prod_1^m \langle 1, s_i \rangle$ with s_i in $\Gamma(Y)$. Since Y satisfies HMP, there exists a signature σ in Y with $|\sigma([E])| = n = \dim_{\bar{R}} \bar{E}$, so that $\bar{R}$ satisfies HMP in the sense of Definition 1.22.

COROLLARY 3.8 [**13**, Theorem 2.12, p. 32; **3**, Satz 25(iv)]. *Let C be a semilocal ring satisfying* (*) *and Y a saturated set of signatures of C. Then the following are equivalent:*

(i) *Y satisfies HMP.*

(ii) *For all a, b in $U(C)$, there exists a Pfister form $P = \prod_1^m \langle 1, t_i \rangle$, t_i in $\Gamma(Y)$ with $\langle 1, a, b, -ab \rangle \otimes_C P$ isotropic, so that if $Y = X(W(C))$, by Lemma* 2.3 (ii), *the space $\langle 1, a, b, -ab \rangle$ is weakly isotropic.*

(iii) *Every semisignature τ of $W(C)$ with $\tau(1) = 1$ and constant on cosets of $T = \Gamma(Y)$ is a signature.* (*Note that if $Y = X(W(C))$ the constancy condition on τ is automatically fulfilled by Remark* 1.10(i).)

Proof. Again let $R = W(C)$ and $\bar{R} = R/I(Y)$. By Corollary 2.12, the Witt ring $\bar{R}$ is dimensional and by Proposition 3.7, the set Y satisfies HMP if and only if $\bar{R}$ does. Applying Proposition 1.23 and Lemma 2.3(ii) then yields the implication (i) $\Rightarrow$ (ii), while Proposition 1.23, Lemma 2.7(iii), and Theorem 2.11 yield the implication (ii) $\Rightarrow$ (i). The equivalence of (i) and (iii) again follows from Proposition 1.23 and the fact that, as noted in Proposition 1.11 and Remark 1.13(i), the semisignatures of $\bar{R}$ are induced by the semisignatures of R constant on the cosets of $T = \Gamma(Y)$ in $U(C)/(U(C))^2$.

Remark 3.9. We shall say that a saturated set of signatures Y of $R = W(C)$ satisfies SAP if every subset of Y closed and open in the induced Zariski topology is of the form $V(u) \cap Y$ for u in $U(C)$. By Proposition 3.7, Lemma 3.5, Corollary 2.12, and Remark 1.24, a saturated subset Y satisfies SAP if and only if it satisfies HMP, provided C satisfies (*). In case $Y = X(R)$ this yields Theorem 3.1 of [**15**] for semilocal rings satisfying (*) and not just for semilocal rings with 2 in $U(R)$. Of course, by using a small amount of quadratic form theory, Knebusch [**9**], has proved [**15**, Theorem 3.1] for arbitrary connected semilocal rings.

THEOREM 3.10. *Let C be a connected semilocal ring satisfying* (*), *Y a saturated set of signatures of $R = W(C)$, $T = \Gamma(Y)$, $\bar{R} = R/I(Y)$, and E a C-space of rank n. If for all semisignatures τ of R constant on cosets of T in $(U(C)/(U(C))^2)$, we have $|\tau([E])| < n$, there exist $t_1, \ldots, t_k$ in T such that $E \otimes (\bigotimes_1^k \langle 1, t_i \rangle)$ is isotropic.*

Proof. As in the proof of Proposition 3.7, we denote the image of $[E]$ in $\bar{R}$ by $\bar{E}$. Let $m = \dim_{\bar{R}} \bar{E}$. Thus there are elements $a_1, \ldots, a_m$ in $U(C)$ such that

$$(3.11) \quad [E] \equiv [\langle a_1, \ldots, a_m \rangle] \quad \text{modulo } I(Y).$$

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By Definition 1.1 we also have

$$\dim_{\bar{R}}\overline{\langle a_1, \ldots, a_m \rangle} = m = \dim_R[\langle a_1, \ldots, a_m\rangle].$$

Suppose first that $m \geqq \operatorname{rank} E$. Now for all semisignatures τ of R constant on cosets of T we have, by Proposition 1.11 and Remark 1.13 (i) that $\tau([\langle a_1, \ldots, a_m\rangle]) = \tau([E])$. Hence the hypothesis of the theorem shows that

$$|\tau([\langle a_1, \ldots, a_m\rangle])| < m = \dim_{\bar{R}}(\overline{[\langle a_1, \ldots, a_m\rangle]}).$$

Therefore, Proposition 1.25 and Corollary 2.12 show that $\dim_R(\langle a_1, \ldots, a_m\rangle) < m$, a contradiction. Therefore $m < \operatorname{rank} E$. But then Lemma 2.3(ii) applied to (3.11) completes the proof.

COROLLARY 3.12. *If, in Theorem* 3.10 *the space* $E = \langle b_1, \ldots, b_n\rangle$, *with* b_i *in* $U(C)$, *then* 0 *lies in* $S(\{b_1, \ldots, b_n\}, T)$.

Proof. By Theorem 3.10 there exist $t_1, \ldots, t_k$ in T with $\langle b_1, \ldots, b_n\rangle \otimes \langle 1, t_1\rangle \otimes \ldots \otimes \langle 1, t_k\rangle$ isotropic. Multiplying out and applying Lemma 2.7(iii) yields the result.

In case $Y = X(R)$, $T = \Gamma(X(R))$ Theorem 3.10 furnishes another proof of [**8**, Theorem 5.13] in the case of trivial involution.

COROLLARY 3.13. *Let C be a semilocal ring satisfying* (*). *If for a C-space E we have* $|\tau(E)| < \operatorname{rank} E$ *for all semisignatures τ of $R = W(C)$, then E is weakly isotropic.*

Proof. By Remark 1.10(i) we may apply Theorem 3.10 for $Y = X(R)$. Hence there are $t_1, \ldots, t_k$ in $\Gamma(X(R))$ such that $E \otimes \langle 1, t_1\rangle \otimes \ldots \otimes \langle 1, t_k\rangle$ is isotropic. By Lemma 2.3(iii), the space E is then weakly isotropic.

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