COMPARISON THEOREMS FOR THE SQUARE INTEGRABILITY OF SOLUTIONS OF $(r(t)y')'+q(t)y=f(t, y)$

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1. Introduction. Bellman [1], [2, p. 116] proved that, if all solutions of the equation

$$y''+q(t)y=0 \tag{1}$$

are in $L^2(a, \infty)$ and $b(t)$ is bounded, then all solutions of

$$y''+(q(t)+b(t))y=0$$

are also in $L^2(a, \infty)$. The purpose of this paper is to present conditions on the function f that guarantee that all solutions of

$$(r(t)y')'+q(t)y=f(t, y) \tag{2}$$

be in the class $L^2(a, \infty)$ whenever all solutions of the equation

$$(r(t)y')'+q(t)y=0 \tag{3}$$

have this property. It is assumed that $r(t)>0$, r and q are continuous on a half line (a, ∞) and f is continuous. Actually the continuity assumptions may be weakened to local integrability and $L^2(a, \infty)$ may be replaced by $L^p(a, \infty)$ for any $p>1$.

The main results are contained in Theorems 1 and 2.

THEOREM 1. *Assume that all solutions of* (3) *are in* $L^2(a, \infty)$ *and that there exist non-negative measurable functions* k_1 *and* k_2 *such that* $|f(t, u)| \leqq k_1(t)+k_2(t)u$. *If* $yk_2^{1/2}$ *is in* $L^2(a, \infty)$ *and* yk_1 *is integrable on* (a, ∞) *for all solutions* y *of* (3), *then all solutions of* (2) *are in* $L^2(a, \infty)$.

If $r(t)\equiv 1$, $f(t, u)=-b(t)u$, then Theorem 1 is contained in a theorem of Halvorsen [4, Theorem 1], and, if b is also bounded, then Theorem 1 reduces to the result of Bellman cited above.

The second result of this paper completely extends Halvorsen's Theorem 1 to a self-adjoint equation of the form (3). This seems interesting since the known transformations for changing (3) to the normal form (1) do not preserve the square integrability of solutions (see §3); it is also easier to find examples of the limit circle case for the self-adjoint form (3). In addition, the proof given here is more straightforward since a Prüfer type transformation is not needed.

The usual meaning of the limit circle and limit point classification for equation (3) is maintained: equation (3) is in the limit circle or limit point case according as all solutions are in $L^2(a, \infty)$ or at most one (linearly independent) solution is in $L^2(a, \infty)$. (See [5].)

THEOREM 2. *If* b *is a real-valued continuous function with the property that* $y|b|^{1/2}$ *is in* $L^2(a, \infty)$ *for all solutions* y *of* (3), *then* $u|b|^{1/2}$ *is in* $L^2(a, \infty)$ *for all solutions* u *of*

$$(r(t)u')'+(q(t)+b(t))u=0. \tag{4}$$

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If $y|b|^{1/2}$ *is in* $L^2(a, \infty)$ *for all solutions* y *of* (3), *then* (4) *is in the limit circle or limit point case according as* (3) *is in the limit circle or limit point case.*

2. Proofs of the theorems. The proofs of Theorems 1 and 2 rely on the following lemma which is a corollary to a version of the Gronwall inequality recently proved by H. E. Gollwitzer [3].

LEMMA. *Let* u, ϕ, g, *and* h *be nonnegative continuous functions on an interval* $[a, b]$, *let* α, β *be positive continuous functions such that* $\alpha(t)+\beta(t) = 1$, *let* $1 \leqq p < \infty$ *and suppose that*

$$u(t) \leqq \phi(t)+g(t)\left[\int_a^t (u(s))^p h(s)\,ds\right]^{1/p} \qquad (a \leqq t \leqq b).$$

Then

$$\int_a^t (u(s))^p h(s)\,ds \leqq \int_a^t \alpha(s)(\phi(s)\alpha^{-1}(s))^p h(s)\exp\left(\int_s^t \beta(x)(g(x)\beta^{-1}(x))^p h(x)\,dx\right)ds, \tag{5}$$

from which it follows that

$$u(t) \leqq \phi(t)+g(t)\left[\int_a^t \alpha(s)(\phi(s)\alpha^{-1}(s))^p h(s)\exp\left(\int_s^t \beta(x)(g(x)\beta^{-1}(x))^p h(x)\,dx\right)ds\right]^{1/p} \tag{6}$$

Proof of Theorem 1. Let y_1 and y_2 be solutions of (3) such that

$$r(t)(y_1(t)y_2'(t)-y_1'(t)y_2(t)) \equiv 1$$

and suppose that y is any solution of (2). Then, on using variation of parameters, $y(t)$ may be expressed as

$$y(t) = c_1 y_1(t)+c_2 y_2(t)+\int_a^t [y_1(s)y_2(t)-y_1(t)y_2(s)]f(s, y(s))\,ds.$$

By hypothesis $k_1 y_1$ and $k_1 y_2$ are integrable, so that there are constants K_1, K_2 such that

$$\int_a^t k_1|y_1| \leqq K_1 \quad \text{and} \quad \int_a^t k_1|y_2| \leqq K_2$$

for all $t \geqq a$. Hence

$$\begin{aligned}|y(t)| &\leqq d_1|y_1(t)|+d_2|y_2(t)|\\ &\quad +|y_2(t)|\int_a^t |y_1(s)|k_2(s)|y(s)|\,ds+|y_1(t)|\int_a^t |y_2(s)|k_2(s)|y(s)|\,ds,\end{aligned} \tag{7}$$

where $d_1 = |c_1|+K_2$ and $d_2 = |c_2|+K_1$. On using the Schwarz inequality, (7) may be written as

$$\begin{aligned}|y(t)| &\leqq d_1|y_1(t)|+d_2|y_2(t)|\\ &\quad +\left[|y_2(t)|\left(\int_a^t y_1^2(s)k_2(s)\,ds\right)^{1/2}+|y_1(t)|\left(\int_a^t y_2^2(s)k_2(s)\,ds\right)^{1/2}\right]\left(\int_a^t k_2(s)y^2(s)\,ds\right)^{1/2}.\end{aligned} \tag{8}$$

By hypothesis there are constants M_1, M_2 such that $\left(\int_a^t y_1^2(s)k_2(s)\,ds\right)^{1/2} \leqq M_1$ and $\left(\int_a^t y_2^2(s)k_2(s)\,ds\right)^{1/2} \leqq M_2$ for $t \geqq a$, so that (8) becomes

$$|y(t)| \leqq \phi(t)+g(t)\left(\int_a^t k_2(s)y^2(s)\,ds\right)^{1/2}, \tag{9}$$

where $\phi(t) = d_1|y_1(t)|+d_2|y_2(t)|$ and $g(t) = M_1|y_2(t)|+M_2|y_1(t)|$. But ϕ is the absolute value of a solution of (3) and therefore $\phi k_2^{1/2}$ is in $L^2(a, \infty)$; similarly $gk_2^{1/2}$ is in $L^2(a, \infty)$. If $\alpha(t) \equiv \beta(t) \equiv \frac{1}{2}$, $h(t) = k_2(t)$, then it follows from the lemma that $y(t)$ is bounded by a linear combination of ϕ and g; therefore y is in $L^2(a, \infty)$. (Perhaps it should be mentioned that the fact y is bounded by a linear combination of ϕ and g also implies that y exists on (a, ∞).)

Proof of Theorem 2. If u is a solution of (4), then (7) holds with y replaced by u, $k_2(t)$ replaced by $b(t)$, $d_1 = |c_1|$, $d_2 = |c_2|$ and y_1, y_2 as in the proof of Theorem 1. After using the Schwarz inequality and this time multiplying by $|b|^{1/2}$, (7) becomes

$$|b(t)|^{1/2}|u(t)| \leqq \phi(t)+g(t)\left(\int_a^t |b|k^2\right)^{1/2}, \tag{10}$$

where $\phi(t) = |b(t)|^{1/2}(d_1|y_1(t)|+d_2|y_2(t)|)$, $g(t) = |b(t)|^{1/2}(M_1|y_2(t)|+M_2|y_1(t)|)$ and M_1, M_2 are as in the proof of Theorem 1. It now follows from (10) and (5), with $h(t) \equiv 1$, $\alpha(t) \equiv \beta(t) \equiv \frac{1}{2}$, that $\int_a^\infty |b(t)|u^2(t) < \infty$. This establishes the first part of Theorem 2; the second part follows from the first part and Theorem 1. Indeed, it is clear that, if (3) is limit-circle and $y|b|^{1/2}$ is in $L^2(a, \infty)$ for all solutions of (3), then (4) is limit-circle. On the other hand, if $y|b|^{1/2}$ is in $L^2(a, \infty)$ for all solutions of (3), then the same is true for all solutions of (4); therefore, since (3) is obtained from (4) by adding $-b(t)$ to the coefficient of y in (4), it follows that (3) is limit-circle when (4) is.

3. Examples and remarks.

The Euler equation

$$(t^6y')'+6t^4y = 0 \tag{11}$$

has the linearly independent solutions $y_1(t) = 1/t^2$ and $y_2(t) = 1/t^3$ which are in $L^2(1, \infty)$. Theorem 1 implies that all solutions of the equation

$$(t^6y')'+(6t^4+t^2)y = d \qquad (d = \text{constant}) \tag{12}$$

are in $L^2(1, \infty)$, which shows that the perturbation $b(t)$ may grow rather fast.

One may attempt to apply Halvorsen's Theorem to equation (11) by writing it as

$$y''+w(t)y'+q(t)y = 0 \tag{13}$$

and then using the transformation

$$u = y\exp\left(\tfrac{1}{2}\textstyle\int w\right), \tag{14}$$

which transforms equation (13) into

$$u''+(q(t)-w'(t)/2-w^2(t)/4)u=0. \tag{15}$$

The equation that results from writing (11) in the form of (13) and applying (14) is the equation $u''=0$, and, since the solutions of this equation are not in $L^2(1,\infty)$, Halvorsen's Theorem does not apply.

Similar difficulties are encountered when attempting to use other well-known transformations to put (3) in the form of (1) and then apply Halvorsen's Theorem.

(The referee made the interesting observation that when the parameter λ is introduced, the standard form of equation (11) is

$$-(t^6y')'-6t^4y=\lambda y, \tag{16}$$

which is in the limit-circle case on $[1,\infty)$. The effect of the transformation (14) is to take (16) into

$$-u''=\lambda t^{-6}u, \tag{17}$$

which, in the weighted integrable-square space with weight function t^{-6}, is still in the limit-circle case; that is,

$$\int_1^\infty t^{-6}|u|^2\,dt<\infty \tag{18}$$

for all solutions u of (17). This follows from the fact that, when $\lambda=0$, the equation (17) has solutions $u_1(t)=1$ and $u_2(t)=t$ and (18) holds for these solutions.)

The following example shows that the perturbation term cannot grow too fast relative to $q(t)$. Two linearly independent solutions of

$$(t^3y')'+ty=0$$

are $y_1(t)=1/t$ and $y_2(t)=(\log t)/t$, and these solutions are in $L^2(1,\infty)$. If $b(t)=-t$, then the perturbed equation becomes $(t^3y')'=0$ which has $y(t)=1$ as a solution.

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