Canad. J. Math. Vol. **62** (2), 2010 pp. 382–399
doi:10.4153/CJM-2010-022-1

Verma Modules over Quantum Torus Lie Algebras

Rencai Lü and Kaiming Zhao

Abstract. Representations of various one-dimensional central extensions of quantum tori (called quantum torus Lie algebras) were studied by several authors. Now we define a central extension of quantum tori so that all known representations can be regarded as representations of the new quantum torus Lie algebras $\mathfrak{L}_q$. The center of $\mathfrak{L}_q$ now is generally infinite dimensional.

In this paper, $\mathbb{Z}$-graded Verma modules $\widetilde{V}(\varphi)$ over $\mathfrak{L}_q$ and their corresponding irreducible highest weight modules $V(\varphi)$ are defined for some linear functions φ. Necessary and sufficient conditions for $V(\varphi)$ to have all finite dimensional weight spaces are given. Also necessary and sufficient conditions for Verma modules $\widetilde{V}(\varphi)$ to be irreducible are obtained.

1 Introduction

In order to better apply Lie theory to various mathematics and physics fields, one of the main tasks is the construction of "good" modules of Lie algebras. The relation to physics is well established in the book [10] on conformal fields theory. Recently there has been substantial activity in developing weight representation theory for higher rank infinite dimensional Lie algebras with a lot of deep results. Here we can only list a few. For representations of toroidal Lie algebras see [4–6, 11, 12, 14, 15, 20]; for extended affine Lie algebras, see [2, 3, 6, 16, 17] and for quantum torus Lie algebras, see [8, 13, 16–18, 21].

Quantum torus algebras were introduced to ring theory in [22] in 1988. They were used in describing extended affine Lie algebras [1]. In the above mentioned studies on representations of quantum torus Lie algebras, except in [8], quantum tori were assumed to have n commutative variables among the $n+1$ variables, that essentially can be considered as two variables. In [16–18] level one (central charge is 1) vertex representations were constructed. In [13], highest weight representations with finite dimensional weight spaces were constructed, where the central charge can be any complex number. In [21], the authors proved that, for exactly the two-variable case with nonzero central charge, $\mathbb{Z}^2$-graded simple modules with all finite dimensional weight spaces are highest weight modules. In [8], the authors constructed vertex representations with positive integral level over the algebras with more variables not commutative.

As in [8], in the present paper, we study $\mathbb{Z}$-graded modules over quantum torus Lie algebras with more variables not commutative. It is natural and interesting to study when we can have highest weight modules with all finite dimensional weight

Received by the editors June 14, 2007.
Published electronically December 4, 2009.
This research was supported by NSERC, Postdoctoral Research Grant, and NSF of China (Grants 10431040, 10601057).
AMS subject classification: **17B10**, 17B65, 17B68.

spaces for any central charge, and when the Verma modules are irreducible. This is the main purpose of the present paper.

Let us recall the construction of highest weight modules over $\mathbb{Z}$-graded Lie algebras constructed in [8]. We shall denote the set of integers, non-negative integers, positive integers, the complex numbers by $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{N}$, $\mathbb{C}$ respectively.

Let L be a complex $\mathbb{Z}^{n+1}$-graded Lie algebra and $L = L_- \oplus L_0 \oplus L_+$ be the generalized triangular decomposition (L_0 can be infinite and non-abelian) relative to a $\mathbb{Z}$-gradation. For any L_0-module A, let L_+ act on A trivially. We introduce the induced module

$$\widetilde{M}(A) := \operatorname{Ind}_{L_0+L_+}^{L} A \simeq U(L^-) \otimes_{\mathbb{C}} A.$$

Then $\widetilde{M}(A)$ is $\mathbb{Z}$-graded. Clearly $\widetilde{M}(A)$ contains a unique maximal proper $\mathbb{Z}$-graded submodule $J(A)$ trivially intersecting with A. Thus we have the $\mathbb{Z}$-graded quotient module $M(A) := \widetilde{M}(A)/J(A)$. In general, $M(A)$ has infinite dimensional weight spaces. So it is meaningful to find necessary and sufficient conditions for $M(A)$ to have all finite dimensional weight spaces.

Let $\varphi\colon L_0 \to \mathbb{C}$ be any Lie homomorphism, and define the associated L_0-module $A = \mathbb{C}v_0$ by $g \cdot v_0 = \varphi(g)v_0, \forall g \in L_0$. We will denote $\widetilde{V}(\varphi) := \widetilde{M}(A)$, $J(\varphi) := J(A)$ and $V(\varphi) := M(A)$.

Now we recall quantum torus Lie algebras.

Let $q = (q_{i,j})_{i,j=0}^{n}$ be an $(n+1) \times (n+1)$ matrix over $\mathbb{C}$ satisfying

$$q_{i,i} = 1, \quad q_{i,j} = q_{j,i}^{-1},$$

where n is a positive integer. The q-quantum torus $\mathbb{C}_q = \mathbb{C}_q[t_0^{\pm 1}, \dots, t_n^{\pm 1}]$ which was studied in [22] is the unital associative algebra over $\mathbb{C}$ generated by $t_0^{\pm 1}, \dots, t_n^{\pm 1}$ and subject to the defining relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1, \quad t_i t_j = q_{i,j} t_j t_i.$$

For any $a \in \mathbb{Z}^{n+1}$ we always write $a = (a(0), \dots, a(n))$, and define $t^a = t_0^{a(0)} \cdots t_n^{a(n)}$. For any $a, b \in \mathbb{Z}^{n+1}$, we define the function $\sigma_q(a, b)$ and $f_q(a, b)$ by

$$t^a t^b = \sigma_q(a, b) t^{a+b}, \quad t^a t^b = f_q(a, b) t^b t^a.$$

Then

$$\sigma_q(a, b) = \prod_{0 \le i < j \le n} q_{j,i}^{a(j)b(i)}, \quad f_q(a, b) = \prod_{i,j=0}^{n} q_{j,i}^{a(j)b(i)},$$

and $f_q(a, b) = \sigma_q(a, b)\sigma_q(b, a)^{-1}$. We define

$$\operatorname{rad} f_q = \{a \in \mathbb{Z}^{n+1} \mid f_q(a, \mathbb{Z}^{n+1}) = 1\},$$

and the Kronecker delta

$$\delta_{a,\operatorname{rad} f_q} = \begin{cases} 1 & \text{if } a \in \operatorname{rad} f_q, \\ 0 & \text{otherwise.} \end{cases}$$

For properties of C_q, f_q, σ_q please refer to [1, 23].

We define the Lie algebra

$$\mathfrak{L}_q = \mathbb{C}_q[t_0^{\pm 1}, \dots, t_n^{\pm 1}] + \mathbb{C}d_0$$

with defining relations:

$$[t^a, t^b] = t^a t^b - t^b t^a + \delta_{a(0)+b(0),0}\delta_{a+b,\mathrm{rad}\, f_q} a(0) t^a t^b \tag{1.1}$$

$$= (\sigma_q(a,b) - \sigma_q(b,a) + \delta_{a(0)+b(0),0}\delta_{a+b,\mathrm{rad}\, f_q}\sigma_q(a,b)a(0))t^{a+b}, \tag{1.2}$$

$$[d_0, t^a] = a(0)t^a, \forall a, b \in \mathbb{Z}^{n+1}, \tag{1.3}$$

which we refer to as *the general quantum torus Lie algebra associated with q*.

Note that if $a+b \in \mathrm{rad}\, f_q$, then $t^a t^b - t^b t^a = 0$ and $\sigma_q(a,b) - \sigma_q(b,a) = 0$. Unlike the Lie algebra constructed directly from the associative algebra $\mathbb{C}_q$, the Lie algebra $\mathfrak{L}_q$ is perfect.

For any Lie algebra L, denote the center of L by $Z(L)$. Clearly we have

$$Z(\mathfrak{L}_q) = \mathrm{span}\{t^a \mid a \in \mathrm{rad}\, f_q \text{ and } a(0) = 0\}.$$

The Lie algebra $\widehat{\mathbb{C}}_q$ defined in [8, (1.6)] and [13, (1.4)] is the quotient algebra

$$\widehat{C}_q = \mathfrak{L}_q / \langle t^a - t^b \mid t^a, t^b \in Z(\mathfrak{L}_q)\rangle.$$

The Lie algebra $\widetilde{\mathbb{C}}_q$ defined in [13, (1.2)] is the quotient algebra

$$\widetilde{C}_q = \mathfrak{L}_q / \langle t^a \mid t^a \in Z(\mathfrak{L}_q), a \neq (0, \dots, 0)\rangle.$$

And the Lie algebra $\widetilde{\mathbb{C}}_q^{(l)}(m)$ defined in [8, (1.6)′] is the quotient algebra

$$\widetilde{C}_q^{(l)}(m) = \mathfrak{L}_q / \langle t^a - t^b, t^c \mid t^a, t^b, t^c \in Z(\mathfrak{L}_q) \text{ with } a(l), b(l) \in m\mathbb{Z} \text{ and } c(l) \notin m\mathbb{Z}\rangle,$$

where m is a nonnegative integer and $l \in \{1, 2, \dots, n\}$.

Note that $\mathfrak{L}_q$ is $\mathbb{Z}^{n+1}$-graded, and $\mathfrak{L}_q$ has a $\mathbb{Z}$-gradation with respect to d_0.

The advantage to introducing our algebra $\mathfrak{L}_q$ is that we can handle all cases at the same time, unlike in [8, 13] where the cases had to be treated separately. Furthermore, we will have a richer representation theory (more representations because of the bigger center) for our algebras $\mathfrak{L}_q$ than the old ones. This is our main motivation for introducing Lie algebras $\mathfrak{L}_q$.

Vertex operator representations and highest weight representations of some of these Lie algebras $\widetilde{C}_q, \widehat{C}_q, \widetilde{C}_q^{(l)}(m)$ were studied in [2, 3, 8, 16, 17, 23].

For $n = 1$, necessary and sufficient conditions for $V(\varphi)$ over $\widetilde{C}_q$ and $\widehat{C}_q$ to have finite weight spaces were obtained in [13]. The nonzero level $\mathbb{Z} \times \mathbb{Z}$ Harish-Chandra modules were studied in [21].

https://doi.org/10.4153/CJM-2010-022-1 Published online by Cambridge University Press

In the present paper, necessary and sufficient conditions for $V(\varphi)$ over $\mathfrak{L}_q$ to have all finite dimensional weight spaces are given in Theorem 2.10, in which case we give the concrete expressions of φ in Theorem 2.11. In proving this we must use the concept of exp-polynomial Lie algebras introduced and studied in [7] and the results therein. Necessary and sufficient conditions for a Verma module to be irreducible is obtained in Theorem 3.1. There the technique we use is the following. We write a vector-valued function defined on $\mathbb{Z}^{n+1}$ into a sum of two parts. If one part is zero at some points in $\mathbb{Z}^{n+1}$, then we can deduce it is identically zero. Therefore the other part must be identically zero.

2 Highest Weight Representations for $\mathfrak{L}_q$

In this section, we shall give necessary and sufficient conditions for irreducible highest weight modules over $\mathfrak{L}_q$ to have all finite dimensional weight spaces. Before starting the proof, we need some preparations.

We will simply denote $\mathfrak{L} = \mathfrak{L}_q$, $f = f_q$, $\sigma = \sigma_q$. Then

$$\mathfrak{L}_i = \bigoplus_{\substack{a\in\mathbb{Z}^{n+1}\\ a(0)=i}} \mathbb{C}t^a + \delta_{i,0}\mathbb{C}d_0.$$

For convenience, we will always use the following symbols:

$$\begin{aligned}
\varepsilon_i &= (\delta_{i,0}, \delta_{i,1}, \dots, \delta_{i,n}), \quad i = 0, \dots, n,\\
\bar{a} &= (a(1), \dots, a(n)) \in \mathbb{Z}^n, \quad \forall a \in \mathbb{Z}^{n+1},\\
\bar{q} &= (q_{i,j})_{i,j=1,\dots,n},\\
\mathbb{C}_{\bar{q}} &= \mathbb{C}_{\bar{q}}[t_1^{\pm}, \dots, t_n^{\pm}] \subseteq \mathbb{C}_q,\\
\operatorname{rad} \bar{f} &:= \{a \in \mathbb{Z}^{n+1} \mid t^a \in Z(\mathfrak{L}_0)\} \subset \{0\} \times \mathbb{Z}^n,\\
\operatorname{rad}_0 f &:= \{a \in \operatorname{rad} f \mid a(0) = 0\} \subseteq \operatorname{rad} \bar{f}.
\end{aligned}$$

Note that in (1.2) $\delta_{a(0)+b(0),0}\delta_{a+b,\operatorname{rad} f} = \delta_{a+b,\operatorname{rad}_0 f}$.

To avoid confusion with the multiplication in $\mathbb{C}_q$, we will denote the associative multiplication in $U(\mathfrak{L})$ by $''\circ''$.

Note that $\mathfrak{L}_0 = \mathbb{C}_{\bar{q}} \oplus \mathbb{C}d_0$. Clearly, we have the following decomposition of ideals:

$$\mathfrak{L}_0 = [\mathfrak{L}_0, \mathfrak{L}_0] \oplus Z(\mathfrak{L}_0).$$

Let $\varphi\colon \mathfrak{L}_0 \to \mathbb{C}$ be any Lie homomorphism. Clearly, $\varphi([\mathfrak{L}_0, \mathfrak{L}_0]) = 0$. We may always assume that $\varphi(d_0) = 0$ (this is only for convenience, since the value does not affect the module structure).

We define the associated $\mathfrak{L}_0$-module $\mathbb{C}v_0$ by $g \cdot v_0 = \varphi(g)v_0, \forall g \in \mathfrak{L}_0$. Let $\mathfrak{L}_+$ act on $\mathbb{C}v_0$ trivially. We introduce the induced module

$$\widetilde{V}(\varphi) := \operatorname{Ind}_{L_0+L_+}^{L} \mathbb{C}v_0 \simeq U(L^-) \otimes_{\mathbb{C}} \mathbb{C}v_0. \tag{2.1}$$

https://doi.org/10.4153/CJM-2010-022-1 Published online by Cambridge University Press

Then $\widetilde{V}(\varphi)$ is $\mathbb{Z}$-graded. Clearly, $\widetilde{V}(\varphi)$ contains a unique maximal proper $\mathbb{Z}$-graded submodule $J(\varphi)$ trivially intersecting with $\mathbb{C}v_0$. Thus we have the $\mathbb{Z}$-graded irreducible quotient module $V(\varphi) := \widetilde{V}(\varphi)/J(\varphi)$.

We shall need the following well-known result frequently.

Lemma 2.1 (**[19, Theorem II. 1.6]**) *If F is a free abelian group of finite rank n and H is a nonzero subgroup of F, then there exists a basis $B_{H,F} = \{b_1, \dots, b_n\}$ of F, an integer $r(1 \leq r \leq n)$ and positive integers $d_1, \dots, d_r$ with $d_1|d_2|\cdots|d_r$ such that H is a free abelian group with basis $\{d_1b_1, \dots, d_rb_r\}$.*

From the definition of $\mathfrak{L}_q$, we have the following.

Lemma 2.2 *Let $B = \{b_1, \dots, b_n\}$ be any basis of $(0, \mathbb{Z}^n)$.*

(i) *There exists a $\mathbb{Z}$-graded associative algebra isomorphism $\rho_B\colon C_{q'} \to C_q$ with $q'_{i,j} = f_q(b_i, b_j)$, and $\rho_B(t^{\varepsilon_i}) = t^{b_i}$ for $i = 0, 1, \dots, n$, where $b_0 = \varepsilon_0$. Moreover,*

$$\rho_B(t^a) = \Big(\prod_{0\leq i<j\leq n} \sigma_q(b_i, b_j)^{a(i)a(j)}\Big)\Big(\prod_{k=0}^{n} \sigma_q(b_k, b_k)^{\frac{a(k)(a(k)-1)}{2}}\Big)\, t^{\sum_{l=0}^{n} a(l)b_l}.$$

(ii) *There exists a $\mathbb{Z}$-graded Lie isomorphism $\varrho_B\colon \mathfrak{L}_{q'} \to \mathfrak{L}_q$ with $q'_{i,j} = f_q(b_i, b_j)$, $\varrho_B(d_0) = d_0$, and $\varrho_B(t^b) = \rho_B(t^b)$ for all $b \in \mathbb{Z}^{n+1}$.*

Proof (i) Clearly, $s_i = t^{b_i}, i = 1, 2, \dots, n$ generate $\mathbb{C}_q$ as an associative algebra, and $s_is_j = q'_{ij}s_js_i$. Then we have the associative algebra $\mathbb{C}_{q'}[s_1, s_2, \dots, s_n]$ that is isomorphic to $\mathbb{C}_q$ via $\rho_B(s^a) = ((t^{b_1})^{\alpha(1)})((t^{b_2})^{\alpha(2)})\cdots((t^{b_n})^{\alpha(n)})$. Then by a simple computation (or from [23, Lemma 6.2]) we can obtain (2.1).

(ii) For any $a \in \mathbb{Z}^{n+1}$, let $a' = \sum_{i=0}^{n} a(i)b_i$. From (i), we know that $\rho_B(\text{rad } f_q) = \text{rad } f_{q'}$. Then $\delta_{a(0)+b(0),0}\delta_{a+b,\text{rad } f_q} = \delta_{a'(0)+b'(0),0}\delta_{a'+b',\text{rad } f_{q'}}$. Using this formula, (1.1) and (i), we can obtain (ii). ■

Definition 2.3 The matrix q is said to be in its *normal form* if

$$\text{rad } \bar{f}_q = \bigoplus_{i=1}^{r} \mathbb{Z}d_i\varepsilon_i, \tag{2.2}$$

where $r = \text{rank}(\text{rad } \bar{f}_q)$ and $d_1, \dots, d_r$ are positive integers.

By using Lemmas 2.1 and 2.2, we see the following.

Lemma 2.4 *For any q, there exists some q' in normal form and a Lie isomorphism $\varrho\colon \mathfrak{L}_{q'} \to \mathfrak{L}_q$ such that $\varrho(d_0) = d_0$.*

It is clear that $t^at^b = t^{a+b}, \forall b \in \text{rad } \bar{f}, a \in \mathbb{Z}^{n+1}$ if q is in normal form. In general we do not have $t^bt^a = t^{a+b}$ if q is not in its normal form.

Now we need to recall some notations from [7].

Definition 2.5 (i) The algebra of *exp-polynomial functions* in r' variables $m_1, m_2, \dots, m_{r'}$ is the algebra of functions $f(m_1, \dots, m_{r'})\colon \mathbb{Z}^{r'} \to \mathbb{C}$ generated as an algebra by functions m_j and a^{m_j} for various constants $a \in \mathbb{C}^\star = \mathbb{C}\setminus\{0\}$, $j = 1, \dots, r'$.

(ii) Let $G = \bigoplus_{a\in\mathbb{Z}^{n+1}} G_a$ be any $\mathbb{Z}^{n+1}$-graded Lie algebra, $\mathcal{K} = \{K_i \mid i \in \mathbb{Z}\}$ a family of finite sets, and $\mathcal{B} = \{g_i^{(k)}(\bar{a}) \mid k \in K_i, (i,\bar{a}) \in \mathbb{Z}^{n+1}\}$ any homogenous spanning set of G with $g_i^{(k)}(\bar{a}) \in G_{i,\bar{a}}$. Then G is said to be a *$\mathbb{Z}^n$-extragraded Lie algebra with respect to* $\mathcal{K}$ *and* $\mathcal{B}$ if there exist a family of exp-polynomial functions $\{f_{i,j}^{k_i,k_j,k_{i+j}}(\bar{a},\bar{b})\}$ in the $2n$ variables $a(l), b(l), l = 1,2,\dots,n$, where $k_i \in K_i, \forall i \in \mathbb{Z}$, such that

$$[g_i^{(k_i)}(\bar{a}), g_j^{(k_j)}(\bar{b})] = \sum_{k\in K_{i+j}} f_{i,j}^{k_i,k_j,k}(\bar{a},\bar{b}) g_{i+j}^{(k)}(\bar{a}+\bar{b}). \tag{2.3}$$

The spanning set $\mathcal{B}$ is called a distinguished spanning set.

Now we are ready to start the study on the modules $V(\varphi)$.

Lemma 2.6 *Suppose that there exists some $\varphi \neq 0$ such that the nontrivial irreducible module $V(\varphi)$ defined in* (2.1) *over $\mathfrak{L}_q$ has all finite dimensional weight spaces. Then* $\operatorname{rank}(\operatorname{rad}\bar{f}) = n$, i.e., *$q_{i,j}$ are roots of unity for all $i, j = 1,\dots,n$.*

Proof Suppose that $\operatorname{rank}(\operatorname{rad}\bar{f}) < n$ and $\varphi(t^a) \neq 0$ for some $a \in \operatorname{rad}\bar{f}$. From the fact that $(0,\mathbb{Z}^n)/\operatorname{rad}\bar{f}$ is an infinite group, we can choose $\{a_i\}_{i\in\mathbb{Z}} \subset (0,\mathbb{Z}^n)$ with $a_i - a_j \notin \operatorname{rad}\bar{f}$ for all $i \neq j$. Suppose that $0 = \sum_i x_i t^{-\varepsilon_1+a_i+a} v_0 \in V(\varphi)$ with $x_i \in \mathbb{C}$ and $x_i = 0$ for all but finitely many i. Since $t^b \in [\mathfrak{L}_0, \mathfrak{L}_0]$ if and only if $b \notin \bar{f}$, and since $\varphi([\mathfrak{L}_0,\mathfrak{L}_0]) = 0$, then $[t^{\varepsilon_1-a_j}, t^{-\varepsilon_1+a_i+a}] \in [\mathfrak{L}_0,\mathfrak{L}_0]$ if $i \neq j$. Then

$$\begin{aligned}0 &= t^{\varepsilon_1-a_j} \circ \Big(\sum_i x_i t^{-\varepsilon_1+a_i+a}\Big) v_0 = x_j \varphi([t^{\varepsilon_1-a_j}, t^{-\varepsilon_1+a_j+a}]) v_0\\ &= x_j \sigma(\varepsilon_1 - a_j, -\varepsilon_1 + a + a_j)\big(1 - f(a, \varepsilon_1 - a_j) + \delta_{a,rad_0 f}\big)\, \varphi(t^a) v_0,\end{aligned}$$

Since $a \in \operatorname{rad}\bar{f}$, we have $a \in \operatorname{rad}_0 f$ if and only if $f(a,\varepsilon_1 - a_j) = 1$. Thus $1 - f(a,\varepsilon_1 - a_j) + \delta_{a,rad_0 f} \neq 0$. We deduce $x_j = 0, \forall j \in \mathbb{Z}$. So we have proved that $\{t^{-\varepsilon_1+a_i+a} v_0\}_{i\in\mathbb{Z}}$ are linearly independent, which implies $\dim V(\varphi)_{-1} = \infty$. ■

Lemma 2.7 *Suppose that $\psi\colon \mathbb{Z}^n \to \mathbb{C}$ is a function,*

$$h_i(t) = \sum_{j=0}^{m_i} x_{i,j} t^j = \prod_{j=1}^{l_i} (t - y_{i,j})^{s_{i,j}}, \quad i = 1,\dots n$$

are polynomials in $\mathbb{C}[t]$ where $s_{i,j}, m_i \in \mathbb{N}$, and $x_{i,j}, y_{i,j} \in \mathbb{C}$ with $x_{i,0}x_{i,m_i} \neq 0$. For $k = 1,2,\dots,n$, let

$$\begin{aligned}\mathcal{F}_k &= \{f_{k,0}(r), f_{k,1}(r), \dots, f_{k,m_k-1}(r)\}\\ &:= \big\{\, y_{k,1}^r, r y_{k,1}^r, \dots, r^{s_{k,1}-1} y_{k,1}^r; y_{k,2}^r, \dots, r^{s_{k,2}-1} y_{k,2}^r; \dots\\ &\qquad \dots; y_{k,l_k}^r, r y_{k,l_k}^r, \dots, r^{s_{k,l_k}-1} y_{k,l_k}^r \big\}\end{aligned}$$

https://doi.org/10.4153/CJM-2010-022-1 Published online by Cambridge University Press

be a set of functions in $r \in \mathbb{Z}$. Then

$$\sum_{j=0}^{m_i} x_{i,j}\psi(\bar{a}+j\bar{\varepsilon}_i)=0,\ \forall \bar{a}\in\mathbb{Z}^n, i=1,2,\dots,n \tag{2.4}$$

if and only if there exist $\prod_{i=1}^{n} m_i$ complex numbers $z_{(b(1),\dots,b(n))}$, $0 \le b(i) \le m_i-1$, $i=1,\dots,n$ such that

$$\psi(\bar{a})=\sum_{b(1)=0}^{m_1-1}\cdots\sum_{b(n)=0}^{m_n-1} z_{(b(1),\dots,b(n))}\prod_{i=1}^{n} f_{i,b(i)}(a(i)),\ \forall \bar{a}\in\mathbb{Z}^n. \tag{2.5}$$

Proof The statement in the lemma for $n=1$ is a well-known combinatorial fact on general solutions of linear homogeneous recurrence relations with constant coefficients [9, Theorem 7.2.2]. Using this lemma for $n=1$, we can easily deduce (2.4) from (2.5) for all n. Now we will prove the other direction by induction on n. From the inductive hypothesis for $n-1$, there exist complex numbers $z'_{(b(1),\dots,b(n))}$, $0 \le b(i) \le m_i-1, i=1,\dots,n, b(n)\in\mathbb{Z}$ such that

$$\psi(\bar{a})=\sum_{b(1)=0}^{m_1-1}\cdots\sum_{b(n-1)=0}^{m_{n-1}-1} z'_{(b(1),\dots,b(n-1),a(n))}\prod_{i=1}^{n-1} f_{i,b(i)}(a(i)),$$

for all $\bar{a}\in\mathbb{Z}^n$. Applying (2.3) for $i=n$, we have

$$\sum_{b(1)=0}^{m_1-1}\cdots\sum_{b(n-1)=0}^{m_{n-1}-1}\sum_{j=0}^{m_n} x_{n,j} z'_{(b(1),\dots,b(n-1),a(n)+j)}\prod_{i=1}^{n-1} f_{i,b(i)}(a(i))=0,$$

for all $0\le b(i)\le m_i-1, i=1,\dots,n-1, a(n)\in\mathbb{Z}$. Consider these $\prod_{i=1}^{n-1} m_i$ linear equations. Noting that the coefficient matrix is invertible (see [7, Lemma 2.1]), we must have

$$\sum_{j=0}^{m_n} x_{n,j} z'_{(b(1),\dots,b(n-1),a(n)+j)}=0, \tag{2.6}$$

for all $0\le b(i)\le m_i-1, i=1,\dots,n-1, a(n)\in\mathbb{Z}$. There exist $\prod_{i=1}^{n} m_i$ complex numbers $z_{(b(1),\cdots,b(n))}$, $0\le b(i)\le m_i-1, i=1,\dots,n$ such that

$$z'_{(b(1),\dots,b(n-1),a(n))}=\sum_{b(n)=0}^{m_n-1} z_{(b(1),\dots,b(n))} f_{n,b(n)}(a(n)).$$

From (2.6) and (2.5) we obtain (2.4). ■

Corollary 2.8 (i) *Let H be any subgroup of $\mathbb{Z}^n$ with* rank $H=n$. *Then $h\colon \mathbb{Z}^n\to\mathbb{C}$ with $h(a(1),\dots,a(n))=\delta_{(a(1),\dots,a(n)),H}$ is an exp-polynomial function.*

https://doi.org/10.4153/CJM-2010-022-1 Published online by Cambridge University Press

(ii) *Let $q_{i,j}, i, j = 1, \dots, n$ be roots of unity. Then $\bar\sigma(\bar a, \bar b) := \prod_{1\le i<j\le n} q_{j,i}^{a(j)b(i)}$ is an exp-polynomial function in $2n$ variables $a(1), \dots, a(n), b(1), \dots, b(n)$*

Proof This corollary follows from applying Lemma 2.7 with $h_i(t) = t^m - 1$, $i = 1, \dots, n$, where $m = |\mathbb{Z}^n/H|$ for(i) and $m = \prod_{i,j=1}^n \operatorname{ord}(q_{i,j})$ for (ii). ■

Denote $\widehat{\mathfrak{L}}_q := \mathbb{C}[t_0^{\pm1}, \dots, t_n^{\pm1}] \subset \mathfrak{L}_q$. The distinguished spanning set of $\widehat{\mathfrak{L}}_q$ in the following lemma will be repeatedly used later.

Lemma 2.9 *Suppose that* $\operatorname{rank}(\operatorname{rad} \bar f) = n$. *Then $\widehat{\mathfrak{L}}_q$ is a $\mathbb{Z}^n$-extragraded Lie algebra with respect to $\mathcal{K}$ and $\mathcal{B}$ defined in Definition* 2.5(ii), *where*

$$\mathcal{K} = \{K_i \mid i \in \mathbb{Z}\}, \quad \mathcal{B} = \{g_i^{(k)}(\bar a) \mid k \in K_i, (i, \bar a) \in \mathbb{Z}^{n+1}\},$$
$$K_0 = \{1, 2\}, \quad K_i = \{1\}\ \forall i \neq 0,$$

and

$$\begin{aligned} g_0^{(1)}(\bar a) &= \delta_{(0,\bar a),\operatorname{rad} \bar f}\Big(1 - \prod_{i=1}^n q_{i,0}^{a(i)} + \delta_{(0,\bar a),\operatorname{rad} f}\Big)\, t^{(0,\bar a)}, \\ g_0^{(2)}(\bar a) &= (1 - \delta_{(0,\bar a),\operatorname{rad} \bar f}) t^{(0,\bar a)}, \\ g_i^{(1)}(\bar a) &= t^{(i,\bar a)}, \forall i \neq 0. \end{aligned} \tag{2.7}$$

Proof By computing $[g_i^{(1)}(\bar a), g_j^{(1)}(\bar b)]$ for all i, j with $ij(i+j) \neq 0$, $[g_0^{(2)}(\bar a), g_j^{(1)}(\bar b)]$ for all $j \neq 0$, and $[g_0^{(1)}(\bar a), g_j^{(1)}(\bar b)]$ for all $j \neq 0$ (note that $[g_0^{(2)}(\bar a), g_j^{(1)}(\bar b)] = 0$ if $(0, \bar a) \in \operatorname{rad} f$), we easily obtain that

$$f_{i,j}^{1,1,1}(\bar a, \bar b) = \sigma((i, \bar a), (j, \bar b)) - \sigma((j, \bar b), (i, \bar a)) \text{ for all } i, j \text{ with } ij(i+j) \neq 0,$$
$$f_{0,j}^{2,1,1}(\bar a, \bar b) = (1 - \delta_{(0,\bar a),\operatorname{rad} \bar f})(\sigma((0, \bar a), (j, \bar b)) - \sigma((j, \bar b), (0, \bar a))), \text{ for all } j \neq 0,$$
$$f_{0,j}^{1,1,1}(\bar a, \bar b) = \delta_{(0,\bar a),\operatorname{rad} \bar f}\Big(1 - \prod_{i=1}^n q_{i,0}^{a(i)}\Big)(\sigma((0, \bar a), (j, \bar b)) - \sigma((j, \bar b), (0, \bar a))) \quad \text{for all } j \neq 0,$$

which are all exp-polynomial functions in $2n$ variable $a(l), b(l)$ for $l = 1, \dots, n$.

It is straightforward to see that if $\bar a + \bar b \in \operatorname{rad}_0 f$, since $t^{(0,\bar a+\bar b)} t_0 = t_0 t^{(0,\bar a+\bar b)}$, then

$$g_0^{(2)}(\bar a + \bar b) = 0,$$
$$1 - \prod_{k=1}^n q_{k,0}^{(a(k)+b(k))i} = \Big(1 - \prod_{k=1}^n q_{k,0}^{(a(k)+b(k))}\Big)\Big(\sum_{j=0}^{i-1} \prod_{k=1}^n q_{k,0}^{(a(k)+b(k))j}\Big),$$
$$i = \sum_{j=0}^{i-1} \prod_{k=1}^n q_{k,0}^{(a(k)+b(k))j}.$$

Since q is in its normal form, it is also clear that $\sigma(\bar a, \bar b) = \sigma(\bar b, \bar a)$ if $\bar a + \bar b \in \operatorname{rad} \bar f$.

https://doi.org/10.4153/CJM-2010-022-1 Published online by Cambridge University Press

If $\bar a+\bar b\in \operatorname{rad}\bar f$ and $i>0$, (we write $a=(i,\bar a)$ and $b=(-i,\bar b)$), we compute

$$
\begin{aligned}
&[g_i^{(1)}(\bar a), g_{-i}^{(1)}(\bar b)]\\
&\quad = [t^{(i,\bar a)}, t^{(-i,\bar b)}]\\
&\quad = (\sigma((i,\bar a),(-i,\bar b)) - \sigma((-i,\bar b),(i,\bar a)) + \delta_{\bar a+\bar b,\operatorname{rad} f}\, i\sigma((i,\bar a),(-i,\bar b)))t^{(0,\bar a+\bar b)}\\
&\quad = \Big(\prod_{j=1}^n q_{j,0}^{-ia(j)} - \prod_{j=1}^n q_{j,0}^{ib(j)} + i\delta_{\bar a+\bar b,\operatorname{rad} f}\prod_{j=1}^n q_{j,0}^{-ia(j)}\Big)\sigma((0,\bar a),(0,\bar b))t^{(0,\bar a+\bar b)}\\
&\quad = \Big(1-\prod_{j=1}^n q_{j,0}^{i(a(j)+b(j))} + \delta_{\bar a+\bar b,\operatorname{rad} f}\sum_{j=0}^{i-1}\prod_{i=1}^n q_{j,0}^{i(a(j)+b(j))}\Big)\\
&\qquad\times\prod_{j=1}^n q_{j,0}^{-ia(j)}\sigma((0,\bar a),(0,\bar b))t^{(0,\bar a+\bar b)}\\
&\quad = f_{i,-i}^{1,1,1} g_0^{(1)}(\bar a+\bar b) + f_{i,-i}^{1,1,2} g_0^{(2)}(\bar a+\bar b),
\end{aligned}
$$

where

(2.8)

$$
\begin{aligned}
f_{i,-i}^{1,1,1}(\bar a,\bar b) &= \Big(\sum_{j=0}^{i-1}\Big(\prod_{k=1}^n q_{k,0}^{j(a(k)+b(k))}\Big)\Big)\Big(\prod_{k=1}^n q_{k,0}^{-ia(k)}\Big)\sigma((0,\bar a),(0,\bar b))\ \forall i\in\mathbb{Z}^+,\\
f_{i,-i}^{1,1,1}(\bar a,\bar b) &= \Big(\sum_{j=0}^{-i-1}\Big(\prod_{k=1}^n q_{k,0}^{-j(a(k)+b(k))}\Big)\Big)\Big(\prod_{k=1}^n q_{k,0}^{-ia(k)}\Big)\sigma((0,\bar a),(0,\bar b))\ \forall i\in -\mathbb{N},\\
f_{i,-i}^{1,1,2}(\bar a,\bar b) &= \sigma((i,\bar a),(-i,\bar b)) - \sigma((-i,\bar b),(i,\bar a)),
\end{aligned}
$$

which are all exp-polynomial functions in $2n$ variable $a(l), b(l)$ for $l=1,\dots,n$. Other cases are simpler. ■

Theorem 2.10 *Suppose that* $\operatorname{rank}(\operatorname{rad}\bar f)=n$ *and q is in its normal form. Then the following statements are equivalent.*

(i) *The module $V(\varphi)$ over $\mathfrak{L}$ has all finite dimensional weight spaces.*

(ii) *There exists a unique nonzero polynomial $P_\varphi(t)=P(t)=\sum_{i=0}^m x_i t^i\in\mathbb{C}[t]$ with lowest degree m, where $x_i\in\mathbb{C}$, $x_0\neq 0$, and $x_m=1$, such that*

$$\sum_{i=0}^m x_i\varphi(g_0^{(1)}(\bar a+id_j\bar\varepsilon_j))=0,\forall a\in\operatorname{rad}\bar f,\ j=1,\dots,n, \tag{2.9}$$

where d_i, $g_0^{(1)}(\bar b)$ are defined as in (2.2) *and* (2.7).

(iii) *There exists an n-variable exp-polynomial function $h\colon\mathbb{Z}^n\to\mathbb{C}$, such that*

$$\varphi(t^a)=\frac{h(\bar a)}{1-\prod_{i=1}^n q_{i,0}^{a(i)}+\delta_{(0,\bar a),\operatorname{rad} f}},\ \forall a\in\operatorname{rad}\bar f.$$

Proof (i) $\Longrightarrow$ (ii). From (2.9) it is easy to see the uniqueness of P_φ. For any $1 \le j \le n$. Since $\dim V_{-1} < \infty$, there exist an $a_j \in \operatorname{rad} \bar f$ and some nonzero polynomial $P_j(t) = \sum_{i=0}^{m_j} x_{j,i} t^i \in \mathbb{C}[t]$ with $x_{j,i} \in \mathbb{C}$, $x_{j,0} \neq 0$ and $x_{m_j} = 1$ such that

$$\sum_{i=0}^{m_j} x_{j,i} t^{-\varepsilon_0 + a_j + id_j\varepsilon_j} v_0 = 0.$$

Applying $t^{\varepsilon_0 + a - a_j}$ for any $a \in \operatorname{rad} \bar f$ to the above equation and noting that $a - a_j \in \operatorname{rad} \bar f$ and $\sigma(e, id_j\varepsilon_j) = 1, \forall e \in \mathbb{Z}^{n+1}$, using the notation in Lemma 2.9 and (2.8), we obtain that

$$\begin{aligned}
0 &= \sum_i x_{j,i} t^{\varepsilon_0 + a - a_j} \circ t^{-\varepsilon_0 + a_j + id_j\varepsilon_j} v_0 = \sum_i x_{j,i} g_1^{(1)}(a - a_j) g_{-1}^{(1)}(a_j + id_j\varepsilon_j) v_0 \\
&= \sum_i x_{j,i} f_{1,-1}^{1,1,1}(a - a_j, a_j + id_j\varepsilon_j) g_0^{(1)}(a + id_j\varepsilon_j) v_0 \\
&= f_{1,-1}^{1,1,1}(a - a_j, a_j) \sum_i x_{j,i} g_0^{(1)}(a + id_j\varepsilon_j) v_0 \quad (\text{since } d_j\varepsilon_j \in \operatorname{rad}(\bar f)) \\
&= f_{1,-1}^{1,1,1}(a - a_j, a_j) \Big(\sum_{i=1}^{m_j} x_{j,i} \varphi(g_0^{(1)}(\bar a + id_j \bar\varepsilon_j)) \Big) v_0,
\end{aligned}$$

where $f_{1,-1}^{1,1,1}(a - a_j, a_j) \neq 0$ is defined in the proof of Lemma 2.9.

Hence

$$\sum_{i=0}^{m_j} x_{j,i} \varphi(g_0^{(1)}(\bar a + id_j\bar\varepsilon_j)) = 0, \ \forall \bar a \in \operatorname{rad} \bar f = (0, d_1\mathbb{Z}, \dots, d_n\mathbb{Z}). \tag{2.10}$$

Let $P'(t) = \prod_{i=1}^n P_j(t) = \sum_{i=0}^{m'} x_i' t^i \in \mathbb{C}[t]$. Then from (2.10) we have

$$\sum_{i=0}^{m'} x_i' \varphi(g_0^{(1)}(\bar a + id_j\bar\varepsilon_j)) = 0, \ \forall \bar a \in \operatorname{rad} \bar f = (0, d_1\mathbb{Z}, \dots, d_n\mathbb{Z}), \ j = 1, 2, \dots, n.$$

So we have proved the existence of $P_\varphi(t)$.

(ii) $\Longrightarrow$ (iii). Let $\psi\colon \mathbb{Z}^n \to \mathbb{C}$ defined by $\psi(\bar a) = \varphi(g_0^{(1)}(\bar a))$, $h_i(t) = P_\varphi(t^{d_i})$. Using Lemma 2.7, we see that ψ is exp-polynomial function, which implies (iii).

(iii) $\Longrightarrow$ (i). From Lemma 2.9 we know that $\widehat{\mathfrak{L}}_q$ is a $\mathbb{Z}^n$-extragraded Lie algebra. The actions of $\widehat{\mathfrak{L}}_0$ on v_0 are $g_0^{(1)}(\bar a) v_0 = h(\bar a) \delta_{(0,\bar a), \operatorname{rad} \bar f} v_0$ and $g_0^{(2)}(\bar a) v_0 = 0$. Hence from [7, Theorem 1.7], we have (i). ∎

Theorem 2.11 *Let $\varphi\colon \mathfrak{L}_0 \to \mathbb{C}$ be any Lie homomorphism with $\varphi|_{\mathbb{C}_{\bar q}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]} \neq 0$. Then the irreducible highest weight module $V(\varphi)$ over $\mathfrak{L}_q$ has finite dimensional weight spaces if and only if $q_{i,j}$ are roots of unity for all $i, j = 1, \dots, n$ and there exists some n-variable exp-polynomial function $\widetilde h\colon \mathbb{Z}^n \to \mathbb{C}$, such that*

$$\varphi(t^a) = \frac{\widetilde h(\bar a)}{1 - \prod_{i=1}^n q_{i,0}^{a(i)} + \delta_{(0,\bar a), \operatorname{rad} f}}, \ \forall a \in \operatorname{rad} \bar f. \tag{2.11}$$

Proof By using Lemma 2.2(i) and Theorem 2.10, we see that $V(\varphi)$ has finite dimensional weight spaces if and only if $q_{i,j}$, $(i, j = 1, \dots, n)$ are roots of unity and there exists an exp-polynomial function $h\colon \mathbb{Z}^n \to \mathbb{C}$ such that

$$\varphi(t^{\sum_{i=1}^n a(i)b_i}) = \frac{h(\bar{a})}{\left(1 - \prod_{i=1}^n f(b_i, \varepsilon_0)^{a(i)} + \delta_{\sum_{i=1}^n a(i)b_i, \mathrm{rad}\, f}\right) g(\bar{a})} \tag{2.12}$$

for all $a \in \bigoplus_{i=1}^n \mathbb{Z} d_i \varepsilon_i$, where $B = \{b_1, \dots, b_n\}$ is a basis of $(0, \mathbb{Z}^n)$, $d_1, \dots, d_n$ are positive integers with $\mathrm{rad}\, \bar{f} = \bigoplus_{i=1}^n \mathbb{Z} d_i b_i$ and

$$g(\bar{a}) = \Big(\prod_{1 \le i < j \le n} \sigma_q(b_i, b_j)^{a(i)a(j)} \Big) \Big(\prod_{i=1}^n \sigma_q(b_i, b_i)^{\frac{a(i)(a(i)-1)}{2}} \Big).$$

Denote $h'(\bar{a}) = \frac{h(\bar{a})}{g(\bar{a})}$. From Lemma 2.7 we see that $\frac{1}{g(\bar{a})}$ is an exp-polynomial function if $q_{i,j}$ are all roots of unity. Hence $h'(\bar{a})$ is an exp-polynomial function. Let $\theta\colon \mathbb{Z}^n \to \mathbb{Z}^n$ be the isomorphism of lattice with $\theta(\bar{b}_i) = \bar{\varepsilon}_i$, and $\widetilde{h}(\bar{a}) = h'(\theta(\bar{a}))$. Now it is easy to check that (2.12) is equivalent to (2.11). ■

3 Verma Modules $\widetilde{V}(\varphi)$

In this section we shall study when Verma modules $\widetilde{V}(\varphi)$ over $\mathfrak{L} = \mathfrak{L}_q$ are irreducible. The answer is the following.

Theorem 3.1 *Suppose that the matrix q is in its normal form. Then $\widetilde{V}(\varphi)$ is not irreducible if and only if one of the following conditions holds:*

(i) *There exist pairwise distinct $a_0, \dots, a_m \in \mathrm{rad}\, \bar{f}$ and a nonzero polynomial $p(t) = \sum_{i=0}^m x_i t^i \in \mathbb{C}[t]$ where $x_i \in \mathbb{C}$ with $x_0 x_m \neq 0$, such that*

$$\sum_{i=0}^m x_i \varphi(g_0^{(1)}(\bar{a} + \bar{a}_i)) = 0, \ \forall a \in \mathrm{rad}\, \bar{f}, \tag{3.1}$$

where $g_0^{(1)}(\bar{b})$ is defined as in (2.7);

(ii) $\mathrm{rad}\, f \neq \mathrm{rad}_0\, f$ *and there exist pairwise distinct $a_0, \dots, a_m \in \mathrm{rad}_0\, f$ and a nonzero polynomial $p(t) = \sum_{i=0}^m x_i t^i \in \mathbb{C}[t]$, where $x_i \in \mathbb{C}$ with $x_0 x_m \neq 0$, such that*

$$\sum_{i=0}^m x_i \varphi(t^{a+a_i}) = 0, \ \forall a \in \mathrm{rad}_0\, f. \tag{3.2}$$

Before proceeding to the proof of this theorem, we need some preparations on the universal enveloping algebra $(U(\mathfrak{L}_-), \circ)$ of $\mathfrak{L}_-$. Denote

$$K_\pm := \sum_{\substack{\pm a(0) > 0 \\ a \in \mathbb{Z}^{n+1} \setminus \mathrm{rad}\, f_q}} \mathbb{C} t^a \quad \text{and} \quad R_\pm := \sum_{\substack{\pm a(0) > 0 \\ a \in \mathrm{rad}\, f_q}} \mathbb{C} t^a.$$

https://doi.org/10.4153/CJM-2010-022-1 Published online by Cambridge University Press

They are Lie subalgebras and any of them can be 0.

Clearly $\mathfrak{L}_\pm = K_\pm \oplus R_\pm$. It is not difficult to verify that $K_\pm$ is generated by $K_{\pm 1} = \sum_{a(0)=\pm 1, a\in\mathbb{Z}^{n+1}\setminus \operatorname{rad} f_q} \mathbb{C}t^a$ as Lie subalgebras, respectively. Then

$$U(\mathfrak{L}_\pm) \cong U(K_\pm) \otimes U(R_\pm) = U(K_\pm) \oplus \big(U(K_\pm) \otimes (U(R_\pm) \circ R_\pm)\big).$$

Let $\mathbb{S}$ be the set of all finite sequences of integers $(i_1, i_2, \ldots, i_t)$. We first define a total ordering $\succ$ on the set $\mathbb{S}$: $(i_1, i_2, \ldots, i_l) \succ (j_1, j_2, \ldots, j_s)$ if and only if $l > s$ or $l = s$, $i_1 = j_1, i_2, = j_2, \ldots, i_{k-1} = j_{k-1}$ and $i_k > j_k$ for some $1 \le k \le s$.

We have the obvious meaning for $\succeq$, $\preceq$, and $\prec$.

We fix a PBW basis $\mathbf{B}$ for $U(\mathfrak{L}_-)$ consisting of the following elements:

$$t^{-a_1} \circ t^{-a_2} \circ \cdots \circ t^{-a_s}, \quad a_i \in \mathbb{N} \times \mathbb{Z}^n,$$

where s is an arbitrary nonnegative integer and $a_i \succeq a_{i+1}$ for all $i = 1, \ldots, s-1$. We call this s the *height* of the element $t^{-a_1} \circ t^{-a_2} \circ \cdots \circ t^{-a_s}$, which is denoted by $\operatorname{ht}(t^{-a_1} \circ t^{-a_2} \circ \cdots \circ t^{-a_s})$. We now define a total ordering on $\mathbf{B}$ as follows:

$$t^{-a_1} \circ t^{-a_2} \circ \cdots \circ t^{-a_s} \succ t^{-b_1} \circ t^{-b_2} \circ \cdots \circ t^{-b_l}$$

if $(a_1, \ldots, a_s) \succ (b_1, b_2, \ldots, b_l)$.

For any nonzero $u \in U(\mathfrak{L}_-)$, we can uniquely write it as a linear combination of elements in $\mathbf{B}$: $u = \sum_{i=1}^m x_i u_i$, where $0 \neq x_i \in \mathbb{C}$, $u_i \in \mathbf{B}$ and $u_1 \succ u_2 \succ \cdots \succ u_m$. We define the *height of* u as $\operatorname{ht}(u_1)$, and the *highest term* of u as $x_1 u_1$, denoted by $\operatorname{hm}(u) = x_1 u_1$. For convenience, we define $\operatorname{ht}(0) = -1$ and $\operatorname{hm}(0) = 0$.

It is clear that $\mathbf{B}v_0 := \{uv_0 \mid u \in \mathbf{B}\}$ is a basis for the Verma module $\widetilde{V}(\varphi)$ where v_0 is again the highest weight vector of $\widetilde{V}(\varphi)$. We define

$$\operatorname{ht}(uv_0) := \operatorname{ht}(u), \quad \operatorname{hm}(uv_0) := \operatorname{hm}(u)v_0, \ \forall u \in U(\mathfrak{L}_-).$$

At last we need to define the notation

$$U_s^l = \{u \in U(\mathfrak{L}_-) \mid [d_0, u] = su, \operatorname{ht}(u) \le l\}.$$

It is easy to see that $U_{-s}^l \circ U_{-s'}^{l'} \subseteq U_{-s-s'}^{l+l'}$ for all $l, l', s, s' \in \mathbb{N}$.

Also we have $t^a \circ U_{-s}^l v_0 \subset U_{-s+a(0)}^l v_0$ for all $l, s \in \mathbb{Z}_+$ and $a \in \mathbb{N} \times \mathbb{Z}^n$ (where we have regarded $U_k^r = 0$ for $k > 0$), and we also have

$$t^{-a_1} \circ t^{-a_2} \circ \cdots \circ t^{-a_s} \in U^s_{-a_1(0)-a_2(0)-\ldots a_s(0)}$$

for any $a_1, a_2, \ldots, a_s \in \mathbb{N} \times \mathbb{Z}^n$.

Proof of Theorem 3.1 Denote $V(\varphi) = \widetilde{V}(\varphi)/J(\varphi)$, where $J(\varphi)$ is the maximal proper submodule of $\widetilde{V}(\varphi)$.

"⇐". Suppose that (i) holds. For all $(1,\bar a)\in\mathbb{Z}^{n+1}$, using Lemma 2.9 and (2.8), we deduce

$$\begin{aligned}t^{(1,\bar a)}\circ\Big(\sum_i x_i t^{(-1,\bar a_i)}\Big)v_0&=\sum_i x_i[t^{(1,\bar a)},t^{(-1,\bar a_i)}]v_0\\&=\delta_{\bar a,\operatorname{rad}\bar f}\sum_i x_i[t^{(1,\bar a)},t^{(-1,\bar a_i)}]v_0\\&=\Big(\sum_i x_i f^{1,1,1}_{1,-1}(\bar a,a_i)\big(g_0^{(1)}(\bar a+\bar a_i)\big)\Big)v_0\\&=\delta((0,\bar a),-\varepsilon_1)\Big(\sum_i x_i\varphi(g_0^{(1)}(\bar a+\bar a_i))\Big)v_0=0,\end{aligned}$$

i.e., $\mathfrak{L}_1\circ(\sum_{i=0}^m x_i t^{(-1,\bar a_i)})v_0=0$. Thus $0\neq(\sum_{i=0}^m x_i t^{-1,\bar a_i})v_0\in J(\varphi)$.

Now suppose (ii) holds. It is not difficult to show that there exists some $c_0\in\mathbb{N}\times\mathbb{Z}^{n-1}$ such that $\operatorname{rad}f=\mathbb{Z}c_0+\operatorname{rad}_0 f$. Now it is easy to check that

$$\mathfrak{L}_+\circ\Big(\sum_{i=0}^m x_i t^{a_i-c_0}\Big)v_0=0\quad\text{and}\quad\mathfrak{L}_0\circ\Big(\sum_{i=0}^m x_i t^{a_i-c_0}\Big)v_0\subset\mathbb{C}\Big(\sum_{i=0}^m x_i t^{a_i-c_0}\Big)v_0,$$

i.e., $U(\mathfrak{L}_-)\circ(\sum_{i=0}^m x_i t^{a_i-c_0})v_0$ is a proper submodule of $\widetilde V(\varphi)$.

"⇒". Since $\widetilde V(\varphi)$ is not irreducible, say $J(\varphi)=\bigoplus_{k=k_0}^{+\infty}J(\varphi)_{-k}$, where $J(\varphi)_{-k_0}\neq 0$ and $k_0\in\mathbb{N}$. Let $0\neq uv_0\in J(\varphi)_{-k_0}$. Write $u=\sum_{i=1}^m x_i'u_i\in U(\mathfrak{L}_-)_{-k_0}$, where $x_i'\in\mathbb{C}^*$ and $u_i\in\mathbf{B}$ with $u_1\succ\cdots\succ u_m$.

We break up the proof into two different cases.

Case 1: $u\notin U(K_-)$. Clearly in this case we have $\operatorname{rad}f\neq\operatorname{rad}_0 f$, and there exists some $c_0\in\operatorname{rad}f$ with $c_0(0)>0$ such that $\operatorname{rad}f=\mathbb{Z}c_0+\operatorname{rad}_0 f$. From the definition of K_-, we know that there exists $u_{i_0}\in U(\mathfrak{L}_-)\circ(\sum_{a\in\operatorname{rad}_0 f}\mathbb{C}t^{-j_0c_0-a})$ for some $1\le i_0\le m$ with $j_0>0$. Now by $0=t^{j_0c_0+b}\circ uv_0\in\widetilde V(\varphi)$ for all $b\in\operatorname{rad}_0 f$, it is easy to deduce (3.2).

Case 2: $u\in U(K_-)$.

Subcase 2.1: $\operatorname{ht}(u)<k_0$. Suppose

$$u_1=t^{(-i_1,-\bar a_1)}\circ t^{(-i_2,-\bar a_2)}\circ\cdots\circ t^{(-i_r,-\bar a_r)}\circ t^{(-1,-\bar a_{r+1})}\circ t^{(-1,-\bar a_{r+2})}\circ\cdots\circ t^{(-1,-\bar a_{r+s})}\in\mathbf{B}$$

with $r>0$ and $i_r\ge 2$. Since $t^{(-i_r,-\bar a_r)}\notin Z(\mathfrak{L})$, there exists some $a\in\mathbb{Z}^n$ with sufficiently large $a(1)$ such that $\sigma((1,-\bar a),(-i_r,-\bar a_r))\neq\sigma((-i_r,-\bar a_r),(1,-\bar a))$. Since $t^{(1,\bar a)}\circ uv_0\in J_{-(k_0-1)}=0$, we have

$$\begin{aligned}\operatorname{hm}(t^{(1,-\bar a)}\circ uv_0)&=\operatorname{hm}([t^{(1,-\bar a)},x_1'u_1]v_0)\\&=l(\sigma((1,-\bar a),(-i_r,-\bar a_r))-\sigma((-i_r,-\bar a_r),(1,-\bar a)))t^{(-i_1,-\bar a_1)}\\&\quad\circ t^{(-i_2,-\bar a_2)}\circ\cdots\circ t^{(-i_r+1,-\bar a_r-\bar a)}\circ t^{(-1,-\bar a_{r+1})}\\&\quad\circ t^{(-1,-\bar a_{r+2})}\circ\cdots\circ t^{(-1,-\bar a_{r+s})}v_0\neq 0,\end{aligned}$$

where l is the number of q such that $(-i_q, -\bar{a}_q) = (-i_r, -\bar{a}_r)$. Then

$$0 \neq t^{(1,-\bar{a})} \circ u v_0 \in J(\varphi)_{-k_0+1},$$

which is a contradiction. So this subcase cannot occur.

Subcase 2.2: $\mathrm{ht}(u) = k_0$. In this case, we may assume that there exist r, s such that $\mathrm{ht}(u_i) = n, 1 \leq i \leq r, \mathrm{ht}(u_i) = n-1, r+1 \leq i \leq s$ and $\mathrm{ht}(u_i) \leq n-2, s+1 \leq i \leq m$.

For $1 \leq i \leq r$, each u_i is of the form $u_i = t^{(-1,-\bar{a}_{i,1})} \circ t^{(-1,-\bar{a}_{i,2})} \circ \cdots \circ t^{(-1,-\bar{a}_{i,k_0})}$. For any $\bar{a} \in \mathbb{Z}^n$ we compute

(3.3)

$$\begin{aligned}
t^{(1,-\bar{a})} \circ u_i v_0 &= [t^{(1,-\bar{a})}, t^{(-1,-\bar{a}_{i,1})} \circ t^{(-1,-\bar{a}_{i,2})} \circ \cdots \circ t^{(-1,-\bar{a}_{i,k_0})}] v_0 \\
&= \sum_{p=1}^{k_0} t^{(-1,-\bar{a}_{i,1})} \circ \cdots \circ t^{(-1,-\bar{a}_{i,p-1})} \circ [t^{(1,-\bar{a})}, t^{(-1,-\bar{a}_{i,p})}] \\
&\qquad \circ t^{(-1,-\bar{a}_{i,p+1})} \circ \cdots \circ t^{(-1,-\bar{a}_{i,k_0})} v_0 \\
&\equiv \sum_{p=1}^{k_0} \varphi([t^{(1,-\bar{a})}, t^{(-1,-\bar{a}_{i,p})}]) t^{(-1,-\bar{a}_{i,1})} \circ \cdots \circ t^{\widehat{(-1,-\bar{a}_{i,p})}} \circ \cdots \circ t^{(-1,-\bar{a}_{i,k_0})} v_0 \\
&\quad + \sum_{p=1}^{k_0} \sum_{q=p+1}^{k_0} t^{(-1,-\bar{a}_{i,1})} \circ t^{(-1,-\bar{a}_{i,2})} \circ \cdots \circ t^{\widehat{(-1,-\bar{a}_{i,p})}} \circ \cdots \\
&\qquad \circ \Big[[t^{(1,-\bar{a})}, t^{(-1,-\bar{a}_{i,p})}], t^{(-1,-\bar{a}_{i,q})} \Big] \circ \cdots \circ t^{(-1,-\bar{a}_{i,k_0})} v_0 \mod (U_{-k_0+1}^{(k_0-2)} v_0) \\
&\equiv \sum_{p=1}^{k_0} \varphi([t^{(1,-\bar{a})}, t^{(-1,-\bar{a}_{i,p})}]) t^{(-1,-\bar{a}_{i,1})} \circ \cdots \circ t^{\widehat{(-1,-\bar{a}_{i,p})}} \circ \cdots \circ t^{(-1,-\bar{a}_{i,k_0})} v_0 \\
&\quad + \sum_{p=1}^{k_0} \sum_{q=p+1}^{k_0} \Big[[t^{(1,-\bar{a})}, t^{(-1,-\bar{a}_{i,p})}], t^{(-1,-\bar{a}_{i,q})} \Big] \circ t^{(-1,-\bar{a}_{i,1})} \circ t^{(-1,-\bar{a}_{i,2})} \circ \cdots \\
&\qquad \circ t^{\widehat{(-1,-\bar{a}_{i,p})}} \circ \cdots \circ t^{\widehat{(-1,-\bar{a}_{i,q})}} \circ \cdots \circ t^{(-1,-\bar{a}_{i,k_0})} v_0 \mod (U_{-k_0+1}^{(k_0-2)} v_0) \\
&\equiv \sum_{p=1}^{k_0} \varphi([t^{(1,-\bar{a})}, t^{(-1,-\bar{a}_{i,p})}]) t^{(-1,-\bar{a}_{i,1})} \circ \cdots \circ t^{\widehat{(-1,-\bar{a}_{i,p})}} \circ \cdots \circ t^{(-1,-\bar{a}_{i,k_0})} v_0 \\
&\quad + \sum_{p=1}^{k_0} \sum_{q=p+1}^{k_0} f_{i,p,q}(\bar{a}) t^{(-1,-\bar{a}-\bar{a}_p-\bar{a}_q)} \circ t^{(-1,-\bar{a}_{i,1})} \circ t^{(-1,-\bar{a}_{i,2})} \circ \cdots \\
&\qquad \circ t^{\widehat{(-1,-\bar{a}_{i,p})}} \circ \cdots \circ t^{\widehat{(-1,-\bar{a}_{i,q})}} \circ \cdots \circ t^{(-1,-\bar{a}_{i,k_0})} v_0 \mod (U_{-k_0+1}^{(k_0-2)} v_0),
\end{aligned}$$

where the $\widehat{\ }$ means the factor is missing and $f_{i,p,q}$ for $i = 1, \ldots, r$ are exp-polynomial

https://doi.org/10.4153/CJM-2010-022-1 Published online by Cambridge University Press

functions defined by

$$[[t^{(1,-\bar{a})}, t^{(-1,-\bar{a}_{i,p})}], t^{(-1,-\bar{a}_{i,q})}] = f_{i,p,q}(\bar{a})t^{(-1,-\bar{a}-\bar{a}_p-\bar{a}_q)}, \ \forall \bar{a} \in \mathbb{Z}^n.$$

For $r+1 \leq i \leq s$, each u_i is of the form $u_i = t^{(-2,-\bar{a}_{i,1})} \circ t^{(-1,-\bar{a}_{i,2})} \circ \cdots \circ t^{(-1,-\bar{a}_{i,k_0-1})}$. For all $\bar{a} \in \mathbb{Z}^n$ we compute

$$\begin{aligned} t^{(1,-\bar{a})} \circ u_i v_0 &= [t^{(1,-\bar{a})}, t^{(-2,-\bar{a}_{i,1})} \circ t^{(-1,-\bar{a}_{i,2})} \circ \cdots \circ t^{(-1,-\bar{a}_{i,k_0-1})}]v_0 \\ &\equiv f_i(\bar{a})t^{(-1,-\bar{a}_{i,1}-\bar{a})} \circ t^{(-1,-\bar{a}_{i,2})} \circ \cdots \circ t^{(-1,-\bar{a}_{i,k_0-1})} v_0 \\ &\qquad \bmod (U^{(k_0-2)}_{-k_0+1} v_0), \end{aligned} \tag{3.4}$$

where f_i for $i = r+1, \dots, s$ are exp-polynomial functions defined by

$$[t^{(1,-\bar{a})}, t^{(-2,-\bar{a}_{i,1})}] = f_i(\bar{a})t^{(-1,-\bar{a}-\bar{a}_{i,1})}, \forall \bar{a} \in \mathbb{Z}^n$$

Using (3.3), (3.4), and the fact that $t^{(1,-\bar{a})} \circ u_i v_0 \in U^{(n-2)}_{-n+1}$ for all $s+1 \leq i \leq m$, we obtain that for all $\bar{a} \in \mathbb{Z}^n$,

(3.5)

$$\begin{aligned} 0 &= t^{(1,-\bar{a})} \circ u v_0 = [t_1 t^{-\bar{a}}, u] v_0 \\ &\equiv \sum_{i=1}^{r} x_i' \sum_{p=1}^{k_0} \varphi([t^{(1,-\bar{a})}, t^{(-1,-\bar{a}_{i,p})}]) t^{(-1,-\bar{a}_{i,1})} \circ \cdots \circ t^{\widehat{(-1,-\bar{a}_{i,p})}} \circ \cdots \circ t^{(-1,-\bar{a}_{i,k_0})} v_0 \\ &\quad + \sum_{i=1}^{r} x_i' \sum_{p=1}^{k_0} \sum_{q=p+1}^{k_0} f_{i,p,q}(\bar{a}) t^{(-1,-\bar{a}-\bar{a}_p-\bar{a}_q)} \circ t^{(-1,-\bar{a}_{i,1})} \circ t^{(-1,-\bar{a}_{i,2})} \circ \cdots \\ &\quad \circ t^{\widehat{(-1,-\bar{a}_{i,p})}} \circ \cdots \circ t^{\widehat{(-1,-\bar{a}_{i,q})}} \circ \cdots \circ t^{(-1,-\bar{a}_{i,k_0})} \\ &\quad + \sum_{i=r+1}^{s} x_i' f_i(\bar{a}) t^{(-1,-\bar{a}_{i,1}-\bar{a})} \circ t^{(-1,-\bar{a}_{i,2})} \circ \cdots \circ t^{(-1,-\bar{a}_{i,k_0-1})} v_0 \\ &\equiv \sum_{i=1}^{r} x_i' \sum_{p=1}^{k_0} \varphi([t^{(1,-\bar{a})}, t^{(-1,-\bar{a}_{i,p})}]) t^{(-1,-\bar{a}_{i,1})} \circ \cdots \circ t^{\widehat{(-1,-\bar{a}_{i,p})}} \circ \cdots \circ t^{(-1,-\bar{a}_{i,k_0})} v_0 \\ &\quad + \sum_{\underline{l} \in J} f_{\underline{l}}(\bar{a}) t^{(-1,-l_1-\bar{a})} \circ t^{(-1,-l_2)} \circ \cdots \circ t^{(-1,-l_{k_0-1})} v_0 \quad \bmod (U^{(k_0-2)}_{-k_0+1} v_0), \end{aligned}$$

https://doi.org/10.4153/CJM-2010-022-1 Published online by Cambridge University Press

where all the elements appeared above (for example $t^{(-1,-l_1+\bar{a})} \circ t^{(-1,-l_2)} \circ \cdots \circ t^{(-1,-l_{k_0-1})}$) are in $\mathbf{B}$ and J is a finite subset of $(\mathbb{Z}^n)^{k_0-1}$, and $f_{\underline{l}}(\bar{a})$ are exp-polynomial functions in $\bar{a}$. Denote

$$w_1 = \sum_{i=1}^{r} x_i' \sum_{p=1}^{k_0} \varphi([t^{(1,-\bar{a})}, t^{(-1,-\bar{a}_{i,p})}]) t^{(-1,-\bar{a}_{i,1})} \circ \cdots \circ \widehat{t^{(-1,-\bar{a}_{i,p})}} \circ \cdots \circ t^{(-1,-\bar{a}_{i,k_0})} v_0,$$

$$w_2 = \sum_{\underline{l}} f_{\underline{l}}(\bar{a}) t^{(-1,-l_1-\bar{a})} \circ t^{(-1,-l_2)} \circ \cdots \circ t^{(-1,-l_{k_0-1})} v_0.$$

Now for $\bar{a}$ with sufficiently large $a(1) \in \mathbb{Z}$, let

$$R = \{t^{(-1,-\bar{a}_{i,1})} \circ \cdots \circ \widehat{t^{(-1,-\bar{a}_{i,p})}} \circ \cdots \circ t^{(-1,-\bar{a}_{i,k_0})} v_0 \mid i = 1, 2, \cdots, r\},$$

which is the set of all possible basis elements in $w_1 \in \widetilde{V}(\varphi)_{-(k_0-1)}$, and

$$T = \{t^{(-1,-l_1-\bar{a})} \circ t^{(-1,-l_2)} \circ \cdots \circ t^{(-1,-l_{k_0-1})} v_0 \mid (l_1, l_2, \cdots, l_{k_0-1}) \in J\},$$

which is the set of all possible basis elements in $w_2 \in \widetilde{V}(\varphi)_{-(k_0-1)}$. Clearly, for $\bar{a} \in \mathbb{Z}^n$ with sufficiently large $a(1) \in \mathbb{Z}$, $R \cup T$ is linearly independent in the vector space $(U^{(k_0-1)}_{-k_0+1} v_0)/(U^{(k_0-2)}_{-k_0+1} v_0)$. Thus $w_1 = 0 = w_2$ for $\bar{a} \in \mathbb{Z}^n$ with sufficiently large $a(1) \in \mathbb{Z}$. Since $f_{\underline{l}}(\bar{a})$ are exp-polynomial functions, using [7, Lemma 2.1] we deduce that $f_{\underline{l}}(\bar{a}) = 0$ for all $\bar{a} \in \mathbb{Z}^n$. Hence from (3.5) we have $0 = w_1 \in \widetilde{V}(\varphi)$ for all $\bar{a} \in \mathbb{Z}^n$.

In terms of linear combination of $\mathbf{B}v_0$, the coefficient of $t^{(-1,-\bar{a}_{1,1})} \circ t^{(-1,-\bar{a}_{1,2})} \circ \cdots \circ t^{(-1,-\bar{a}_{1,k_0-1})} v_0$ in the expression of w_1 is 0, i.e.,

$$\sum_{i \in I} p_i x_i' \varphi([t^{(1,-\bar{a})}, t^{(-1,-\bar{a}_{i,k_0})}]) = 0, \forall \bar{a} \in \mathbb{Z}^n$$

where $I = \{1 \le i \le r \mid \bar{a}_{i,p} = \bar{a}_{1,p}, \forall 1 \le p \le k_0 - 1)\}$, p_i is the number of q such that $\bar{a}_{i,q} = \bar{a}_{i,k_0}$. Let $I_1 = \{i \in I \mid \bar{a}_{i,k_0} - \bar{a}_{1,k_0} \in \operatorname{rad} \bar{f}\}$. Noting that $\varphi(t^b) = 0, \forall b \in (0, \mathbb{Z}^n) \setminus \operatorname{rad} \bar{f}$, we have

$$\sum_{i \in I_1} p_i x_i' \varphi([t^{(1,\bar{a}_{1,k_0}+\bar{a})}, t^{(-1,-\bar{a}_{i,k_0})}]) = 0, \forall \bar{a} \in \operatorname{rad} \bar{f}. \tag{3.6}$$

From (1.1) we know that $[t^{(1,\bar{a}_{1,k_0}+\bar{a})}, t^{(-1,-\bar{a}_{i,k_0})}] \neq 0$. Using Lemma 2.9 and (2.8), we have

$$\begin{aligned}[t^{(1,\bar{a}_{1,k_0}+\bar{a})}, t^{(-1,-\bar{a}_{i,k_0})}] &= f^{1,1,1}_{1,-1}((\bar{a}_{1,k_0} + \bar{a}), (-1, -\bar{a}_{i,k_0})) g_0^{(1)}(\bar{a} - \bar{a}_{i,k_0} + \bar{a}_{1,k_0}) \\ &= \prod_{j=1}^{n} q_{k,0}^{-\bar{a}(k) - \bar{a}_{1,k_0}(k)} \sigma(\bar{a}_{1,k_0}, \bar{a}_{i,k_0}) g_0^{(1)}(\bar{a} - \bar{a}_{i,k_0} + \bar{a}_{1,k_0}).\end{aligned}$$

Now it is a straightforward computation to see that (3.6) implies (3.1). This completes the proof of the theorem. ∎

https://doi.org/10.4153/CJM-2010-022-1 Published online by Cambridge University Press

We can apply this theorem to some special cases.

Corollary 3.2 (i) *If* $\operatorname{rad} \bar{f} = \{(0,\dots,0)\}$, *then the Verma module* $\widetilde{V}(\varphi)$ *is irreducible if and only if* $\varphi(t^{(0,\dots,0)}) \neq 0$.

(ii) *If* $n = 1$ *and* $q_{0,1}$ *is generic, then the Verma module* $\widetilde{V}(\varphi)$ *is not irreducible if and only if* φ *satisfies* (2.11).

(iii) *If* $n = 1$ *and* $q_{0,1}$ *is the* m*-th primitive root of unity, then the Verma module* $\widetilde{V}(\varphi)$ *is not irreducible if and only if there exists some exp-polynomial function* $h\colon \mathbb{Z} \to \mathbb{C}$ *such that* $\varphi(t_1^{mi}) = h(i), \forall i \in \mathbb{Z}$.

Remark 3.3 Corollary 3.2(ii) was obtained in [13]. From part (iii) and Theorem 2.11, we know that there exist some φ so that $\widetilde{V}(\varphi)$ is not irreducible and $V(\varphi)$ has some infinite dimensional weight spaces.

We would like to conclude this paper with two easy examples.

Let w_k be a primitive k-th root of unity, $q_{1,0}$ a fixed generic complex number, *i.e.*, $q_{1,0}$ is not a root of unity, and

$$q = \begin{pmatrix} 1 & q_{1,0}^{-1} & 1 \\ q_{1,0} & 1 & w_k^{-1} \\ 1 & w_k & 1 \end{pmatrix} \tag{3.7}$$

It is easy to see that $\operatorname{rad}_0 f_q = \operatorname{rad} f_q = (0,0,k\mathbb{Z})$, $\operatorname{rad} \bar{f}_q = (0,k\mathbb{Z},k\mathbb{Z})$. By Theorem 2.11 an irreducible highest weight module $V(\varphi)$ over $\mathfrak{L}_q$ has finite dimensional weight spaces if and only if there exists a 2-variable exp-polynomial function $h\colon \mathbb{Z}^2 \to \mathbb{C}$, such that $\varphi(t_1^{ki}t_2^{kj}) = \frac{h(i,j)}{1-q_{1,0}^{ki}+\delta_{i,0}}$.

Example 3.4 Let q be the same as in (3.7) and let $\varphi(t_1^{ki}t_2^{kj}) = \delta_{i,0}\delta_{j,0}$. It is easy to see that there does not exist a polynomial $p(t) \in \mathbb{C}[t]$ satisfying the conditions in Theorem 3.1(i). Hence from Theorem 3.1, the Verma module $\widetilde{V}(\varphi)$ is irreducible.

Example 3.5 Let q be the same as defined in (3.7) and let $\varphi(t_1^{ki}t_2^{kj}) = \frac{\delta_{i,0}}{1-q_{1,0}^{ki}+\delta_{i,0}} = \delta_{i,0}$. It is easy to check that $g(i,j) := \delta_{i,0}$ is not an exp-polynomial function. Hence from Theorem 2.11, the highest weight module $V(\varphi)$ has an infinite dimensional weight space. Clearly, $\varphi(t_1^{ki}t_2^{kj}(1-t_2)) = 0$ for all $i,j \in \mathbb{Z}$. From Theorem 3.1(i), we know that the Verma module $\widetilde{V}(\varphi)$ is not irreducible.

Acknowledgement The authors would like to thank the referee for pointing out several inaccuracies, and making valuable suggestions to improve the paper.

References

[1] S. Berman, Y. Gao, and Y. S. Krylyuk, *Quantum tori and the structure of elliptic quasi-simple Lie algebras.* J. Funct. Anal. **135**(1996), no. 2, 339–389. doi:10.1006/jfan.1996.0013

[2] S. Berman and Y. Billig, *Irreducible representations for toroidal Lie algebras.* J. Algebra **221**(1999), no.1, 188–231. doi:10.1006/jabr.1999.7961

[3] S. Berman and J. Szmigielski, *Principal realization for the extended affine Lie algebra of type* sl_2 *with coordinates in a simple quantum torus with two generators.* In: Recent Developments in Quantum Affine Algebras and Related Topics. Contemp. Math. 248, American Mathematical Society, Providence, RI, 1999, pp. 39–67.
[4] Y. Billig, *Representations of toroidal Lie algebras and Lie algebras of vector fields.* Resenhas **6**(2004), no. 2-3, 111–119.
[5] ———, *A category of modules for the full toroidal Lie algebra.* Int. Math. Res. Not. **2006** Art. ID 68395.
[6] ———, *Representations of toroidal extended affine Lie algebras.* J. Algebra **308**(2007), no. 1, 252–269. doi:10.1016/j.jalgebra.2006.09.010
[7] Y. Billig and K. Zhao, *Weight modules over exp-polynomial Lie algebras.* J. Pure Appl. Algebra **191**(2004), no. 1-2, 23–42. doi:10.1016/j.jpaa.2003.12.004
[8] ———, *Vertex operator representations of quantum tori at roots of unity.* Commun. Contemp. Math. **6**(2004), no. 1, 195–220. doi:10.1142/S0219199704001252
[9] R.A. Brualdi, *Introductory Combinatorics*, Second edition. North-Holland Publishing, New York, 1992.
[10] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory.* Springer-Verlag, New York, 1997.
[11] S. Eswara Rao, *Irreducible representations for toroidal Lie-algebras.* J. Pure Appl. Algebra **202**(2005), no. 1-3, 102–117. doi:10.1016/j.jpaa.2005.01.011
[12] S. Eswara Rao and R. V. Moody, *Vertex representations for n-toroidal Lie algebras and a generalization of the Virasoro algebra.* Comm. Math. Phys. **159**(1994), no. 2, 239–264. doi:10.1007/BF02102638
[13] S. Eswara Rao and K. Zhao, *Highest weight irreducible representations of rank 2 quantum tori.* Math. Res. Lett. **11**(2004). no.5-6, 615–628.
[14] S. Eswara Rao and C. Jiang, *Classification of irreducible integrable representations for the full toroidal Lie algebras.* J. Pure Appl. Algebra **200**(2005), no.1-2, 71–85. doi:10.1016/j.jpaa.2004.12.051
[15] J. Fu and C. Jiang, *Integrable representations for the twisted full toroidal Lie algebras.* J. Algebra **307**(2007), no. 2, 769–794. doi:10.1016/j.jalgebra.2006.08.031
[16] Y. Gao, *Representations of extended affine Lie algebras coordinatized by certain quantum tori.* Compositio Math. **123**(2000), no. 1, 1–25. doi:10.1023/A:1001830622499
[17] ———, *Vertex operators arising from the homogeneous realization for* $\widehat{\mathrm{gl}}_N$. Comm. Math. Phys. **211**(2000), no. 3, 745–777. doi:10.1007/s002200050834
[18] M. Golenishcheva-Kutuzova and D. Lebedev, *Vertex operator representation of some quantum tori Lie algebras.* Comm. Math. Phys. **148**(1992), no. 2, 403–416. doi:10.1007/BF02100868
[19] T. W. Hungerford, *Algebra.* Graduate Texts in Mathematics 73. Springer-Verlag, New York, 1980.
[20] C. Jiang and Q. Jiang, *Irreducible representations for the abelian extension of the Lie algebra of diffeomorphisms of tori in dimensions greater than 1.* Pacific J. Math. **231**(2007), no. 1, 85–101.
[21] W. Lin and S. Tan, *Harish-Chandra modules for the q-analog Virasoro-like algebra.* J. Algebra **297**(2006), no. 1, 254–272. doi:10.1016/j.jalgebra.2005.09.010
[22] J. C. McConnell and J. J. Pettit, *Crossed products and multiplicative analogues of Weyl algebras.* J. London Math. Soc. **38**(1988), no. 1, 47–55. doi:10.1112/jlms/s2-38.1.47
[23] K. Zhao, *Weyl type algebras from quantum tori.* Commun. Contemp. Math. **8**(2006), no. 2, 135–165. doi:10.1142/S0219199706002064

Department of Mathematics, Suzhou university, Suzhou 215006, Jiangsu, P.R. China
e-mail: rencail@amss.ac.cn

Department of Mathematics, Wilfrid Laurier University, Waterloo, ON, Canada N2L 3C5

and

Institute of Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100190, P.R. China
e-mail: kzhao@wlu.ca