ON FACTORIZATION OF ELLIPTIC FUNCTIONS

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I. Introduction. In this paper we shall be concerned with the following problem: If h is an elliptic function and $h(z) = f(g(z))$, what can be said about the functions f and g? In order to simplify the discussion we introduce some basic definitions.

Definition 1. A meromorphic function $h(z) = f(g(z))$ is said to have $f(z)$ and $g(z)$ as left and right factors, respectively, provided that either $f(z)$ is non-linear and meromorphic and $g(z)$ is non-linear and entire or $f(z)$ is rational and $g(z)$ is meromorphic.

Definition 2. $h(z)$ is said to be prime if every factorization $h(z) = f(g(z))$ implies that one of the functions $f(z)$ or $g(z)$ is linear.

Definition 3. $h(z)$ is said to be pseudo-prime if every factorization $h(z) = f(g(z))$ implies that either $f(z)$ is rational or $g(z)$ is a polynomial.

All factorizations fall into one of the three following categories:
(a) f rational and g meromorphic,
(b) f meromorphic and g entire and transcendental,
(c) f meromorphic, g polynomial.
We shall have occasion to refer to these categories (a), (b), and (c) in what follows.

II. Theorems about right factors.

LEMMA 1 (the inverse function theorem). *If $w:f(\xi)$ is meromorphic in $|\xi - \xi_0| < R$, then it is possible to find a ρ, $\delta > 0$ and a positive integer q such that all solutions ξ of the equation*

$$f(\xi) = w \qquad (|w - f(\xi_0)| < \rho)$$

which satisfy $|\xi - \xi_0| < \delta$ are given by

$$\xi = \phi((w - f(\xi_0))^{1/q}),$$

where $\phi(t)$ is a Laurant series in t with at most a finite number of negative powers.

Using Lemma 1, one can easily prove

Received January 13, 1967.
The author is indebted to the referee for useful remarks including the statement and proof of Lemma 2.

https://doi.org/10.4153/CJM-1968-047-5 Published online by Cambridge University Press

LEMMA 2. *Let* $h(z) = f(g(z))$ *be an elliptic function and* $\{c_n\}$ *a sequence of complex numbers tending to* ∞. *Choose a period parallelogram* P *of* $h(z)$ *and determine the period* w_n *of* $h(z)$ *so that* $c_n - w_n \in P$. *If* $\{g(c_n)\}$ *is bounded, then one of the numbers* $w_k - w_l$ *is a period of* $g(z)$.

Proof. We may suppose that

$$c_n - w_n \to c_0 \in P;$$

then $g(c_n - w_n) \to g_0 = g(c_0)$.

By the inverse function theorem there is a neighbourhood N of $f(g_0)$ so that for all $w \in N$, all solutions of

$$f(\zeta) = w$$

which lie in $|\zeta - g_0| < \delta$ are given by the formula

$$(1) \qquad \begin{aligned} \zeta &= \phi[(w - f(g_0))^{1/q}], \qquad & f(g_0) &\neq \infty, \\ \zeta &= \phi[w^{-1/q}] & f(g_0) &= \infty, \end{aligned}$$

and ϕ is a Laurent series with a finite number of negative terms, at most.

Let

$$w = h(z - w_n) \quad (= h(z))$$

and let

$$\zeta = g(z - w_n).$$

Then, for all sufficiently large n and $|z - c_n| < \epsilon$, say, $g(z - w_n)$ is given by one of the determinations of (1). Since there are only a finite number of these,

$$g(z - w_k) \equiv g(z - w_l)$$

for two indices k and l. This completes the proof of the lemma.

Immediate consequences are Theorems 1 and 2 below.

THEOREM 1. *In case* (a), $g(z)$ *is elliptic.*

Consider from now on only factorizations (b) and (c).

THEOREM 2. *In case* (b) $g(z)$ *is a periodic function; there is a number* $\alpha \neq 0$ *and an integer* M *such that*

$$(2) \qquad g(z) = \sum_{k=-M}^{M} c_k e^{\alpha k z}.$$

At least one c_k *with* $k > 0$ *and one* c_k *with* $k < 0$ *is different from* 0.

Proof. It suffices to show that the functions of the form (2) are the only periodic functions tending to ∞ as $x \to \infty$ $(z = x + iy)$. Suppose that $g(z)$ is periodic, of period $2\pi i$, $g(x + yi) \to \infty$ as $x \to \infty$. Let $\zeta = e^z$; then $g(z)$ is an analytic function of ζ in $0 < |\zeta| < \infty$. Therefore

$$(3) \qquad g(z) = \sum_{-\infty}^{\infty} c_n \zeta^n$$

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and if g tends to limit as $x \to \infty$, then $\zeta = 0$ and $\zeta = \infty$ are either removable singularities or poles of (3), i.e.,

$$g(z) = \sum_{-M}^{M} c_n \zeta^n.$$

THEOREM 3. *A right factor of an elliptic function of order 2 (i.e., 2 poles in its period parallelogram) must be either a polynomial of degree 2, of the form $A\cos(cz + \tau) + B$ or elliptic where A, B, c, and τ are constants. The latter case, clearly, can only occur when the corresponding left factor is rational.*

Proof. If f is any transcendental meromorphic function and $g(z)$ is a transcendental entire function, then

$$T(r, f(g))/T(r, g) \to \infty$$

as $r \to \infty$ (see Hayman **5**). Any elliptic function, h, of order 2 satisfies an equation of the form

$$(h')^2 = P(h)$$

where P is a polynomial. Suppose that $h = f(g)$; then $(g'f'(g))^2 = P(f(g))$.

We may assume that f is transcendental. (This follows from the inverse function theorem.)

By the opening remark in our proof, we see that

$$(g')^2 = Q_1(g)/Q_2(g),$$

where $Q_1(w)$ and $Q_2(w)$ are relatively prime polynomials. Let

$$Q_1(w) = c_1(w - a_1)^{n_1} \dots (w - a_k)^{n_k},$$
$$Q_2(w) = c_2(w - b_1)^{t_1} \dots (w - b_m)^{t_m};$$

c_1, c_2 constants. Then

$$\frac{(g - a_1)^{n_1}(g - a_2)^{n_2} \dots (g - a_k)^{n_k}}{(g - b_1)^{t_1}(g - b_2)^{t_2} \dots (g - b_m)^{t_m}} = c(g')^2, \tag{4}$$

c constant.

Since g' has no poles, the denominator of (4) cannot vanish. Thus we may assume that $t_i = 0$ for $i > 1$. If $t_1 \neq 0$, then g must be of the form $e^\alpha + b_1$, where α is entire. This implies that g cannot have any additional completely ramified values. Hence in this case (4) implies that

$$c(\alpha' e^\alpha)^2 = e^{-t_1\alpha},$$

which is impossible.

We, therefore, conclude that

$$(g - a_1)^{n_1}(g - a_2)^{n_2} \dots (g - a_k)^{n_k} = c(g')^2. \tag{5}$$

Since g can have at most two completely ramified values in the finite plane, we get

$$(g - a_1)^{n_1}(g - a_2)^{n_2} = c(g')^2. \tag{6}$$

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By Theorem 2 we may assume that g attains a_1. If $n_1 > 1$, then g' must have a double zero, so that g'' vanishes whenever $g(z) = a_1$. Thus g must attain a_1 triply everywhere. Repeating this argument we would arrive at the ridiculous conclusion that g attains a_1 with infinite multiplicity.

The only possible values for n_1 and n_2, therefore, are 0 and 1.

Suppose that g is a polynomial of degree n. Then $(g')^2$ has degree $2n - 2$ while the left side of (6) has degree n or $2n$. It follows, in this case, that $n = 2$ is the only possibility.

When g is transcendental, n_1 and n_2 must both be equal to 1. Integrating the resulting equation one gets

$$g = A\cos(cz + \tau) + B,$$

where A, B, c, and τ are constants. This completes the proof.

With respect to Theorem 3, we note that the Weierstrass $\wp$-function is even and can be expressed as the ratio of two even entire functions. Thus, it can be written as $f(z^2)$, where f is meromorphic. There also exist meromorphic functions $f_n(z)$ with the property that $f_n(z^n)$ is elliptic for $n = 3, 4$, and 6. We prove this for $n = 6$. A similar proof exists for the case $n = 4$.

The Weierstrass $\wp$-function can be expressed as

$$\wp(z) = \left(1 + \sum_{k=2}^{\infty} c_k z^{2k}\right) \Big/ z^2,$$

where the coefficients c_k, for $k \geqslant 4$, satisfy the recursive relation

$$c_k = \frac{3}{(2k+1)(k-3)} \sum_{m=2}^{k-2} c_m c_{k-m}$$

and

$$c_2 = g_2/20, \qquad c_3 = g_3/28$$

(see N.B.S. Handbook **4**).

One can easily verify, by induction, that when $g_2 = 0$, $c_k = 0$ if and only if $k \not\equiv 0 \pmod 3$.

Thus, the corresponding $\wp$-function is of the form $f(z^6)/z^2$. Consequently $G(z) = \wp(z)^3$ may be written as $h(z^6)$, where h is a meromorphic function.

On the other hand, a circle of least positive radius containing a period can have only 2, 4, or 6 periods lying on it. It follows that for $n = 5$ or $n \geqslant 7$, z^n cannot be a right factor of an elliptic function.

This leads to the following conjecture about factorizations (c) which the author has not been able to resolve.

Conjecture. If g is a polynomial of degree n, where $n = 5$ or $n \geqslant 7$, and f is any meromorphic function, then $f(g)$ is not elliptic.

Particular functions having transcendental right factors of the type described in Theorem 3 are

$$\wp, \text{ sn, dn, cn, and } \sqrt{(\wp - e_1)}.$$

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In fact, explicit factorizations can easily be found. For example (using the notation used by Hille in **(6)**).

$$\operatorname{sn}(2Kz/\pi) = 2q^{1/4}k^{-1/2} \sin z \prod_{n=1}^{\infty} \left(\frac{1 - 2q^{2n} \cos 2z + q^{4n}}{1 - 2q^{2n-1} \cos 2z + q^{4n-2}}\right).$$

Thus,

$$\operatorname{sn}(2Kz/\pi) = f(\sin z),$$

where

$$f(w) = cw \prod_{n=1}^{\infty} \left(\frac{1 - 2q^{2n}(1 - 2w^2) + q^{4n}}{1 - 2q^{2n-1}(1 - 2w^2) + q^{4n-2}}\right), \quad c \text{ a constant,} \tag{7}$$

is a meromorphic function.

More generally we remark that every elliptic function of order 2 has a right factor of the form $\cos(az + b)$, where a and b are constants. This follows from the fact that every even elliptic function is a rational function of a $\wp$-function (see Hille **6**) and from the fact that every elliptic function of order 2 has the property that for some constant γ, $\phi(2\gamma - z) = \phi(z)$ (see Neville **7**). Simply let $z' = z - \gamma$ and $\bar{\phi}(z') = \phi(z' + \gamma)$ and observe that $\bar{\phi}(z')$ is an even elliptic function.

While the above functions have transcendental right factors, there are some functions for which this is not the case.

THEOREM 4. *Let*

$$\wp = \frac{1}{z^2} + \frac{g_3}{28} z^4 + \ldots \qquad \text{(i.e., take } g_2 = 0).$$

$\wp'$ is pseudo-prime. The only possible non-elliptic right factor is a cubic polynomial.

Proof. $\wp'$ satisfies a differential equation of the form

$$(w')^3 = P(w),$$

where $P(w)$ is a polynomial.

Suppose $\wp' = f(g)$.

As in the proof of Theorem 3 we conclude that

$$(g')^3 = c(g - a_1)^{n_1}(g - a_2)^{n_2} \ldots (g - a_k)^{n_k},$$

and that g attains every value. If $n_1 > 0$ and $n_2 > 0$, then g must attain a_1 and a_2, each with multiplicity 3. This is impossible. Hence we may assume that for $i > 1$, $n_i = 0$. One sees easily that $n_1 \leqslant 2$ and that g must be a polynomial satisfying the condition

$$(g')^3 = c(g - a_1)^{n_1}.$$

If g has degree K, then $(g')^3$ has degree $3(K - 1)$, while $(g - a_1)^{n_1}$ has degree $n_1 K$. Hence

$$3(K - 1) = n_1 K$$

and $K = 3$, $n_1 = 2$. Our proof is complete.

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It is clear from the series for $\wp$ that $\wp'(z) = f(z^3)$ is a valid factorization and one can find f by looking at the series for $\wp'(z)$. It is interesting to note, however, that by using the condition established in the proof of Theorem 4

$$P(f(w))/(f'(w))^3 = c(w - a_1)^2, \tag{8}$$

P a polynomial, one can express f in closed form. This has the advantage that many of the properties of the possible polynomial right factors might be easier to analyse when f is expressed in this form.

Using (8) one can show that when z^3 is replaced by an arbitrary cubic polynomial of third degree,

$$f(w) = \pm\sqrt{[9^3(\wp_1(c_0(w - a_1)^{1/3} + c))^3 - g_3]},$$

c_0 a constant, where $\wp_1(u)$ is a Weierstrass elliptic function with its g_2 constant equal to zero.

We note that when $g = z^3$, f corresponds to the values $a_1 = c = 0$.

It would be interesting to know whether prime elliptic functions also exist. Unfortunately, the author has not been able to resolve this problem.

By the method used in the proof of Theorem 3 one can also prove

THEOREM 5. *An elliptic function η has a common right factor with an elliptic function ϕ of order 2 if and only if η is a rational function of ϕ.*

COROLLARY. *Any two elliptic functions of order 2 having a common right factor must be linear transformations of each other.*

III. Theorems about left factors.

THEOREM 6. *Let h be elliptic and suppose it has the factorization*

$$h = f \circ g.$$

If, in addition, for some non-rational function ϕ, f has the factorization $f = \phi \circ \chi$, then either χ is of order $1/2$ and g is a quadratic polynomial or χ is a polynomial.

Proof. By Theorem 2, either g is a polynomial or g is transcendental and both g and $\chi(g)$ are of exponential type with the type of g, $\tau(g)$, positive. Suppose that the latter holds if χ is also transcendental. Then

$$T(r, \chi(g))/T(r, g) \to \infty \qquad \text{as } r \to \infty$$

(see Hayman **5**). Consequently, in this case, $\tau(\chi(g)) = \infty$ and we have a contradiction. Thus, if χ is transcendental, we may assume that g is a polynomial and $\chi(g)$ is periodic transcendental of order 1. It follows (see **2**, Theorem 6) that g must be a quadratic polynomial and χ of order $1/2$. This completes the proof.

For particular functions one can say more about the left factor; for example,

THEOREM 7. *f defined by* (7) *must be pseudo-prime.*

Remark. Though this follows from Theorem 6, we give a somewhat more illustrative direct proof.

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Proof. For suppose that $f = l(g)$, g entire and transcendental. Then $l(g(\cos(cz + \tau)))$ is an elliptic function of order 2 which has $g(\cos cz + \tau)$ as a right factor. $g(\cos(cz + \tau))$ must, therefore, be of the form $\cos(c'z + \tau')$. But for any $\bar{K} > 0$,

$$T(r, g(\cos(cz + \tau))) > \bar{K}T(r, \cos(cz + \tau))$$

for sufficiently large r. One can easily verify that

$$M_{\cos \pi z}(r) \sim c_1 e^{c_2 r}, \qquad c_1, c_2 \text{ constant.}$$

It follows from the well-known fact

$$T(r,f) \leqslant \log^+ M(r,f) \leqslant \frac{R+r}{R-r} T(R,f) \qquad (0 \leqslant r \leqslant R)$$

that

$$T(r, \cos cz) \sim cr, \qquad c \text{ a constant.}$$

Thus,

$$T(r, \cos(c'z + \tau')) < c''r < \bar{K}'cr < \bar{K}T(r, \cos(z + \tau))$$

for sufficiently large $\bar{K}'$ and $\bar{K}$. Our assertion therefore follows.

By a method similar to the proof of Theorem 7 in **(3)** we prove

THEOREM 8. *Let h be an elliptic function with left and right factors f and g respectively. If g is entire, then f has no deficient values.*

Proof. Applying Lemma 1 to entire functions, one can easily verify that f cannot be a rational function. Furthermore, h, being an elliptic function, has no deficient values. Thus, if g is a polynomial, one can easily verify that f has no deficient values either. Let us, therefore, assume that both f and g are transcendental and that g is not elliptic. Let $\tau = re^{i\theta}$ be a period of h. We may assume that g is not periodic with a period having argument θ. Let L be equal to the half-line $re^{i\theta}$ everywhere except near poles of h, where we let L loop around them with radius ϵ, ϵ a small positive number. h is bounded on L. If g is also bounded on L, then it follows that g is periodic with a period having argument θ. Since this contradicts our hypothesis, we may assume that $g(L)$ is a curve extending to infinity. Furthermore, since g is assumed to be transcendental, f must be of zero order.* By a well-known extension of Wiman's theorem **(**see Hayman **5)**, a meromorphic function of order zero which has a deficient value, a, satisfies the condition that for some sequence $r_n \to +\infty$, $f(r_n e^{i\alpha}) \to a$ as $n \to \infty$, uniformly for $0 \leqslant \alpha \leqslant 2\pi$. Thus ∞ is not a deficient value of f. Similarly, by considering for arbitrary points, a, the function $1/(h - a)$ one sees that a is not a deficient value of f and our theorem follows.

While right factors are either polynomials or periodic, (Theorems 1 and 2) left factors are never periodic.

THEOREM 9. *An elliptic function cannot have a periodic left factor.*

We shall need the following.

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*See Lemma 3 below.

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LEMMA 3 (Edrei and Fuchs **1**). *If f is any meromorphic function and g is entire, then $f(g)$ is of finite order implies that either f is of finite order and g is a polynomial or f is of zero order.*

Proof of theorem. Let $h = f(g)$ be an elliptic function. If g is transcendental, then f must be of order zero. This is an immediate consequence of Lemma 3. Since any periodic function is at least of order 1, it follows that f cannot be periodic. Before proceeding with the case when g is a non-linear polynomial we prove the following fact:

Let $P(z)$ be any non-linear polynomial and let S be the locus of points satisfying at least one of the equations $P(z) = m + c$, c a constant, $m = 0, \pm 1, \pm 2, \ldots$. Then S has an infinite sequence of elements z_i such that $|z_{i+j_i} - z_i|, j_i > 0$, takes on arbitrary small values as i approaches infinity.

The proof of this fact is fairly simple. Let

$$P(z) = A_0 + A_1 z + \ldots + A_k z^k.$$

Suppose that

$$A_0 + A_1 z_m + \ldots + A_k z_m^k = m + c,$$
$$A_0 + A_1 z_{m+j} + \ldots + A_k z_{m+j}^k = m + j + c.$$

Then

$$(9) \quad |z_{m+j} - z_m|\ |A_k|\ |z_{m+j}^{k-1} + z_m\, z_{m+j}^{k-2} + \ldots + z_m^{k-1} + P_{k-2}(z_m, z_{m+j})| = j,$$

where P_{k-2} is a polynomial in z_m, z_{m+j} of degree $k - 2$.

One can easily verify that for every m there exist j_1 and j_2, greater than or equal to m, whose difference is less than $4k^2$ and which satisfy the condition that

$$|\arg z_{j_1} - \arg z_{j_2}| \leqslant 2\pi/4k^2.$$

Thus, we may assume in (9) that the arguments of z_{m+j} and z_m differ by no more than $2\pi/4k^2$. It follows that

$$|z_{m+j}^{k-1} + \ldots + P_{k-2}(z_m, z_{m+j})| > \max\,(|z_{m+j}|^{k-1}, |z_m|^{k-1}) - |P_{k-2}|.$$

This factor must approach infinity through an infinite sequence of integers m. Thus, it follows from (9) that $|z_{m+j} - z_m|$ takes on arbitrarily small values as m approaches infinity.

We now proceed with the remainder of the proof of Theorem 9. Suppose that f is periodic with period 1 and $f(g)$ has periods τ_1 and τ_2. Then, we have

$$(10) \qquad f(g(z + n_1\tau_1 + n_2\tau_2) + m) = f(g(z)).$$

For any fixed z_0 the locus of zeros of

$$\{g(z + n_1\tau_1 + n_2\tau_2) + m - g(z_0)\,;\, n_1, n_2, m \text{ integers}\}$$

has a limit point, by virtue of what we proved above. Thus, from (10), one easily concludes that f is a constant. This completes the proof of the theorem.

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Bellcomm, Inc.,
Washington, D.C.

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