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AN ABSTRACT DAUNS-HOFMANN-KAPLANSKY MULTIPLIER THEOREM

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1. Introduction. The present investigation was stimulated by a theorem of Alfsen and Effros (4.9 of [**1**]) concerning a real Banach space, its M-ideals, and its primitive M-ideals (these are defined in [**1**]). This theorem states that a real Banach space is in the natural way a module over the ring of bounded continuous real-valued functions on the space of primitive M-ideals with the Jacobson topology.

It was shown in 6.18 and 6.19 of [**1**] that the M-ideals of the self-adjoint part of a C^*-algebra are the self-adjoint parts of the closed two-sided ideals of the C^*-algebra, and that the primitive M-ideals are the self-adjoint parts of the primitive ideals. In this specialization, then, the theorem 4.9 of [**1**] becomes a theorem which was proved by Dauns and Hofmann (III.5.2 and III.8.16 of [**2**]), after a special case had been established by Kaplansky (Theorem 3.3 of [**4**]).

In [**3**], a proof was given of the Dauns-Hofmann theorem which is in fact at the same time a proof of the Alfsen-Effros theorem referred to above. This is of interest both because the argument in [**3**] is considerably simpler than the argument in [**1**], and because the argument in [**3**] also proves the Alfsen-Effros theorem for complex Banach spaces, indeed, for a Banach space over any finite-dimensional real algebra.

The contribution of the present paper is to show that the methods of [**3**] can be refined to yield a result which is, at least formally, a considerable generalization of the Alfsen-Effros multiplier theorem. The principal feature of this result is the absence of the assumption that the norm of an element is the supremum of its norms in primitive quotients. It is the sacrifice of this property which is the source of difficulty; if it were assumed then the methods of [**3**] would not need modification.

We shall formulate our result axiomatically; the axioms are presented in § 2 and the main result is proved in § 3. In § 4 the boundedness of the module structure is investigated, and also the related possibility of extending the main result to more general scalars.

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2. The axioms. Let A be a real Banach space. Let there be given a collection of closed subspaces of A, which we shall call G-ideals, such that 0 and A are

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G-ideals, and such that if J_1 and J_2 are G-ideals then $J_1 + J_2$ and $J_1 \cap J_2$ are G-ideals and the canonical linear isomorphism

$$J_1/J_1 \cap J_2 \to (J_1 + J_2)/J_2$$

is isometric. (The last condition can be expressed by saying that G-ideals are mutually orthogonal modulo intersections.) Let there be given a collection of proper G-ideals, which we shall call G-primitives, such that every G-ideal is an intersection of G-primitives. In these circumstances we shall say that A has a G-algebra structure.

If G-primitives are prime, and if the closure of a sum of G-ideals is again a G-ideal, then it is easy to show that those sets of G-primitives which are hulls (a hull being the set of all G-primitives containing some fixed G-ideal) form the closed sets of a topology. We shall not make these assumptions, howevers but work instead with an abstract notion of continuity. A real-valued function on G-primitives will be called continuous if the inverse image of an open interval is the complement of a hull.

3. The module structure.

3.1. THEOREM. *Let A be a real Banach space with a G-algebra structure. Let f be a bounded continuous real-valued function on G-primitives. Let x be an element of A. Then there exists a unique element fx of A such that for every G-primitive t,*

$$(fx)(t) = f(t)x(t).$$

Proof. The conclusion follows from 3.3 and 3.6.

3.2. *Remark.* If the norm on A satisfies

$$||y|| = \sup_{t\, G\text{-primitive}} ||y(t)||, \quad \text{all } y \in A,$$

then the conclusion of 3.1 can be established by the simple argument given in [**3**]. The entire purpose of the lemmas which follow is to show that this assumption on the norm is not essential.

3.3. LEMMA. *Let A be a Banach space with a G-algebra structure. Let $J_1, \ldots, J_n$ be G-ideals such that $J_1 + \ldots + J_n = A$, and let x be an element of A. Assume that there exist a bounded real-valued continuous function f on G-primitives and open intervals $O_1, \ldots, O_n$ such that* hull $J_1 = f^{-1}(O_1)^c, \ldots,$ hull $J_n = f^{-1}(O_n)^c$. *Then there exist $x_1 \in J_1, \ldots, x_n \in J_n$ such that $x = x_1 + \ldots + x_n$ and such that for any positive real $\lambda_1, \ldots, \lambda_n$,*

$$||\lambda_1 x_1 + \ldots + \lambda_n x_n|| \leqq (7/2) \sup_n |\lambda_n| \, ||x||.$$

Hence if $\lambda_1, \ldots, \lambda_n$ are arbitrary real numbers,

$$||\lambda_1 x_1 + \ldots + \lambda_n x_n|| \leqq 7 \sup_n |\lambda_n| \, ||x||.$$

Proof. We may suppose that the cover $\{O_1, \ldots, O_n\}$ of the range of f is minimal. Then the O's can be renumbered so that for $j \geqq i + 2$, O_j consists of real numbers strictly greater than those of O_i.

For each $i = 1, \ldots, n - 1$, denote by μ_i the point midway between the upper endpoint of O_i and the lower endpoint of O_{i+1}. By the definition of continuity of f there exist G-ideals $I_1, \ldots, I_n$ such that hull $I_1 = f^{-1}(]-\infty, \mu_1[)^c$, hull $I_2 = f^{-1}(]\mu_1, \mu_2[)^c, \ldots$, hull $I_n = f^{-1}(]\mu_{n-1}, \infty[)^c$. Set $I_1 + \ldots + I_n = I$. Then hull $I =$ hull $I_1 \cap \ldots \cap$ hull $I_n = f^{-1}(\{\mu_1, \ldots, \mu_{n-1}\})$.

For each $i = 1, \ldots, n - 1$, $O_i \cap O_{i+1}$ is an open interval (possibly empty) and therefore by continuity of f there exists a G-ideal K_i such that hull $K_i = f^{-1}(O_i \cap O_{i+1})^c$. Set $K_1 + \ldots + K_{n-1} = K$. Then hull $K =$ hull $K_1 \cap \ldots \cap$ hull $K_{n-1} = (\bigcup_{i=1}^{n-1} f^{-1}(O_i \cap O_{i+1}))^c$.

By construction, hull $(I + K) =$ hull $I \cap$ hull $K = \emptyset$. Since $I + K$ is a G-ideal and therefore the intersection of G-primitives containing it, $I + K = A$. In fact, $I_1 + K_1 = J_1$, $I_2 + K_1 + K_2 = J_2$, $I_3 + K_2 + K_3 = J_3, \ldots$, $I_n + K_{n-1} = J_n$. This follows from the calculations

$$\text{hull } I_1 \cap \text{hull } K_1 = f^{-1}(]-\infty, \mu_1[\cup O_1 \cap O_2)^c = f^{-1}(O_1)^c,$$
$$\text{hull } I_1 \cap \text{hull } K_1 \cap \text{hull } K_2$$
$$= f^{-1}(]\mu_1, \mu_2[\cup O_1 \cap O_2 \cup O_2 \cap O_3)^c = f^{-1}(O_2)^c, \ldots,$$
$$\text{hull } I_n \cap \text{hull } K_{n-1} = f^{-1}(]\mu_{n-1}, \infty[\cup O_{n-1} \cap O_n)^c = f^{-1}(O_n)^c.$$

There exist $y \in I$ and $z \in K$ such that $y + z = x$. Since by hypothesis the canonical linear isomorphism $I/I \cap K \to (I + K)/K$ is isometric, it is possible to choose y such that $||y|| \leqq (1 + 1/4)\ ||x||$. There exist $y_1 \in I_1, \ldots, y_n \in I_n$ such that $y = y_1 + \ldots + y_n$, and $z_1 \in K_1, \ldots, z_n \in K_n$ (where K_n denotes 0) such that $z = z_1 + \ldots + z_n$. Set $y_1 + z_1 = x_1, \ldots, y_n + z_n = x_n$. Then $x_1 \in J_1, \ldots, x_n \in J_n$ and $x = x_1 + \ldots + x_n$.

Let S be a subset of $\{1, \ldots, n\}$. Write $-\infty = \mu_0$, $+\infty = \mu_n$, and $\emptyset = O_{n+1}$. Then

$$(\textstyle\bigcap_{i\in S} \text{hull } I_i \cup \bigcap_{i\notin S} \text{hull } I_i)^c = f^{-1}\ (\bigcup_{i\in S}]\mu_{i-1}, \mu_i[\cap \bigcup_{i\notin S}]\mu_{i-1}, \mu_i[)$$
$$= \emptyset,$$

$$(\textstyle\bigcap_{i\in S} \text{hull } K_i \cup \bigcap_{i\notin S} \text{hull } K_i)^c = f^{-1}(\bigcup_{i\in S} O_i \cap O_{i+1} \cap \bigcup_{i\notin S} O_i \cap O_{i+1})$$
$$= \emptyset,$$

whence $\sum_{i\in S} I_i \cap \sum_{i\notin S} I_i$ and $\sum_{i\in S} K_i \cap \sum_{i\notin S} K_i$ are each contained in every G-primitive and are therefore both 0. By assumption two G-ideals with intersection 0 are orthogonal; in particular

$$||\textstyle\sum_{i\in S} y_i|| \leqq ||\sum_{i\in S} y_i + \sum_{i\notin S} y_i|| = ||y||,$$
$$||\textstyle\sum_{i\in S} z_i|| \leqq ||\sum_{i\in S} z_i + \sum_{i\notin S} z_i|| = ||z||.$$

Hence

$$||\textstyle\sum_{i\in S} x_i|| \leqq ||y|| + ||z|| \leqq ||y|| + ||x|| + ||y||$$
$$\leqq (2(1 + 1/4) + 1)||x|| = (7/2)\ ||x||.$$

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Since the characteristic functions of subsets S of $\{1, \ldots, n\}$ are the extreme points of the n-tuples of positive reals $(\lambda_1, \ldots, \lambda_n)$ with $\sup_n \lambda_n \leqq 1$, it follows by convexity that

$$||\lambda_1 x_1 + \ldots + \lambda_n x_n|| \leqq (7/2) \sup_n \lambda_n \, ||x||, \ \lambda_1, \ldots, \lambda_n \geqq 0.$$

If $\lambda_1, \ldots, \lambda_n$ are arbitrary real numbers,

$$||\lambda_1 x_1 + \ldots + \lambda_n x_n|| = ||\textstyle\sum_{\lambda_i \geqslant 0} \lambda_i x_i + \sum_{\lambda_i \leqslant 0} \lambda_i x_i|| \leqq 2(7/2) \sup_n |\lambda_n| \, ||x||.$$

3.4. *Problems.* It is not clear whether in 3.3 the assumption of the existence of f and open intervals $O_1, \ldots, O_n$ can be omitted. (Cf. Lemma 1 of [**3**].)

It would be of interest also to determine the best possible constants in the inequalities in 3.3. The constant in the inequality for positive λ's can certainly be no smaller than 1, and a two-dimensional example shows that in the inequality for arbitrary real λ's the best constant must be at least 2.

3.5. LEMMA. *Let A be a Banach space with a G-algebra structure. If J_1, J_2 are G-ideals and $x \in A$ then*

$$||x(J_1 \cap J_2)|| \leqq 2||x(J_1)|| + ||x(J_2)||.$$

Proof. Fix $\epsilon > 0$. Then

$$x = x_1 + y_1 \text{ with } y_1 \in J_1 \text{ and } ||x_1|| \leqq (1 + \epsilon)||x(J_1)||;$$
$$x = x_2 + y_2 \text{ with } y_2 \in J_2 \text{ and } ||x_2|| \leqq (1 + \epsilon)||x(J_2)||.$$

Using $x(J_1) = x_1(J_1) = x_2(J_1) + y_2(J_1)$, we deduce that

$$\begin{aligned}||y_2(J_1 \cap J_2)|| = ||y_2(J_1)|| = ||x(J_1) - x_2(J_1)|| &\leqq ||x(J_1)|| + ||x_2|| \\ &\leqq ||x(J_1)|| + (1 + \epsilon)||x(J_2)||.\end{aligned}$$

Hence

$$\begin{aligned}||x(J_1 \cap J_2)|| &\leqq ||x_2(J_1 \cap J_2)|| + ||y_2(J_1 \cap J_2)|| \\ &\leqq (1 + \epsilon)||x(J_2)|| + ||x(J_1)|| + (1 + \epsilon)||x(J_2)||.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary the desired inequality follows.

3.6. LEMMA. *Let A be a Banach space with a G-algebra structure. Let f be a bounded continuous real-valued function on G-primitives, and let x be an element of A. Let $J_1, \ldots, J_n$ be G-ideals such that $J_1 + \ldots + J_n = A$ and there exist open intervals $O_1, \ldots, O_n$ with* hull $J_1 = f^{-1}(O_1)^c, \ldots,$ hull $J_n = f^{-1}(O_n)^c$. *Let $x_1 \in J_1, \ldots, x_n \in J_n$ be such that $x_1 + \ldots + x_n = x$ and the inequalities in* 3.3 *are satisfied. Let $J'_1, \ldots, J'_{n'}, O'_1, \ldots, O'_{n'}, x'_1, \ldots, x'_{n'}$ satisfy the same conditions with primes added. Assume that the families (J_i) and $(J'_{i'})$ are minimal with sum A. Assume that all lengths $|O_i|$ are equal and all lengths $|O'_{i'}|$ are equal. Then for any $\lambda_i \in O_i$ and $\lambda'_{i'} \in O'_{i'}$,*

$$||\textstyle\sum_{i=1}^{n} \lambda_i x_i - \sum_{i'=1}^{n'} \lambda'_{i'} x'_{i'}|| \leqq 196 \sup \{|O_i|, |O'_{i'}|\} \, ||x||.$$

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Proof. We shall suppose that $|O'_{i'}| \leqq |O_i|$, and denote $|O_i|$ by ϵ. We shall denote $O_1 \cup \ldots \cup O_n$ by O; since $J_1 + \ldots + J_n = A$, $O \supset$ range f.

For $k = 1, \ldots, n$, denote by S_k the set of $i' = 1, \ldots, n'$ such that $J'_{i'} \not\subset \sum_{i \neq k} J_i$, so that $x(\sum_{i \neq k} J_i) = \sum_{i' \in S_k} x'_{i'}(\sum_{i \neq k} J_i)$. We shall suppose that the families (O_i) and $(O'_{i'})$ are labelled so that within each family intersections occur only for consecutive indices. Then $S_{k_1} \cap S_{k_2} = \emptyset$ if $|k_1 - k_2| > 1$. ($J'_{i'} \not\subset \sum_{i \neq k} J_i$ is equivalent to (hull $J'_{i'})^c \not\subset (\bigcap_{i \neq k}$ hull $J_i)^c$, which implies $O \cap O'_{i'} \not\subset \bigcup_{i \neq k} O_i$, i.e., $O'_{i'} \cap (O \backslash \bigcup_{i \neq k} O_i) \neq \emptyset$, and the distance between $O \backslash \bigcup_{i \neq k_1} O_i$ and $O \backslash \bigcup_{i \neq k_2} O_i$ is at least ϵ if $|k_1 - k_2| > 1$.)

Fix $k = 1, \ldots, n$. Then

$$x_k(\textstyle\sum_{i \neq k} J_i) = x(\sum_{i \neq k} J_i) = \sum_{i' \in S_k} x'_{i'}(\sum_{i \neq k} J_i).$$

Hence, with y denoting $\sum_{i=1}^{n} \lambda_i x_i - \sum_{i'=1}^{n'} \lambda'_{i'} x'_{i'}$,

$$\begin{aligned} y(\textstyle\sum_{i \neq k} J_i) &= (\lambda_k x_k - \sum_{i' \in S_k} \lambda'_{i'} x'_{i'})(\sum_{i \neq k} J_i) \\ &= \textstyle\sum_{i' \in S_k} (\lambda_k - \lambda'_{i'}) x'_{i'}(\sum_{i \neq k} J_i). \end{aligned}$$

Suppose that k is even. Since by what precedes $S_{k_1} \cap S_{k_2} = \emptyset$ for distinct even $k_1, k_2 = 1, \ldots, n$, if m is even and $m \neq k$ then $\sum_{i' \in S_m} J'_{i'} \subset \sum_{i \neq k} J_i$. Hence,

$$y(\textstyle\sum_{i \neq k} J_i) = \sum_{m \text{ even}} \sum_{i' \in S_m} (\lambda_m - \lambda'_{i'}) x'_{i'}(\sum_{i \neq k} J_i).$$

Now take the intersection over all even k. We have

$$y(\textstyle\bigcap_{k \text{ even}} \sum_{i \neq k} J_i) = \sum_{m \text{ even}} \sum_{i' \in S_m} (\lambda_m - \lambda'_{i'}) x'_{i'}(\bigcap_{k \text{ even}} \sum_{i \neq k} J_i).$$

We remark that if $i' \in S_m$ then $|\lambda_m - \lambda'_{i'}| < 2\epsilon$. ($J'_{i'} \not\subset \sum_{i \neq m} J_i$ is equivalent to (hull $J'_{i'})^c \not\subset (\bigcap_{i \neq m}$ hull $J_i)^c$, which implies $O \cap O'_{i'} \not\subset \bigcup_{i \neq m} O_i$, in particular, $O'_{i'} \cap O_m \neq \emptyset$.) Again bearing in mind that $S_{k_1} \cap S_{k_2} = \emptyset$ if $|k_1 - k_2| > 1$, by 3.3 we have

$$\|\textstyle\sum_{m \text{ even}} \sum_{i' \in S_m} (\lambda_m - \lambda'_{i'}) x'_{i'}\| \leqq 14\epsilon \|x\|.$$

Hence

$$\|y(\textstyle\bigcap_{k \text{ even}} \sum_{i \neq k} J_i)\| \leqq 14\epsilon \|x\|.$$

In the same way we obtain the inequality

$$\|y(\textstyle\bigcap_{k \text{ odd}} \sum_{i \neq k} J_i)\| \leqq 14\epsilon \|x\|.$$

Now define open intervals $O''_0, \ldots, O''_n$ as follows. We shall suppose that the interval O_1 is on the left in O. Denote by O''_i, $i = 0, \ldots, n$, the open interval of which the left endpoint is the right endpoint of O_{i-1}, and the right endpoint is the left endpoint of O_{i+2}. Here O_{-1}, O_0 and O_{n+1}, O_{n+2} denote some nonempty intervals outside O to the left and to the right respectively. The

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properties of $O''_0, \ldots, O''_n$ that we shall use are:

$$O \subset O''_0 \cup \ldots \cup O''_n;$$
$$O \backslash \bigcup_{i \neq p} O''_i = (O_p \cap O_{p+1})^-, \; p = 1, \ldots, n-1;$$
$$\text{for } |p_1 - p_2| > 2, \text{ distance } (O''_{p_1}, O''_{p_2}) \geqq \epsilon.$$

By continuity of f there exist G-ideals $K_0, \ldots, K_n$ such that hull $K_0 = f^{-1}(O''_0)^c, \ldots,$ hull $K_n = f^{-1}(O''_n)^c$. For $p = 1, \ldots, n-1$ denote by T_p the set of $i' = 1, \ldots, n'$ such that $J'_{i'} \not\subset \sum_{i \neq p} K_i$, so that $x(\sum_{i \neq p} K_i) = \sum_{i' \in T_p} x'_{i'}(\sum_{i \neq p} K_i)$. Then $T_{p_1} \cap T_{p_2} = \emptyset$ if $|p_1 - p_2| > 2$. ($J'_{i'} \not\subset \sum_{i \neq p} K_i$ is equivalent to (hull $J'_{i'})^c \not\subset (\bigcap_{i \neq p}$ hull $K_i)^c$, which implies $O \cap O'_{i'} \not\subset \bigcup_{i \neq p} O''_i$, in particular, $O'_{i'} \cap O''_p \neq \emptyset$, and the distance between O''_{p_1} and O''_{p_2} is at least ϵ if $|p_1 - p_2| > 2$.)

Fix $r = 1, 2, 3$ and set

$$\sum_{O_{r+3m} \cap O_{r+3m+1} \neq \emptyset} (\lambda_{r+3m} - \lambda_{r+3m+1}) x_{r+3m+1} = z_r.$$

Then all $\lambda_{r+3m} - \lambda_{r+3m+1}$, $m = 0, 1, \ldots, O_{r+3m} \cap O_{r+3m+1} \neq \emptyset$, are of absolute value less than 2ϵ, so by the second inequality of 3.3,

$$||z_r|| \leqq 7(2\epsilon)||x|| = 14\epsilon||x||.$$

Since $p_1 \neq p, \; p+1$ implies $O_{p_1} \subset O \backslash (O_p \cap O_{p+1})^-$, equivalently, $J_{p_1} \subset \sum_{i \neq p} K_i$, we have, for any $p = 1, \ldots, n-1$,

$$(x_p + x_{p+1})(\textstyle\sum_{i \neq p} K_i) = x(\sum_{i \neq p} K_i) = \sum_{i' \in T_p} x'_{i'}(\sum_{i \neq p} K_i),$$

and, for $p \equiv r \pmod 3$,

$$(y + z_r)(\textstyle\sum_{i \neq p} K_i) = \sum_{i' \in T_p} (\lambda_p - \lambda'_{i'}) x'_{i'} (\sum_{i \neq p} K_i).$$

(Note that if $O_p \cap O_{p+1} = \emptyset$ then $O \subset \bigcup_{i \neq p} O''_i$, so $\sum_{i \neq p} K_i = A$.) Using the fact that $T_{p_1} \cap T_{p_2} = \emptyset$ if $|p_1 - p_2| > 2$ we deduce that for each $p \equiv r \pmod 3$,

$$(y + z_r)(\textstyle\sum_{i \neq p} K_i) = \sum_{q \equiv r \pmod 3} \sum_{i' \in T_q} (\lambda_p - \lambda'_{i'}) x'_{i'} (\sum_{i \neq p} K_i),$$

and hence that

$$(y + z_r)(\textstyle\bigcap_{p \equiv r \pmod 3} \sum_{i \neq p} K_i)$$
$$= \textstyle\sum_{q \equiv r \pmod 3} \sum_{i' \in T_q} (\lambda_p - \lambda'_{i'}) \, x'_{i'} \, (\bigcap_{p \equiv r \pmod 3} \sum_{i \neq p} K_i).$$

Using again the fact that $T_{p_1} \cap T_{p_2} = \emptyset$ if $|p_1 - p_2| > 2$ and also using the fact that if $i' \in T_q$ then $|\lambda_q - \lambda'_{i'}| < 2\epsilon$ ($J'_{i'} \not\subset \sum_{i \neq q} K_i$ is equivalent to (hull $J'_{i'})^c \not\subset (\bigcap_{i \neq q}$ hull $K_i)^c$, which implies $O \cap O'_{i'} \not\subset \bigcup_{i \neq q} O''_i$, in particular, $O'_{i'} \cap O_q \neq \emptyset$), we have by 3.3

$$||\textstyle\sum_{q \equiv r \pmod 3} \sum_{i' \in T_q} (\lambda_q - \lambda'_{i'}) x'_{i'}|| \leqq 14\epsilon \, ||x||.$$

Therefore

$$||(y + z_r)(\textstyle\bigcap_{p \equiv r \pmod 3} \sum_{i \neq p} K_i)|| \leqq 14\epsilon \, ||x||,$$

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and, since $||z_r|| \leqq 14\epsilon\,||x||$,

$$||y(\bigcap_{p\equiv r(\mathrm{mod}\ 3)}\sum_{i\neq p}K_i)|| \leqq (14 + 14)\ \epsilon||x|| = 28\epsilon\,||x||.$$

Now consider the five G-ideals

$$I_s = \bigcap_{k\equiv s(\mathrm{mod}\ 2)}\sum_{i\neq k}J_i, \quad s = 1, 2,$$
$$I_{2+r} = \bigcap_{p\equiv r(\mathrm{mod}\ 3)}\sum_{i\neq p}K_i, \quad r = 1, 2, 3.$$

We have $\bigcup_{s=1}^5$ hull $I_s \subset$ hull $\bigcap_{s=1}^5 I_s$. Computation shows that

$$\text{hull } I_1 \cup \text{hull } I_2 \supset f^{-1}(\bigcup_{i=1}^{n-1} O_i \cap O_{i+1})^c,$$
$$\text{hull } I_3 \cup \text{hull } I_4 \cup \text{hull } I_5 \supset f^{-1}\,(\bigcup_{i=1}^{n-1}(O_i \cap O_{i+1})^-).$$

This shows that every G-primitive lies in hull $\bigcap_{s=1}^5 I_s$, whence $\bigcap_{s=1}^5 I_s = 0$. By four applications of 3.5, to the pairs (I_1, I_3), $(I_1 \cap I_3, I_2)$, $(I_1 \cap I_2 \cap I_3, I_4)$ and $(I_1 \cap I_2 \cap I_3 \cap I_4, I_5)$ respectively, it follows that

$$||y|| \leqq 7(2\cdot 2 + 4 + 2\cdot 2 + 2\cdot 4 + 2\cdot 4)\ \epsilon||x|| = 196\epsilon||x||.$$

4. The Banach module structure.

4.1. *Remarks.* It follows by the closed graph theorem that for each fixed f as in 3.1 the map $x \mapsto fx$ is continuous. If M_f denotes this bounded operator, then a second application of the closed graph theorem shows that the map $f \mapsto M_f$ is continuous, with respect to the supremum norm $||f||_\infty$ on the space of functions f. This shows that there exists a constant $k \geqq 1$ such that for all f and x,

$$||fx|| \leqq k||f||_\infty||x||.$$

As a matter of fact it follows from 3.3 that $k \leqq 7$, and that the inequality holds with $k = 7/2$ if $f \geqq 0$. Analysis of the proof of 3.3 shows that these numbers can be replaced by 6 and 3 respectively.

4.2. THEOREM. *Let A be a Banach space with a G-algebra structure. Suppose that for every decreasing sequence of G-ideals $J_1 \supset J_2 \supset \ldots$ with intersection* 0 *the identity*

$$||x|| = \sup_n ||x(J_n)||$$

holds. Then for every bounded continuous real-valued function f on G- primitives and every $x \in A$,

$$||fx|| \leqq 2||f||_\infty||x||,$$

and if $f \geqq 0$,

$$||fx|| \leqq ||f||_\infty||x||.$$

Proof. Fix $\epsilon > 0$. Choose mutually disjoint nonempty open intervals $O^1{}_1, \ldots, O^1{}_n$ such that the distance between the midpoints of any two is at

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most ϵ and such that any point in the range of f is between the midpoints of $O^1{}_2$ and $O^1{}_{n-1}$. For each $i = 1, \ldots, n$ choose a decreasing sequence of nonempty open intervals $O^1{}_i \supset O^2{}_i \supset \ldots$ with empty intersection. Denote by $J^k{}_i$ the G-ideal the hull of which is $f^{-1}(O^k{}_i)^c$, $i = 1, \ldots, n$, $k = 1, 2, \ldots$. For each $k = 1, 2, \ldots$ set $\sum_{i=1}^n J^k{}_i = J^k$. Then

$$\text{hull } \bigcap_k J^k \supset \bigcup_k \text{hull } J^k = \bigcup_k \bigcap_{i=1,\ldots,n} \text{hull } J^k{}_i = \bigcup_k \bigcap_{i=1,\ldots,n} f^{-1}(O^k{}_i)^c = \bigcap_{i=1,\ldots,n} (\bigcap_k f^{-1}(O^k{}_i))^c = \emptyset^c,$$

so $\bigcap_k J^k = 0$.

Let x be an element of A and let f be a bounded continuous real-valued function on G-primitives. Then

$$||x|| = \sup_k ||x(J^k)||, \quad ||fx|| = \sup_k ||fx(J^k)||,$$

so it is enough to show for each $k = 1, 2, \ldots$ that

$$||fx(J^k)|| \leqq 2||f||_\infty ||x(J^k)||,$$

and that if $f \geqq 0$,

$$||fx(J^k)|| \leqq ||f||_\infty ||x(J^k)||.$$

Passing to the quotient of A by a fixed J^k, $k = 1, 2, \ldots$, we may suppose that $J^k = 0$. In other words, the range of f is disjoint from $O^k{}_1 \cup \ldots \cup O^k{}_n$. If g denotes the real-valued function such that $g(t)$ is the average of the midpoints of the two open intervals $O^k{}_i$ and $O^k{}_{i+1}$ between which $f(t)$ lies, then g is continuous and $||f - g||_\infty < \epsilon$. Moreover, by the orthogonality of a pair of G-ideals with intersection 0 (cf. proof of 3.3)

$$||gx|| \leqq ||g||_\infty ||x|| \quad \text{if } g \geqq 0$$

and in general (as a consequence)

$$||gx|| \leqq 2||g||_\infty ||x||.$$

Hence

$$\begin{aligned} ||fx|| &\leqq ||(f-g)x|| + ||gx|| \\ &\leqq 3||f-g||_\infty ||x|| + 2||g||_\infty ||x|| \\ &\leqq 3||f-g||_\infty ||x|| + 2||f-g||_\infty ||x|| + 2||f||_\infty ||x|| \\ &\leqq 5\epsilon ||x|| + 2||f||_\infty ||x||. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary,

$$||fx|| \leqq 2||f||_\infty ||x||.$$

If $f \geqq 0$, then $O_1, \ldots, O_n$ may be chosen so that $g \geqq 0$, and a similar calculation using the inequality $||gx|| \leqq ||g||_\infty ||x||$ yields the inequality

$$||fx|| \leqq ||f||_\infty ||x||.$$

4.3. *Remark*. A two-dimensional example shows that in 4.2 the estimate $||fx|| \leqq 2||f||_\infty||x||$ is the best possible for real-valued f not assumed to be positive. On the other hand, under the additional assumption on G-ideals that for $x_1 \in J_1$, $x_2 \in J_2$ (J_1 and J_2 G-ideals),

$$||(x_1 + x_2)(J_1 \cap J_2)|| = ||(x_1 - x_2)(J_1 \cap J_2)||,$$

the proof of 4.2 shows that the inequality

$$||fx|| \leqq ||f||_\infty||x||$$

holds for all real-valued f.

It is not clear whether the assumption on the projective limit behaviour of the norm made in 4.2 is necessary or not.

4.4. *Remark*. If A is a Banach space with a G-algebra structure, then for each bounded continuous real-valued function f on G-primitives the map $x \mapsto fx$ is (see 4.1) a bounded operator on A. This map leaves G-primitives invariant, and therefore leaves all G-ideals invariant.

It would be interesting to determine if, conversely, every bounded operator on A which leaves G-ideals invariant and induces a homothety in each G-primitive quotient must be the map $x \mapsto fx$ for some bounded continuous real-valued function f on G-primitives. It is not difficult to show this under the additional assumptions that the hulls of G-ideals form the closed subsets of a topology on G-primitives, and that for each $x \in A$ the set of all G-primitives t such that $||x(t)|| \leqq 1$ is a hull. (It is necessary to show that if f is a real-valued function on G-primitives such that for every $x \in A$ there exists $fx \in A$ with $(fx)(t) = f(t)x(t)$ for every G-primitive t, then f is continuous. Adding a constant and scaling reduces the question to showing that the set $f^{-1}([-1, 1])$ is a hull. But this set is the intersection over all $x \in A$ of the set of G-primitives t such that $||(fx)(t)|| \leqq 1$.)

4.5. *Problem*. The question arises how general the scalars can be in 3.1. Of course the G-ideals must be invariant under the scalars. If we assume that the hulls form the closed sets of a topology on the set of G-primitives, so that continuity of functions can be defined in terms of this topology, then it is clear that if the range of f lies in a finite-dimensional algebra of operators on A which leaves G-ideals invariant then the conclusion of 3.1 holds also for f (it is understood that $f(t)$ should denote also the operator induced in the quotient A/t by $f(t)$).

The situation is not quite so clear if the range of f is allowed to be an arbitrary bounded subset of operators leaving G-ideals invariant. Simple examples show that some restriction is necessary, and it seems reasonable to ask if total boundedness of the range of f is sufficient. Under either the hypothesis that the norm of an element of A is the supremum of its norms in G-primitive quotients (e.g. in the M-ideal situation of Alfsen and Effros [**1**]) or the hypothesis that for the range of f in a finite-dimensional subspace there is a constant k independent of the dimension such that $||fx|| \leqq k||f||_\infty||x||$, the argument which follows shows that total boundedness of the range of f suffices for fx to exist as in 3.1. (Note that by the closed graph theorem, if the conclusion is true then the second hypothesis must be satisfied.)

Fix $\epsilon > 0$. Since the closure of the range of f, say K, is compact, there exist open balls $O_1, \ldots, O_n$ of radius ϵ covering K. Choose $\alpha_1 \in O_1, \ldots, \alpha_n \in O_n$, and choose a partition of unity $(f_1, \ldots, f_n)$ on the compact space K subordinate to the cover $(O_1, \ldots, O_n)$. Denote the function $t \mapsto f_1(f(t))\alpha_1 + \ldots + f_n(f(t))\alpha_n$ by f_ϵ. Then $||f - f_\epsilon||_\infty \leqq \epsilon$, and by 3.1 there exists $f_\epsilon x \in A$ such that $(f_\epsilon x)(t) = f_\epsilon(t)x(t)$ for every G-primitive t. Either of the hypotheses described in the preceding paragraph implies that as $\epsilon \to 0$, $f_\epsilon x$ converges.

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