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FULL COACTIONS ON HILBERT C^*-MODULES

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Abstract

We introduce a natural notion of full coactions of a locally compact group on a Hilbert C^*-module, and associate each full coaction in a natural way to an ordinary coaction. We also introduce a natural notion of strong Morita equivalence of full coactions which is sufficient to ensure strong Morita equivalence of the corresponding crossed product C^*-algebras.

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Introduction

Coactions of a Hopf C^*-algebra on a Hilbert C^*-module were introduced by Baaj and Skandalis in their study of equivariant Kasparov theory in [1]. A coaction of a locally compact group G on a Hilbert C^*-module X is then defined to be a coaction of the Hopf C^*-algebra $(C_r^*(G), \delta_G)$ on X. On the other hand, Raeburn introduced in [6] the notion of full coactions of G on C^*-algebras and the crossed products by such full coactions. He showed that each full coaction ϵ of G on B is associated to an ordinary coaction δ of G on a quotient B/I, and the full crossed product $B \times_\epsilon G$ is isomorphic to the crossed product $(B/I) \times_\delta G$.

In this paper, we introduce a notion of full coactions of a locally compact group G on a Hilbert C^*-module X which is an analogue of the notion of Baaj and Skandalis' coactions and a generalization of the notion of Raeburn's full coactions. Each full coaction of G on X is then associated to an ordinary coaction of G on a quotient of X. Applying this result and [1, Proposition 6.9] we obtain criteria for strong Morita equivalence of crossed products by full coactions.

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Our work is organized as follows. In Section 1 we recall some definitions concerning crossed products by coactions and full coactions, and Hilbert C^*-modules. In Section 2 we study the 'maximal tensor product' of Hilbert C*-modules. We then define the notion of full coactions on a Hilbert C^*-module, and establish some basic properties for them. In Section 3 we present Theorem 3.4, concerning strong Morita equivalence of crossed products by full coactions.

1. Preliminaries

Throughout this paper G will be a locally compact group, and λ denotes the left regular representation of G. We will denote by $\otimes$ the minimal tensor product.

Let $C_r^*(G)$ denote the reduced group C^*-algebra. The *comultiplication* δ_G on $C_r^*(G)$ is the integrated form of the representation $s \mapsto \lambda(s)\otimes\lambda(s)$. Let $C^*(G)$ denote the full group C^*-algebra, and $i_G : G \to UM(C^*(G))$ denote the natural strictly continuous homomorphism. The *comultiplication* ϵ_G on $C^*(G)$ is the integrated form of the homomorphism $s \mapsto i_G(s) \otimes_{\max} i_G(s)$. We denote by $\tilde{W}_G \in UM(C_0(G) \otimes C^*(G))$ the *multiplier* determined by $\tilde{W}_G(s) = i_G(s)$, $\forall s \in G$. If A and B are C^*-algebras, and ν denotes the minimal C^*-norm or the maximal C^*-norm, we put

$$\tilde{M}(A \otimes_\nu B) = \{m \in M(A \otimes_\nu B) : m(1 \otimes_\nu b), (1 \otimes_\nu b)m \in A \otimes_\nu B,\ \forall b \in B\}.$$

Let B be a C^*-algebra. A *coaction* of G on B is an injective non-degenerate homomorphism $\delta : B \to \tilde{M}(B \otimes C_r^*(G))$ such that

$$(\delta \otimes \mathrm{id})^- \circ \delta = (\mathrm{id} \otimes \delta_G)^- \circ \delta.$$

See [5, Definition 2.1]. The *crossed product* $B \times_\delta G$ of (B, G, δ) is the C^*-subalgebra of $M(B \otimes \mathscr{K}(L^2(G)))$ generated by the set

$$\{\delta(b)(1 \otimes M_f) : b \in B, f \in C_0(G)\}.$$

See [5, Definition 2.4].

A *full coaction* of G on B is a non-degenerate homomorphism $\epsilon : B \to \tilde{M}(B \otimes_{\max} C^*(G))$ such that

$$(\epsilon \otimes_{\max} \mathrm{id})^- \circ \epsilon = (\mathrm{id} \otimes_{\max} \epsilon_G)^- \circ \epsilon.$$

See [6, Definition 2.1]. A covariant representation of (B, G, ϵ) is a pair of non-degenerate representations $\pi : B \to \mathscr{B}(\mathscr{H})$ and $\mu : C_0(G) \to \mathscr{B}(\mathscr{H})$, such that for all $b \in B$

$$(\pi \otimes \mathrm{id})^- \circ \epsilon(b) = (\mu \otimes \mathrm{id})^-(\tilde{W}_G)(\pi(b) \otimes 1)(\mu \otimes \mathrm{id})^-(\tilde{W}_G)^*,$$

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as elements of $M(\mathcal{K}(\mathcal{H}) \otimes C^*(G))$. See [6, Definition 2.4]. A *full crossed product* for (B, G, ϵ) is a C^*-algebra $\mathcal{B}$ together with non-degenerate homomorphisms $\hat{j}_B : B \to M(\mathcal{B})$ and $\hat{j}_{C(G)} : C_0(G) \to UM(\mathcal{B})$ satisfying

(i) for every non-degenerate representation ρ of $\mathcal{B}$, the pair $(\rho \circ \hat{j}_B, \rho \circ \hat{j}_{C(G)})$ is a covariant representation of (B, G, ϵ);

(ii) for every covariant representation (π, μ) of (B, G, ϵ), there is a non-degenerate representation $\pi \times \mu$ of $\mathcal{B}$ such that

$$\pi = (\pi \times \mu)^- \circ \hat{j}_B \quad \text{and} \quad \mu = (\pi \times \mu)^- \circ \hat{j}_{C(G)};$$

$\pi \times \mu$ is called the *integrated form* of (π, μ).

(iii) the linear span of $\{\hat{j}_B(b)\hat{j}_{C(G)}(f) : b \in B, f \in C_0(G)\}$ is dense in $\mathcal{B}$.

See [6, Definition 2.8]. There always exists a full crossed product for each system (B, G, ϵ), unique up to isomorphism. See [6, Proposition 2.13].

Suppose that ϵ_B is a full coaction of G on a C^*-algebra B. Let $\varrho_B : B \otimes_{\max} C^*(G) \to B \otimes_{\min} C^*(G)$ be the canonical quotient map. We define

$$\delta_B^1 = (\mathrm{id}_B \otimes_{\min} \lambda)^- \circ \bar{\varrho}_B \circ \epsilon_B.$$

Then δ_B^1 is a non-degenerate homomorphism of B into $\tilde{M}(B \otimes_{\min} C_r^*(G))$. Put $I_B = \ker(\delta_B^1)$ and $\dot{B} = B/I_B$, and let $q_B : B \to \dot{B}$ denote the canonical quotient map. We define

$$\delta_{\dot{B}}(q_B(b)) = (q_B \otimes_{\min} \mathrm{id}_{C_r^*(G)})^- \circ \delta_B^1(b) = (q_B \otimes_{\min} \lambda)^- \circ \bar{\varrho}_B \circ \epsilon_B(b),$$

for all $b \in B$. Then $\delta_{\dot{B}} : \dot{B} \to \tilde{M}(\dot{B} \otimes_{\min} C_r^*(G))$ is a coaction of G on $\dot{B}$. See [6, Lemma 3.1].

Let $(B \times_\epsilon G, \hat{j}_B, \hat{j}_{C(G)})$ be the full crossed product for (B, G, ϵ). We represent $B \times_\epsilon G$ on a Hilbert space by a faithful non-degenerate representation. By [6, Proposition 3.4], there is a non-degenerate representation π of $\dot{B}$ such that $\hat{j}_B = \pi \circ q$ and $(\pi, \hat{j}_{C(G)})$ is a covariant representation of $(\dot{B}, G, \delta)$. The integrated form $\Psi = \pi \times \hat{j}_{C(G)}$ is called the *reduction map*.

THEOREM 1.1. *The reduction map Ψ is an isomorphism of the crossed product $\dot{B} \times_{\delta_{\dot{B}}} G$ onto the full crossed product $B \times_{\epsilon_B} G$.*

PROOF. See [6, Theorem 4.1].

Let B_0 be a dense *-subalgebra of a C^*-algebra B, and X_0 a complex vector space. A right (respectively, left)-prehilbert B_0-module is a right (respectively, left) B_0-module X_0 equipped with a B_0-valued pre-inner product $\langle \cdot | \cdot \rangle_{B_0}$ (respectively, ${}_{B_0}\langle \cdot | \cdot \rangle$) such that

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(i) $\langle\cdot|\cdot\rangle_{B_0}$ (respectively, ${}_{B_0}\langle\cdot|\cdot\rangle$) is linear in the second (respectively, first) variable;
(ii) $\langle y|xb\rangle_{B_0} = \langle y|x\rangle_{B_0} b$ (respectively, ${}_{B_0}\langle bx|y\rangle = b\,{}_{B_0}\langle x|y\rangle$) for all $x, y \in X_0$ and $b \in B_0$.

We will say that X_0 or $\langle\cdot|\cdot\rangle_{B_0}$ is *full* if the linear span of $\{\langle y|x\rangle_{B_0} : x, y \in X_0\}$ is dense in B_0. Note that a full right-prehilbert B_0-module is a right B_0-rigged space in the sense of [7, Definition 2.8]. A right-prehilbert B-module X is called a *right-Hilbert B-module* if $\langle\cdot|\cdot\rangle_B$ is definite and X is complete under the norm $x \mapsto \|\langle x|x\rangle_B\|^{1/2}$. Left-Hilbert B-modules are defined similarly.

Let X and Y be right-Hilbert B-modules. We will denote by $\mathscr{L}(X, Y)$ the set of maps $T : X \to Y$ which admit an adjoint $T^* : Y \to X$ such that $\langle Tx|y\rangle_B = \langle x|T^*y\rangle_B$, $\forall x \in X, \forall y \in Y$. Put $\mathscr{L}(X) = \mathscr{L}(X, X)$. For any $x \in X$ and $y \in Y$, put

$$\theta_{x,y}(y') = x\langle y|y'\rangle_B, \qquad \forall y' \in Y.$$

Then $\theta_{x,y} \in \mathscr{L}(Y, X)$ and $\theta^*_{x,y} = \theta_{y,x}$. We will denote by $\mathscr{K}(Y, X)$ the closure in $\mathscr{L}(Y, X)$ of the linear span of $\{\theta_{x,y} : x \in X, y \in Y\}$. Put $\mathscr{K}(X) = \mathscr{K}(X, X)$.

For more information on Hilbert C^*-modules we refer the reader to [2, Chapter VI, §13], [4] and [9, Chapter 1].

2. Full coactions on Hilbert C^*-modules

Suppose that B_0 and D_0 are dense *-subalgebras of C^*-algebras B and D. Let ν be a C^*-norm on the algebraic tensor product $B \odot D$. We will denote by $B_0 \odot_\nu D_0$ the *-algebra $B_0 \odot D_0$ equipped with the norm ν. Suppose that $(X_0, \langle\cdot|\cdot\rangle_{B_0})$ is a right-prehilbert B_0-module and $(Y_0, \langle\cdot|\cdot\rangle_{D_0})$ is a right-prehilbert D_0-module. Then $X_0 \odot Y_0$ becomes a right-prehilbert $B_0 \odot_\nu D_0$-module in the natural way:

$$\langle x \odot y|x' \odot y'\rangle_{B_0\odot_\nu D_0} = \langle x|x'\rangle_{B_0} \odot \langle y|y'\rangle_{D_0},$$
$$(x \odot y)(b \odot d) = xb \odot yd,$$

for all $x, x' \in X_0$, $y, y' \in Y_0$, $b \in B_0$ and $d \in D_0$. We will denote by $X_0\hat{\odot}_\nu Y_0$ the quotient of $X_0 \odot Y_0$ by the subspace of vectors of length zero. The Hausdorff-completion $X_0\hat{\otimes}_\nu Y_0$ of $X_0 \odot Y_0$ is a right-Hilbert $B \otimes_\nu D$-module. The image of each element $\sum_i x_i \odot y_i$ under the canonical quotient map is denoted by $\sum_i x_i\hat{\odot}_\nu y_i$. Suppose that X and $\mathscr{V}$ are right-Hilbert B-modules, and Y and $\mathscr{W}$ are right-Hilbert D-modules. Then for any $S \in \mathscr{L}(X, \mathscr{V})$ and $T \in \mathscr{L}(Y, \mathscr{W})$, there is a $S\hat{\otimes}_\nu T \in \mathscr{L}(X\hat{\otimes}_\nu Y, \mathscr{V}\hat{\otimes}_\nu\mathscr{W})$ such that

$$S\hat{\otimes}_\nu T(x\hat{\odot}_\nu y) = Sx\hat{\odot}_\nu Ty, \qquad \forall x \in X,\ \forall y \in Y.$$

The proof of the above assertions can be found in [9, 1.1.14(d)] when ν is the minimal C^*-norm. The general case is proved in the same way.

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LEMMA 2.1. *Suppose that X is a right-Hilbert B-module, and Y is a right-Hilbert D-module. Let ν denote the minimal C^*-norm or the maximal C^*-norm. Then there is a homomorphism $\Psi_\nu : \mathscr{L}(X) \otimes_\nu \mathscr{L}(Y) \to \mathscr{L}(X \hat{\otimes}_\nu Y)$ such that*

$$(1) \qquad \Psi_\nu(S \odot T) = S \hat{\otimes}_\nu T, \qquad \forall S \odot T \in \mathscr{L}(X) \odot \mathscr{L}(Y).$$

PROOF. If ν is the minimal C^*-norm, the result is well-known; see [2, 13.5], [9, 1.1.14(d)]. Assume that ν is the maximal C^*-norm. Put

$$\phi(S) = S \hat{\otimes}_{\max} I, \qquad \varphi(T) = I \hat{\otimes}_{\max} T,$$

Then ϕ and φ are homomorphisms with commuting ranges, and hence there is a homomorphism $\Psi_{\max} : \mathscr{L}(X) \otimes_{\max} \mathscr{L}(Y) \to \mathscr{L}(X \hat{\otimes}_{\max} Y)$ satisfying (1).

Let A_0 and B_0 be dense *-subalgebras of C^*-algebras A and B, respectively. A right-prehilbert B_0-module X_0 is called a *right-prehilbert A_0, B_0-bimodule* if X_0 is an A_0, B_0-bimodule and

(i) $\langle ax|y\rangle_{B_0} = \langle x|a^* y\rangle_{B_0}, \qquad \forall a \in A_0, \forall x, y \in X_0;$

(ii) $\langle ax|ax\rangle_{B_0} \le \|a\|^2 \langle x|x\rangle_{B_0}, \qquad \forall a \in A_0, \forall x \in X_0.$

Similarly, we can define left-prehilbert A_0, B_0-bimodules.

COROLLARY 2.2. *Let A, B, C and D be C^*-algebras. Suppose that X is a right-Hilbert A, B-bimodule, and Y is a right-Hilbert C, D-bimodule. Let ν denote the minimal C^*-norm or the maximal C^*-norm. Then $X \odot Y$ is a right-prehilbert $A \odot_\nu C, B \odot_\nu D$-bimodule. Furthermore, if X is an A, B-imprimitivity bimodule and Y is a C, D-imprimitivity bimodule, then $X \hat{\otimes}_\nu Y$ is an $A \otimes_\nu C, B \otimes_\nu D$-imprimitivity bimodule.*

PROOF. The proof follows from Lemma 2.1 and some routine computations.

COROLLARY 2.3. *Suppose that X is a right-Hilbert B-module, and Y is a right-Hilbert D-module. Let ν denote the minimal C^*-norm or the maximal C^*-norm. Then the map $\theta \odot \theta' \mapsto \theta \hat{\otimes}_\nu \theta'$ is an isomorphism from $\mathscr{K}(X) \otimes_\nu \mathscr{K}(Y)$ onto $\mathscr{K}(X \hat{\otimes}_\nu Y)$.*

PROOF. Put $A = \mathscr{K}(X)$ and $C = \mathscr{K}(Y)$. Observe that X is a left- and right-Hilbert A, B-bimodule and Y is a left and right-Hilbert C, D-bimodule. By Corollary 2.2, $X \odot Y$ is a left and right-Hilbert $A \odot_\nu C, B \odot_\nu D$-bimodule. Put $E = A \otimes_\nu C$ and $F = B \otimes_\nu D$. Then $X \hat{\otimes}_\nu Y$ is a left- and right-Hilbert E, F-bimodule. Since ${}_A\langle\cdot|\cdot\rangle$ and ${}_B\langle\cdot|\cdot\rangle$ are full, it follows that ${}_E\langle\cdot|\cdot\rangle$ is full. Therefore the natural map $\theta_{x,y} \odot \theta_{x'y'} \mapsto \theta_{x\hat{\odot}_\nu y, x'\hat{\odot}_\nu y'}$ extends to an isomorphism from E onto $\mathscr{K}(X \hat{\otimes}_\nu Y)$. Since $\theta_{x\hat{\odot}_\nu y, x'\hat{\odot}_\nu y'} = \theta_{x,y} \hat{\otimes}_\nu \theta_{x',y'}$ for all $x, x' \in X$ and $y, y' \in Y$, we get the desired result.

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Let X be a right-Hilbert B-module, and $\mathcal{V}$ and $\mathcal{W}$ right-Hilbert C-modules. Suppose that $f : B \to \mathcal{L}(\mathcal{V})$ is a homomorphism, and $\phi : X \to \mathcal{L}(\mathcal{V}, \mathcal{W})$ is a linear map. We say that ϕ is *compatible with* f (or f*-compatible*) if

(i) $\phi(xb) = \phi(x)f(b), \qquad \forall x \in X, \forall b \in B,$
(ii) $\phi(x)^*\phi(x') = f(\langle x|x'\rangle_B), \qquad \forall x, x' \in X.$

We say that ϕ is *non-degenerate* is the linear span of $\{\phi(x)\xi : x \in X, \xi \in \mathcal{V}\}$ is dense in $\mathcal{W}$.

Recall from [3, Proposition 2.2] that if $\phi : X \to \mathcal{L}(\mathcal{V}, \mathcal{W})$ is an f-compatible non-degenerate linear map, then there is a unique unital homomorphism $h : \mathcal{L}(X) \to \mathcal{L}(\mathcal{W})$ such that

$$h(T)\phi(x) = \phi(Tx), \qquad \forall T \in \mathcal{L}(X), \forall x \in X.$$

We will refer to h as the *natural homomorphism* corresponding to ϕ.

Furthermore if f is non-degenerate, then there is a unique linear map $\bar{\phi} : \mathcal{L}(B, X) \to \mathcal{L}(\mathcal{V}, \mathcal{W})$ such that

$$\bar{\phi}(P)f(b) = \phi(Pb), \qquad \forall P \in \mathcal{L}(B, X), \forall b \in B.$$

See [3, Proposition 2.3].

LEMMA 2.4. *Let X be a right-Hilbert A-module, Y a right-Hilbert B-module, $\mathcal{V}$ and $\mathcal{X}$ right-Hilbert C-modules, and $\mathcal{W}$ and $\mathcal{Y}$ right-Hilbert D-modules. Assume that $f : A \to \mathcal{L}(\mathcal{V})$ and $g : B \to \mathcal{L}(\mathcal{W})$ are homomorphisms. Suppose that $\phi : X \to \mathcal{L}(\mathcal{V}, \mathcal{X})$ is an f-compatible linear map, and $\varphi : Y \to \mathcal{L}(\mathcal{W}, \mathcal{Y})$ is a g-compatible linear map. Let ν denote the minimal C^*-norm or the maximal C^*-norm. Then there is a homomorphism $f\hat{\otimes}_\nu g : A \otimes_\nu B \to \mathcal{L}(\mathcal{V}\hat{\otimes}_\nu\mathcal{W})$, and an $f\hat{\otimes}_\nu g$-compatible linear map $\phi\hat{\otimes}_\nu\varphi : X\hat{\otimes}_\nu Y \to \mathcal{L}(\mathcal{V}\hat{\otimes}_\nu\mathcal{W}, \mathcal{X}\hat{\otimes}_\nu\mathcal{Y})$ such that*

$$(f\hat{\otimes}_\nu g)(a \odot b) = f(a)\hat{\otimes}_\nu g(b), \qquad \forall a \in A, \forall b \in B, \tag{1}$$

$$(\phi\hat{\otimes}_\nu\varphi)(x\hat{\odot}_\nu y) = \phi(x)\hat{\otimes}_\nu\varphi(y), \qquad \forall x \in X, \forall y \in Y. \tag{2}$$

PROOF. Let $f \otimes_\nu g : A \otimes_\nu B \to \mathcal{L}(\mathcal{V}) \otimes_\nu \mathcal{L}(\mathcal{W})$ be the natural homomorphism and let $\Psi_\nu : \mathcal{L}(\mathcal{V}) \otimes_\nu \mathcal{L}(\mathcal{W}) \to \mathcal{L}(\mathcal{V}\hat{\otimes}_\nu\mathcal{W})$ be the homomorphism defined in Lemma 2.1. Put $f\hat{\otimes}_\nu g = \Psi_\nu \circ (f \otimes_\nu g)$. Then $f\hat{\otimes}_\nu g : A \otimes_\nu B \to \mathcal{L}(\mathcal{V}\hat{\otimes}_\nu\mathcal{W})$ is a homomorphism satisfying (1). Let $\Lambda_\nu : \mathcal{L}(\mathcal{V}, \mathcal{X}) \odot \mathcal{L}(\mathcal{W}, \mathcal{Y}) \to \mathcal{L}(\mathcal{V}\hat{\otimes}_\nu\mathcal{W}, \mathcal{X}\hat{\otimes}_\nu\mathcal{Y})$ be defined by

$$\Lambda_\nu(S \odot T) = S\hat{\otimes}_\nu T, \qquad \forall S \in \mathcal{L}(\mathcal{V}, \mathcal{X}), \forall T \in \mathcal{L}(\mathcal{W}, \mathcal{Y}).$$

Let $\phi \otimes \varphi : X \odot Y \to \mathcal{L}(\mathcal{V}, \mathcal{X}) \odot \mathcal{L}(\mathcal{W}, \mathcal{Y})$ denote the linear map defined by

$$(\phi \otimes \varphi)(x \odot y) = \phi(x) \odot \varphi(y), \qquad \forall x \in X, \forall y \in Y.$$

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Then $\Phi_0 = \Lambda_\nu \circ (\phi \otimes \varphi)$ is linear and compatible with $h_0 = (f\hat{\otimes}_\nu g)|A \odot B$, and hence $\|\Phi_0(z)\| \leq \|z\|_\nu, \forall z \in X \odot Y$. Thus we can define a linear map $\phi\hat{\otimes}_\nu\varphi : X\hat{\otimes}_\nu Y \to \mathscr{L}(\mathscr{V}\hat{\otimes}_\nu\mathscr{W}, \mathscr{X}\hat{\otimes}_\nu\mathscr{Y})$ satisfying (2). Since Φ_0 is compatible with h_0, the map $\phi\hat{\otimes}_\nu\varphi|X\hat{\odot}Y$ is compatible with h_0, and hence $\phi\hat{\otimes}_\nu\varphi$ is compatible with $f\hat{\otimes}_\nu g$.

Let X be a right-Hilbert B-module and Y a right-Hilbert D-module. Suppose that $f : B \to D$ is a homomorphism, and $\phi : X \to Y$ is a linear map. We say that ϕ is *compatible with* f (or f*-compatible*) if

(i) $\phi(xb) = \phi(x)f(b), \qquad \forall x \in X, \forall b \in B,$
(ii) $\langle\phi(x)|\phi(x')\rangle_D = f(\langle x|x'\rangle_B), \qquad \forall x, x' \in X.$

We say that ϕ is *non-degenerate* is the linear span of $\{\phi(x)d : x \in X, d \in D\}$ is dense in Y.

LEMMA 2.5. *Let X, Y, $\mathscr{V}$ and $\mathscr{W}$ be right-Hilbert modules over C^*-algebras A, B, C and D, respectively. Suppose that $f : A \to C$ and $g : B \to D$ are homomorphisms, $\phi : X \to \mathscr{V}$ is an f-compatible linear map, and $\varphi : Y \to \mathscr{W}$ is a g-compatible linear map. Let ν denote the minimal C^*-norm or the maximal C^*-norm, and $f \otimes_\nu g : A \otimes_\nu B \to C \otimes_\nu D$ the natural homomorphism. Then there is an $f \otimes_\nu g$-compatible linear map*

$$\phi\hat{\otimes}_\nu\varphi : X\hat{\otimes}_\nu Y \to \mathscr{V}\hat{\otimes}_\nu\mathscr{W}$$

such that

(1) $$(\phi\hat{\otimes}_\nu\varphi)(x\hat{\odot}_\nu y) = \phi(x)\hat{\odot}_\nu\varphi(y), \qquad \forall x \in X, \forall y \in Y.$$

PROOF. Apply similar arguments as in Lemma 2.4.

We put

$$\begin{aligned}\tilde{M}(X\hat{\otimes}_{\max}C^*(G)) = \{T \in \mathscr{L}(B \otimes_{\max} C^*(G), X\hat{\otimes}_{\max}C^*(G)) : \\ (1_X\hat{\otimes}_{\max}s)T, T(1_B \otimes_{\max} s) \in \mathscr{K}(B \otimes_{\max} C^*(G), X\hat{\otimes}_{\max}C^*(G)), \\ \forall s \in C^*(G)\}.\end{aligned}$$

DEFINITION 2.6. Let $\epsilon_B : B \to \tilde{M}(B \otimes_{\max} C^*(G))$ be a full coaction of G on B. An *ϵ_B-compatible full coaction of G on X* is a linear map $\epsilon_X : X \to \tilde{M}(X\hat{\otimes}_{\max}C^*(G))$ such that

(i) $\epsilon_X(xb) = \epsilon_X(x)\epsilon_B(b), \qquad \forall x \in X, \forall b \in B,$
$\epsilon_X(y)^*\epsilon_X(x) = \epsilon_B(\langle y|x\rangle_B), \quad \forall x, y \in X;$
(ii) the linear span of $\{\epsilon_X(x)\gamma : x \in X, \gamma \in B \otimes_{\max} C^*(G)\}$ is dense in $X\hat{\otimes}_{\max}C^*(G)$;

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(iii) $(\epsilon_X \hat{\otimes}_{\max} \mathrm{id})^- \circ \epsilon_X = (\mathrm{id} \hat{\otimes}_{\max} \epsilon_G)^- \circ \epsilon_X$ as maps from X into $\mathscr{L}(B \otimes_{\max} C^*(G) \otimes_{\max} C^*(G), X \hat{\otimes}_{\max} C^*(G) \hat{\otimes}_{\max} C^*(G))$.

We note that the existence of $(\epsilon_X \hat{\otimes}_{\max} \mathrm{id})^-$ and $(\mathrm{id} \hat{\otimes}_{\max} \epsilon_G)^-$ follows from Lemma 2.4 and [3, Proposition 2.3].

Let X be a right-Hilbert B-module. We define maps P_1 from $X \oplus B$ into X and P_2 from $X \oplus B$ into B by

$$P_1(x \oplus b) = x, \qquad P_2(x \oplus b) = b, \qquad \forall x \in X, \forall b \in B.$$

Next we define maps $\bar{c}_{ij}$ from $P_i \mathscr{L}(X \oplus B) P_j^*$ into $\mathscr{L}(X \oplus B)$ by

$$\bar{c}_{ij}(T_{ij}) = P_i^* T_{ij} P_j, \qquad \forall T_{ij} \in P_i \mathscr{L}(X \oplus B) P_j^*.$$

We will denote by c_{ij} the restriction of $\bar{c}_{ij}$ to $P_i \mathscr{K}(X \oplus B) P_j^*$.

PROPOSITION 2.7. *Suppose that $\epsilon_X : X \to \tilde{M}(X \hat{\otimes}_{\max} C^*(G))$ is an ϵ_B-compatible full coaction of G on X. Then there is a unique full coaction $\epsilon_{\mathscr{K}(X)} : \mathscr{K}(X) \to \tilde{M}(\mathscr{K}(X) \otimes_{\max} C^*(G))$ of G on $\mathscr{K}(X)$ satisfying the following equivalent conditions:*

(i) $\epsilon_X(\theta x) = \epsilon_{\mathscr{K}(X)}(\theta) \epsilon_X(x), \qquad \forall \theta \in \mathscr{K}(X), \forall x \in X;$

(ii) $\epsilon_{\mathscr{K}(X)}(\theta_{x,y}) = \epsilon_X(x) \epsilon_X(y)^*, \qquad \forall x, y \in X.$

PROOF. Apply similar arguments as in [3, Proposition 2.8].

PROPOSITION 2.8. *Suppose that $\epsilon_X : X \to \tilde{M}(X \hat{\otimes}_{\max} C^*(G))$ is an ϵ_B-compatible full coaction of G on X. Then there is a unique full coaction $\epsilon_{\mathscr{K}(X \oplus B)} : \mathscr{K}(X \oplus B) \to \tilde{M}(\mathscr{K}(X \oplus B) \otimes_{\max} C^*(G))$ of G on $\mathscr{K}(X \oplus B)$ such that*

(i) $\epsilon_{\mathscr{K}(X \oplus B)} \circ c_{2,2} = (c_{2,2} \otimes_{\max} \mathrm{id})^- \circ \epsilon_B;$

(ii) $\epsilon_{\mathscr{K}(X \oplus B)} \circ c_{1,2} = (c_{1,2} \hat{\otimes}_{\max} \mathrm{id})^- \circ \epsilon_X.$

PROOF. Apply similar arguments as in [3, Proposition 2.9].

Suppose that ϵ_B is a full coaction of G on B. As in Section 1 we get a coaction $\delta_{\dot{B}} : \dot{B} \to \tilde{M}(\dot{B} \otimes C_r^*(G))$ of G on $\dot{B}$.

Now we want to generalize this result to the context of Hilbert C^*-modules. Suppose that $\epsilon_X : X \to \tilde{M}(X \hat{\otimes}_{\max} C^*(G))$ is an ϵ_B-compatible full coaction of G on X. Let $\varrho_X : X \hat{\otimes}_{\max} C^*(G) \to X \hat{\otimes}_{\min} C^*(G)$ denote the canonical quotient map. We put

$$\delta_X^1 = (\mathrm{id}_X \hat{\otimes}_{\min} \lambda)^- \circ \bar{\varrho}_X \circ \epsilon_X; \qquad V_X = \{x \in X : \delta_X^1(x) = 0\};$$
$$\dot{X} = X / V_X, \qquad q_X : X \to \dot{X} \text{ the canonical quotient map;}$$
$$\delta_{\dot{X}}(q_X(x)) = (q_X \hat{\otimes}_{\min} \mathrm{id}_{C_r^*(G)})^- \circ \delta_X^1(x) = (q_X \hat{\otimes}_{\min} \lambda)^- \circ \bar{\varrho}_X \circ \epsilon_X(x).$$

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LEMMA 2.9. *With the above notation, we have*

(i) $\delta^1_{\dot X}$ *is compatible with* $\delta^1_{\dot B}$ *and* $V_X = \{x \in X : \langle x|x\rangle_B \in I_B\}$;
(ii) $\dot X$ *is a right-Hilbert* $\dot B$*-module in the obvious way;*
(iii) $\delta_{\dot X}$ *is a linear map from* $\dot X$ *into* $\tilde M(\dot X \hat\otimes_{\min} C^*_r(G))$.

PROOF. (i) Since ϵ_X, $\bar\varrho_X$ and $(\mathrm{id}_X \hat\otimes_{\min}\lambda)^-$ are compatible with ϵ_B, $\bar\varrho_B$ and $(\mathrm{id}_B \otimes_{\min}\lambda)^-$, respectively, it follows that $(\mathrm{id}_X \hat\otimes_{\min}\lambda)^- \circ \bar\varrho_X \circ \epsilon_X$ is compatible with $(\mathrm{id}_B \otimes_{\min}\lambda)^- \circ \bar\varrho_B \circ \epsilon_B$. The other assertion follows from the fact that $\|\delta^1_B(\langle x|x\rangle_B)\| = \|\delta^1_X(x)\|^2$.

(ii) This follows from routine computations.

(iii) Observe that $\delta_{\dot X} \circ q_X = (q_X \hat\otimes_{\min}\lambda)^- \circ \bar\varrho_X \circ \epsilon_X$, and ϵ_X maps X into $\tilde M(X \hat\otimes_{\max} C^*(G))$. Thus it is enough to show that

$$(1) \qquad \bar\varrho_X(S) \in \tilde M(X \hat\otimes_{\min} C^*_r(G)), \qquad \forall S \in \tilde M(X \hat\otimes_{\max} C^*_r(G));$$

$$(2) \qquad (q_X \hat\otimes_{\min}\lambda)^-(T) \in \tilde M(\dot X \hat\otimes_{\min} C^*_r(G)), \qquad \forall T \in \tilde M(X \hat\otimes_{\min} C^*(G)).$$

Let $g : \mathscr{L}(X \hat\otimes_{\max} C^*(G)) \to \mathscr{L}(X \hat\otimes_{\min} C^*(G))$ be the natural unital homomorphism corresponding to the ϱ_B-compatible non-degenerate linear map $\varrho_X : X \hat\otimes_{\max} C^*(G) \to X \hat\otimes_{\min} C^*(G)$. For any $u \in C^*(G)$, we have

$$\begin{aligned}(1_X \hat\otimes_{\min} u)\bar\varrho_X(S) &= g(1_X \hat\otimes_{\max} u)\bar\varrho_X(S)\\ &= \bar\varrho_X((1_X \hat\otimes_{\max} u)S) \in X \hat\otimes_{\min} C^*(G);\\ \bar\varrho_X(S)(1_B \otimes_{\min} u) &= \bar\varrho_X(S)\bar\varrho_B(1_B \otimes_{\max} u)\\ &= \bar\varrho_X(S(1_B \otimes_{\max} u)) \in X \hat\otimes_{\min} C^*(G).\end{aligned}$$

Thus $\bar\varrho_X(S) \in \tilde M(X \hat\otimes_{\min} C^*(G))$, and hence (1) is proved. Assertion (2) can be proved in a very similar way.

PROPOSITION 2.10. $\delta_{\dot X}$ *is a* $\delta_{\dot B}$*-compatible coaction of* G *on* $\dot X$.

PROOF. It is clear that $\delta_{\dot X} \circ q_X$ is compatible with $\delta_{\dot B} \circ q_B$, and hence $\delta_{\dot X}$ is compatible with $\delta_{\dot B}$. Since $(q_X \hat\otimes_{\min}\lambda) \circ \varrho_X : X \hat\otimes_{\max} C^*(G) \to \dot X \hat\otimes_{\min} C^*_r(G)$ is surjective and ϵ_X is non-degenerate, it follows that $\delta_{\dot X}$ is non-degenerate.

Now it remains to check the coaction identity

$$(\delta_{\dot X} \hat\otimes_{\min} \mathrm{id}_{C^*_r(G)})^- \circ \delta_{\dot X} = (\mathrm{id}_{\dot X} \hat\otimes_{\min}\delta_G)^- \circ \delta_{\dot X}.$$

Put $R = C^*_r(G)$ and $F = C^*(G)$. Let

$$\begin{aligned}&\upsilon : (X \hat\otimes_{\max} F)\hat\otimes_{\max} F \to (X \hat\otimes_{\max} F)\hat\otimes_{\min} F,\\ &\chi : F \otimes_{\max} F \to F \otimes_{\min} F,\\ &\omega : X \hat\otimes_{\max}(F \otimes_{\max} F) \to X \hat\otimes_{\min}(F \otimes_{\max} F)\end{aligned}$$

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be the canonical quotient maps. Then we have

$$\begin{aligned}
(\delta_{\dot X}\hat{\otimes}_{\min}\mathrm{id}_R)^{-}\circ\delta_{\dot X}\circ q_X &= (\delta_{\dot X}\hat{\otimes}_{\min}\mathrm{id}_R)^{-}\circ(q_X\hat{\otimes}_{\min}\lambda)^{-}\circ\bar{\varrho}_X\circ\epsilon_X\\
&= ([(q_X\hat{\otimes}_{\min}\lambda)^{-}\circ\bar{\varrho}_X\circ\epsilon_X]\hat{\otimes}_{\min}[\bar{\lambda}\circ\mathrm{id}_F])^{-}\circ\bar{\varrho}_X\circ\epsilon_X\\
&= ((q_X\hat{\otimes}_{\min}\lambda)\hat{\otimes}_{\min}\lambda)^{-}\circ(\varrho_X\hat{\otimes}_{\min}\mathrm{id}_F)^{-}\circ(\epsilon_X\hat{\otimes}_{\min}\mathrm{id}_F)^{-}\circ\bar{\varrho}_X\circ\epsilon_X\\
&= ((q_X\hat{\otimes}_{\min}\lambda)\hat{\otimes}_{\min}\lambda)^{-}\circ(\varrho_X\hat{\otimes}_{\min}\mathrm{id}_F)^{-}\circ\bar{\upsilon}\circ(\epsilon_X\hat{\otimes}_{\max}\mathrm{id}_F)^{-}\circ\epsilon_X\\
&= (q_X\hat{\otimes}_{\min}(\lambda\hat{\otimes}_{\min}\lambda))^{-}\circ(\mathrm{id}_X\hat{\otimes}_{\min}\chi)^{-}\circ\bar{\omega}\circ(\mathrm{id}_X\hat{\otimes}_{\max}\epsilon_G)^{-}\circ\epsilon_X\\
&= (q_X\hat{\otimes}_{\min}(\lambda\hat{\otimes}_{\min}\lambda))^{-}\circ(\mathrm{id}_X\hat{\otimes}_{\min}\chi)^{-}\circ(\mathrm{id}_X\hat{\otimes}_{\min}\epsilon_G)^{-}\circ\bar{\varrho}_X\circ\epsilon_X\\
&= ([\bar{q}_X\circ\mathrm{id}_X]\hat{\otimes}_{\min}[(\lambda\otimes_{\min}\lambda)^{-}\circ\bar{\chi}\circ\epsilon_G])^{-}\circ\bar{\varrho}_X\circ\epsilon_X\\
&= (\mathrm{id}_{\dot X}\hat{\otimes}_{\min}\delta_G)^{-}\circ(q_X\hat{\otimes}_{\min}\lambda)^{-}\circ\bar{\varrho}_X\circ\epsilon_X\\
&= (\mathrm{id}_X\hat{\otimes}_{\min}\delta_G)^{-}\circ\delta_{\dot X}\circ q_X.
\end{aligned}$$

3. Morita equivalence of crossed products by full coactions

In this section X is a Banach A, B-imprimitivity bimodule, and ϵ_A and ϵ_B are full coactions of G on A and B, respectively. If ϵ_D is a full coaction of G on a C^*-algebra D, then we get a coaction $\delta_{\dot D} : \dot D \to \tilde M(\dot D \otimes C_r^*(G))$ of G on $\dot D$ as described in Section 1.

DEFINITION 3.1. Let ϵ_X be an ϵ_B-compatible full coaction of G on X. We say that ϵ_X is an ϵ_A, ϵ_B-*compatible full coaction of* G *on* X if

$$\epsilon_X(x)\epsilon_X(y)^* = (\vartheta\hat{\otimes}_{\max}\mathrm{id}_{C^*(G)})^{-}\circ\epsilon_A({}_A\langle x|y\rangle), \qquad \forall x, y \in X,$$

where $\vartheta : A \to \mathscr{K}(X)$ is the natural isomorphism. The full coactions ϵ_A and ϵ_B, or the dynamical systems (A, G, ϵ_A) and (B, G, ϵ_B), are said to be *strongly Morita equivalent* by means of the imprimitivity system (X, ϵ_X).

LEMMA 3.2. *Suppose that* ϵ_X *is an* ϵ_A, ϵ_B*-compatible full coaction of* G *on* X. *Then we have*

(i) $\delta_X^1(x)\delta_X^1(y)^* = (\vartheta\hat{\otimes}_{\min}\mathrm{id}_{C_r^*(G)})^{-}\circ\delta_A^1({}_A\langle x|y\rangle), \qquad \forall x, y \in X,$
where $\vartheta : A \to \mathscr{K}(X)$ *is the natural isomorphism.*

(ii) I_A *is the ideal of* A *corresponding to* I_B *via the* A, B*-imprimitivity bimodule* X. *Therefore* $\dot X$ *is a Banach* $\dot A, \dot B$*-imprimitivity bimodule.*

PROOF. (i) Put $R = C_r^*(G)$ and $F = C^*(G)$. Let $g : \mathscr{L}(X\hat{\otimes}_{\max}F) \to \mathscr{L}(X\hat{\otimes}_{\min}F)$, $k : \mathscr{L}(X\hat{\otimes}_{\min}F) \to \mathscr{L}(X\hat{\otimes}_{\min}R)$ be the natural unital homomorphisms corresponding to the non-degenerate linear maps $\varrho_X : X\hat{\otimes}_{\max}F \to X\hat{\otimes}_{\min}F$ and $\mathrm{id}_X\hat{\otimes}_{\min}\lambda$:

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$X\hat{\otimes}_{\min}F \to X\hat{\otimes}_{\min}R$, respectively. Then it is easy to show that

$$g \circ (\vartheta\hat{\otimes}_{\max}\mathrm{id}_F) = (\vartheta\hat{\otimes}_{\min}\mathrm{id}_F) \circ \varrho_A;$$
$$k \circ (\vartheta\hat{\otimes}_{\min}\mathrm{id}_F) = (\vartheta\hat{\otimes}_{\min}\mathrm{id}_R) \circ (\mathrm{id}_A \otimes_{\min}\lambda);$$
$$k \circ g \circ (\vartheta\hat{\otimes}_{\max}\mathrm{id}_F)^- = (\vartheta\hat{\otimes}_{\min}\mathrm{id}_R)^- \circ (\mathrm{id}_A \otimes_{\min}\lambda)^- \circ \bar{\varrho}_A.$$

Now we have

$$\begin{aligned}
\delta_X^1(x)\delta_X^1(y)^* &= \left[(\mathrm{id}_X \hat{\otimes}_{\min}\lambda)^- \circ \bar{\varrho}_X \circ \epsilon_X(x)\right]\left[(\mathrm{id}_X \hat{\otimes}_{\min}\lambda)^- \circ \bar{\varrho}_X \circ \epsilon_X(y)\right]^* \\
&= k(\bar{\varrho}_X(\epsilon_X(x))\bar{\varrho}_X(\epsilon_X(y))^*) \\
&= (k \circ g)(\epsilon_X(x)\epsilon_X(y)^*) \\
&= k \circ g \circ (\vartheta\hat{\otimes}_{\max}\mathrm{id}_F)^- \circ \epsilon_A({}_A\langle x|y\rangle) \\
&= (\vartheta\hat{\otimes}_{\min}\mathrm{id}_R)^- \circ (\mathrm{id}_A \otimes_{\min}\lambda)^- \circ \bar{\varrho}_A \circ \epsilon_A({}_A\langle x|y\rangle) \\
&= (\vartheta\hat{\otimes}_{\min}\mathrm{id}_R)^- \circ \delta_A^1({}_A\langle x|y\rangle).
\end{aligned}$$

(ii) Recall from [8, Theorem 3.1] that the closed A, B-submodule of X corresponding to the ideal I_A is $Y = \{x \in X : {}_A\langle x|x\rangle \in I_A\}$. Recall from Lemma 2.9(i) that the closed A, B-submodule of X corresponding to the ideal I_B is V_X. By (i), we have

$$\|\delta_A^1({}_A\langle x|x\rangle)\| = \|\delta_X^1(x)\|^2, \qquad \forall x \in X.$$

Hence, $Y = V_X$. This proves (ii).

THEOREM 3.3. *Suppose that ϵ_X is an ϵ_A, ϵ_B-compatible full coaction of G on X. Then we have*

$$(1) \qquad \delta_{\dot{X}}(\dot{x})\delta_{\dot{X}}(\dot{y})^* = (\dot{\vartheta}\hat{\otimes}_{\min}\mathrm{id}_{C_r^*(G)})^- \circ \delta_{\dot{A}}({}_{\dot{A}}\langle \dot{x}|\dot{y}\rangle), \qquad \forall x, y \in X,$$

where $\dot{\vartheta} : \dot{A} \to \mathcal{K}(\dot{X})$ is the natural isomorphism. Therefore if ϵ_A and ϵ_B are strongly Morita equivalent then the corresponding ordinary coactions $\delta_{\dot{A}}$ and $\delta_{\dot{B}}$ are strongly Morita equivalent.

PROOF. The proof of (1) is very similar to that in Lemma 3.2(i). The last assertion is a consequence of Proposition 2.10, Lemma 3.2(ii) and Condition (1).

THEOREM 3.4. *Suppose that the full coactions ϵ_A and ϵ_B are strongly Morita equivalent. Then the full crossed products $A \times_{\epsilon_A} G$ and $B \times_{\epsilon_B} G$ are strongly Morita equivalent.*

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PROOF. By Theorem 3.3, the coactions $\delta_{\dot{A}}$ and $\delta_{\dot{B}}$ are strongly Morita equivalent. It then follows from [1, Proposition 6.9] (or [3, Theorem 2.16]) that the ordinary crossed products $\dot{A} \times_{\delta_{\dot{A}}} G$ and $\dot{B} \times_{\delta_{\dot{B}}} G$ are strongly Morita equivalent. We then deduce from Raeburn's theorem (Theorem 1.1) that the full crossed products $A \times_{\epsilon_A} G$ and $B \times_{\epsilon_B} G$ are strongly Morita equivalent.

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