Bull. Aust. Math. Soc. **80** (2009), 202–204
doi:10.1017/S0004972709000173

ADDENDUM

DIAGRAMS OF AN ABELIAN GROUP – ADDENDUM

THEODORE G. FATICONI

doi:10.1017/S0004972708001238, Published by The Australian Mathematical Society, 6 July 2009

The results in this addendum extend [1, Theorems 1.1 and 8.7].

Let $h > 0$ be an integer. We characterize algebraic number fields possessing class number h in terms of the sequence of rational primes.

Using the notation of [1], let $\mathbf{k}$ be an algebraic number field, let $[\mathbf{k} : \mathbb{Q}] = f$, and let $h(\mathbf{k})$ denote the class number of $\mathbf{k}$. Let $\overline{E}$ be the ring of algebraic integers in $\mathbf{k}$. Then $\overline{E}$ is a ring whose additive group $\overline{E}, +$ is a free Abelian group of finite rank f. For each rational prime p let $E(p) = \mathbb{Z} + p\overline{E}$. Let $G(p)$ be a reduced torsion-free rank-f Abelian group such that $\mathrm{End}(G(p)) \cong E(p)$. These groups exist by Butler's theorem [3, Theorem I.2.6]. There is a torsion-free reduced group $\overline{G}(p)$ of rank f such that $\overline{G}(p)/G(p)$ is finite, and $\mathrm{End}(\overline{G}(p)) = \overline{E}$.

Let $L(p) = \mathrm{card}(u(\overline{E})/u(E(p)))$ where $u(R)$ is the group of units in the ring R. For an Abelian group H let $h(H)$ be the number of isomorphism classes of groups L that are *locally isomorphic* to H. (See [3].) Sequences s_n and t_n are *asymptotically equal* if $\lim_{n\to\infty} s_n/t_n = 1$.

The main theorem of this paper follows.

THEOREM 1. *Let* $\mathbf{k}$ *be an algebraic number field, let* $[\mathbf{k} : \mathbb{Q}] = f$*, and let* $h(\mathbf{k}) = h$*. Then* $\{L(p)h(G(p)) \mid$ *rational primes* $p\}$ *is asymptotically equal to the sequence* $\{hp^{f-1} \mid$ *rational primes* $p\}$.

PROOF. In addition to the the stated notation we let:

(1) $\widehat{m}_p = \mathrm{card}(u(\overline{E}/p\overline{E}))$;

(2) $\widehat{n}_p = \mathrm{card}(u(E(p)/p\overline{E}))$;

(3) $L(p) = \mathrm{card}(u(\overline{E})/u(E(p)))$.

There are at most finitely many rational primes that ramify in $\mathbf{k}$, so let us avoid those primes. By [2, Theorem 8.4],

$$L(p)h(G(p))\frac{\widehat{n}_p}{\widehat{m}_p} = h(\overline{G}(p)). \tag{1}$$

Because $\mathrm{End}(\overline{G}(p)) = \overline{E}$, [2, Corollary 3.2] implies that $h(\overline{G}(p)) = h(\overline{E}) = h(\mathbf{k}) = h$. Hence

$$L(p)h(G(p))\frac{\widehat{n}_p}{\widehat{m}_p} = h. \tag{2}$$

Since p does not ramify in $\mathbf{k}$, there are distinct prime ideals $I_1, \ldots, I_g$ in $\overline{E}$ and integers $f_1, \ldots, f_g$ such that $\Sigma_{i=1}^{g} f_i = f$,

$$p\overline{E} = I_1 \cap \cdots \cap I_g,$$

and $[\overline{E}/I_i : \mathbb{Z}/p\mathbb{Z}] = f_i$ for each $i = 1, \ldots, g$. Then

$$\overline{E}/p\overline{E} = \frac{\overline{E}}{I_1} \times \cdots \times \frac{\overline{E}}{I_g}$$

so that

$$u(\overline{E}/p\overline{E}) = u\left(\frac{\overline{E}}{I_1}\right) \times \cdots \times u\left(\frac{\overline{E}}{I_g}\right).$$

Since $\overline{E}/I_i$ is a finite field of characteristic p,

$$\widehat{m}_p = (p^{f_1} - 1) \cdots (p^{f_g} - 1). \tag{3}$$

Since $E(p)/p\overline{E} \cong \mathbb{Z}/p\mathbb{Z}$, $\widehat{n}_p = p - 1$.

Form the polynomial of degree $f - 1$,

$$x^{f-1} + Q_p(x) = \frac{(x^{f_1} - 1) \cdots (x^{f_g} - 1)}{x - 1}. \tag{4}$$

The coefficients of $(x^{f_2} - 1) \cdots (x^{f_g} - 1)$ are multinomial coefficients $\binom{f-1}{r_1, \ldots, r_t}$ for some partitions $r_1, \ldots, r_t$ of $f - 1$. These coefficients are bounded above by $(f - 1)!$. The coefficients of $Q_p(x)$ in (4) are then bounded above by $f!$. Thus $Q_p(x)$ has degree $\leq f - 2$, and the coefficients of $Q_p(x)$ are bounded above by $f!$. Hence

$$\lim_p \frac{p^{f-1} + Q_p(p)}{p^{f-1}} = 1 + \lim_p \frac{Q_p(p)}{p^{f-1}} = 1. \tag{5}$$

Now, $p^{f-1} + Q_p(p) = \widehat{m}_p/\widehat{n}_p$ when p replaces x in (4), so by (2),

$$\frac{L(p)h(G(p))}{p^{f-1} + Q_p(p)} = L(p)h(G(p))\frac{\widehat{n}_p}{\widehat{m}_p} = h. \tag{6}$$

Furthermore,

$$\begin{aligned}\frac{L(p)h(G(p))}{p^{f-1}} &= \frac{(L(p)h(G(p))/p^{f-1})}{(L(p)h(G(p))/p^{f-1}+Q_p(p))}\cdot\frac{L(p)h(G(p))}{p^{f-1}+Q_p(p)}\\ &= \frac{p^{f-1}+Q_p(p)}{p^{f-1}}\cdot h\end{aligned}$$

by (6). Using the limit in (5) we see that

$$\lim_p \frac{L(p)h(G(p))}{hp^{f-1}} = 1.$$

Therefore, $\{L(p)h(G(p)) \mid \text{rational primes } p\}$ is asymptotically equal to $\{hp^{f-1} \mid \text{rational primes } p\}$. □

COROLLARY 2. *Let* $\mathbf{k}$ *be a quadratic number field, and let* $h(\mathbf{k}) = h$. *Then* $\{L(p)h(G(p)) \mid$ *rational primes* $p\}$ *is asymptotically equal to the sequence* $\{hp \mid$ *rational primes* $p\}$.

THEOREM 3. *Let* $\mathbf{k}$ *be an algebraic number field and let* $h > 0$ *be an integer. The following are equivalent.*

(1) $h(\mathbf{k}) = h$.
(2) *The sequence* $\{L(p)h(G(p)) \mid$ *rational primes* $p\}$ *is asymptotically equal to the sequence* $\{hp^{f-1} \mid$ *rational primes* $p\}$.

PROOF. $1 \Rightarrow 2$. This is Theorem 1.

$2 \Rightarrow 1$. The sequence $\{L(p)h(G(p)) \mid \text{rational primes } p\}$ is asymptotically equal to the sequence $\{hp^{f-1} \mid \text{rational primes } p\}$ for some integer $h > 0$. Then by Theorem 1 and part 2,

$$\lim_p \frac{L(p)h(G(p))}{h(\mathbf{k})p^{f-1}} = 1 = \lim_p \frac{L(p)h(G(p))}{hp^{f-1}}.$$

Hence $h(\mathbf{k}) = h$ which completes the proof. □

COROLLARY 4. *Let* $\mathbf{k}$ *be a quadratic number field and let* $h > 0$ *be an integer. The following are equivalent.*

(1) $h(\mathbf{k}) = h$.
(2) *The sequence* $\{L(p)h(G(p)) \mid$ *rational primes* $p\}$ *is asymptotically equal to the sequence* $\{hp \mid$ *rational primes* $p\}$.

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https://doi.org/10.1017/S0004972709000173 Published online by Cambridge University Press